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A new geometry of a Lorentzian manifold

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In memory of Albert Crumeyrolle
Professor at "Paul Sabatier" University, Toulouse, France
who helped many mathematicians from Eastern Europe
with an admirable generosity during the time of the "iron-curtain"

Introduction

In General Relativity Theory a mathematical structure named Lorentzian manifold is fundamental. This structure is denoted by $[M, g]$ where M is a real C^∞-differentiable 4-dimensional paracompact manifold without boundary and g is a global C^∞-differentiable field of two times covariant symmetric tensors with diagonal form $(---+)$. This g is called the Lorentzian metric of M.

The geometry of $[M, g]$ is generally based on the concept of Minkowski norm. In a local chart of M, this norm is:

$$\forall X \in T_xM \;\Big|\; |X| = (g_{ab}(x)X^aX^b)^{1/2} = (g(x)(X,X))^{1/2},\ a,b = 1,2,3,4.$$

The Minkowski norm of T_xM and the distance on M defined by it are important concepts of Lorentzian geometry (in Beem and Ehrlich's monograph [1], 1981, consisting of eleven chapters five are dedicated to the study of this norm and distance). But in spite of their definitions, which are very similar to those of Riemannian geometry, the first is not at all a norm nor is the second a distance. Obviously, here the terminology is improperly used. The efforts to draw a parallelism between this geometry of $[M, g]$ and the geometry of a Riemannian manifold in such a way are ineffectual (see [1]). However, both functions are important and useful in General Relativity Theory.

In the present paper, for a time oriented Lorentzian manifold $[M, g]$, a new norm of T_xM ($\forall x \in M$) and consequently a new distance on M are proposed, these providing a new geometry of $[M, g]$. This new geometry is similar to the Riemannian geometry and it is also justified from a physical point of view. Some elements of this new geometry (the fundamental ones) are presented here.

The ideas come from the theory of Krein spaces (defined and studied by M. Krein in 1936). A Krein space is a mathematical structure $[B, K]$, where B is a Banach space and K is a convex pointed closed cone with nonempty interior in B (Cone = a subset K of B for which the relation $X \in K$ implies $\rho X \in K$ ($\forall \rho \in \mathbb{R},\ \rho \geq 0$). K is pointed iff X and $-X$ belong to K then $X = 0$). The Krein spaces are fundamental structures for a theory of positive operators developed after World War II by M.A. Krasnosel'skij and his school at Voronezh (Russia). Two important monographs were published ([3], M.A. Krasnosel'skij, 1964 and [4], M.A. Krasnosel'skij, J.A. Lifshits, A.V. Sobolev, 1989). Using some ideas of these works, we studied in three papers ([5], 1988; [6], 1988; [7], 1990; D.I. Papuc) the geometry of manifolds endowed with a field of tangent cones. The theory of vector bundles endowed with cone fields was studied in two other works ([8], 1992; [9], 1994; D.I. Papuc).

1. A new local geometry of a Lorentzian manifold

I.1. *Definition 1.* A Lorentzian manifold is a structure, denoted by $[M, g]$, where M is a real $(n+1)$-dimensional C^∞-differentiable connected paracompact manifold without boundary, and g is a C^∞-differential global field of tangents to M tensors, these tensors being two-times covariant symmetric and with diagonal form $(- \cdots - +)$.

For a Lorentzian manifold a tangent vector $X \in T_xM$ can be: *timelike* if $g(x)(X, X) > 0$, *isotropic* if $g(x)(X, X) = 0$, and *nonspacelike* (*nontimelike*) if $g(x)(X, X) \geq 0$ (≤ 0).

Given a Lorentzian manifold $[M, g]$, $C(x)$ denotes the set of nonspacelike tangent vectors from T_xM, i.e.

$$(1) \qquad C(x) = \{X \in T_xM \mid g(x)(X, X) \geq 0\}, \quad \forall x \in M.$$

Every set $C(x)$ is determined by a degenerate hyperquadric $g(x)(X,X)=0$ and it is composed of two subsets, denoted by $C^+(x)$ and $C^-(x)$, which are convex pointed closed cones with nonempty interior in the topological vector space T_xM. These cones satisfy the relations: $C^+(x)\cup C^-(x) = C(x)$, $C^+(x)\cap C^-(x)=\{0\}$, $X\in C^+(c)$ iff $-X\in C^-(x)$.

2. *Definition 2.* A locally time-normalized Lorentzian manifold is a structure $[M,g]$ for which an open covering $\{U\}$ of M is fixed, every subset U being a connected geometric zone of a local chart of M, and for every U a C^∞-differentiable vector field Z defined on U of tangent timelike vectors is also fixed.

A locally time-normalized Lorentzian manifold will be denoted by $[M,g;\{(U,Z)\}]$. Obviously, every Lorentzian manifold can be endowed with a structure of locally time-normalized Lorentzian manifold.

One has:

$$\forall x\in U \mid g(x)(Z(x),Z(x))>0. \tag{2}$$

3. *The local cone field of nonspacelike tangent vectors*. Given a locally time-normalized Lorentzian manifold $[M,g;\{(U,Z)\}]$, for every pair (U,Z), in each point of U, the field Z determines one of the cones $C^+(x)$, $C^-(x)$. This is the cone $C^+(x)$ or $C^-(x)$ which contains the vector $Z(x)$. This cone denoted by $K(x)$ is defined by:

$$\begin{aligned} K(x)=\{X\in T_xM \mid (X\neq 0\wedge g(X,X)\geq 0 \\ \wedge\, g(Z(x),X)>0)\vee(X=0)\}. \end{aligned} \tag{3}$$

In such a way a local map

$$K: x\in U\to K(x)\subset T_xM\subset T(M) \tag{4}$$

is defined. This map verifies the axioms A_1, A_2 from [8], p. 39:

A_1. For every $x\in U$ the set $K(x)\subset T_xM$ is a convex pointed closed cone with a non-empty interior (in the topological vector space T_xM);

A_2. The sets $\bigcup_{x\in U}\operatorname{Int}K(x)$ and $\bigcup_{x\in U}(T_xM\setminus K(x))$ are open subsets of $T(M)$.

