

**Title:** Trigonometric identities and quadratic residues

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In this paper, we obtain some novel identities involving trigonometric functions. Let  $n$  be any positive odd integer. We mainly show that

$$\sum_{r=0}^{n-1} \frac{1}{1 + \sin 2\pi \frac{x+r}{n} + \cos 2\pi \frac{x+r}{n}} = \frac{(-1)^{(n-1)/2} n}{1 + (-1)^{(n-1)/2} \sin 2\pi x + \cos 2\pi x}$$

for any complex number with  $x + 1/2, x + (-1)^{(n-1)/2}/4 \notin \mathbb{Z}$ , and

$$\sum_{j,k=0}^{n-1} \frac{1}{\sin 2\pi \frac{x+j}{n} + \sin 2\pi \frac{y+k}{n}} = \frac{(-1)^{(n-1)/2} n^2}{\sin 2\pi x + \sin 2\pi y}$$

for all complex numbers  $x$  and  $y$  with  $x + y, x - y - 1/2 \notin \mathbb{Z}$ . We also determine the values of  $\prod_{k=1}^{(p-1)/2} (1 + \tan \pi \frac{k^2}{p})$  and  $\prod_{k=1}^{(p-1)/2} (1 + \cot \pi \frac{k^2}{p})$  for any odd prime  $p$ .

In addition, we pose several conjectures on the values of  $\prod_{k=1}^{(p-1)/2} (x - e^{2\pi i k^2/p})$  with  $p$  an odd prime and  $x$  a root of unity.

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