Consequently the open submanifold U of M is endowed with a field of tangent cones. One can consider the structure $[(T(U), p, U); K]$, where $T(U), p, U)$ is the tangent vector bundle of U and K is the map defined by (4). All results exposed in [8] for an arbitrary structure $[(E, p, M); K]$ can be applied for $(T(U), p, U); K]$. Thus for every pair (U, Z), a new geometry of the Lorentzian manifold $[U, g_{|U}]$ is obtained. In Section II some fundamental elements of this new geometry will be presented.

Remark 1. It is easy to see that for every local pair (U, Z) the structure $[(T(U), p, U); K]$ is uniquely determined by M, g, U and Z.

Remark 2. The same field K can be determined by any element of the set of all Lorentzian metrics globally conformal to g, i.e. by an arbitrary element of the set

$$C(M, g) = \{\rho g : M \to (0, \infty),\ \rho \text{ is } C^{\infty}\text{-differentiable}\}.$$

Remark 3. In order to study the structures $[(T(U), p, U); K]$ determined by the pairs $\{(U, Z)\}$, a preferential atlas of the manifold M may be used, its local charts being characterised by the following: the geometric zones of these local charts are open sets $\{U\}$ and in every such chart $Z = \partial/\partial x^{n+1}$ (the local Frobenius theorem).

Obviously, in virtue of (2), in a preferential local chart $g_{n+1\,n+1} > 0$ holds.

The transformation of coordinates between two preferential local charts having the same open set U as geometric zone is given by:

$$\begin{aligned} x'^{i} &= x'^{i}(x^1, x^2, \dots, x^n), \quad i = 1, 2, \dots, n \\ x'^{n+1} &= x'^{n+1}(x^1, x^2, \dots, x^n) + x^{n+1}. \end{aligned} \tag{5}$$

Remark 4. In order to study a fixed structure $[(T(U), p, U); K]$ in any of the local preferential charts specified above, the Lorentzian metric tensor g (which together with U and Z determines the cone field K) will be replaced by the local tensor $g \cdot (g_{n+1\,n+1})^{-1} = g^*$. Obviously $g^*_{n+1\,n+1} = 1$.

Remark 5. In a local preferential chart associated to the pair (U, Z) one has

$$g^*(Z, Z) = g^*_{n+1\ n+1} = 1,\ g^*(Z, X) = g^*_{n+1\,i} X^i + X^{n+1},\ i = 1, \dots, n \tag{6}$$

II. Using the results concerning the geometry of a structure $[M, K; Z]$, ([5], [6], [7]) for a locally time-normalized Lorentzian manifold, i.e. for a structure $[M, g; \{(U, Z)\}]$, a new geometry is obtained. We are going to present some elements of this new geometry.

For an arbitrary fixed pair (U, Z) the following geometric elements are considered:

a) The structure $[(T(U), p, U); K]$ which was presented in I.3. $(T(U), p, U)$ is the tangent vector bundle of the manifold U and K is the cone field determined by (3) and (4).

The interior of the cone $K(x)$ is given by

$$\text{Int } K(x) = \{X \in T_xM \mid (g(X,X) > 0 \wedge g(Z(x), X) > 0)\}. \tag{7}$$

b) An ordering relation for the elements of T_xM, $\forall x \in U$:

$$\forall X, Y \in T_xM \mid X \leq Y \Rightarrow Y - X \in K(x). \tag{8}$$

$$X \leq Y \Leftrightarrow (X = Y) \vee \Big[(X \neq Y) \wedge \big(g(X,X) + g(Y,Y) \geq 2\, g(X,Y)\big) \wedge \big(g(Z(x), Y\big) > g\big(Z(x), X)\big)\Big].$$

The pair $(T_xM, \leq)$ is an ordered vector space, directed on both sides.

c) A pseudonorm of T_xM.
$\forall X \in T_xM$ and $\forall Y \in \text{Int}\, K(x)$ we have:

$$X \notin \{\rho Y \mid \rho \in \mathbb{R}\} \Leftrightarrow (g(Y,X))^2 - g(Y,Y)g(X,X) > 0. \tag{9}$$

If $X \in T_xM$ one puts

$$D(X) = [(g(Z(x), X))^2 - g(Z(x), Z(x))g(X,X)]^{1/2}(g(Z(x), Z(x)))^{-1}, \tag{10}$$

then the map

$$D(.) : X \in T_xM \to D(X) \in \mathbb{R} \tag{10'}$$

is a pseudonorm on T_xM, i.e.

$$D(X) \geq 0,\ D(\rho X) = |\rho| D(X),\ D(X) + D(Y) \geq D(X+Y). \tag{11}$$

Notice that $D(X) = 0 \Leftrightarrow X \in \{\rho\, Z(x) \mid \rho \in \mathbb{R}\}$.

In a local preferential chart of the pair (U, Z) (see (6))

$$D(X) = ((g^*(Z(x), X))^2 - g^*(X, X))^{1/2} \tag{12}$$

holds.

d) By means of the ordering relation defined by (8) the fundamental function ν is defined ([7], p. 44, 1992, D.I. Papuc):

$$\nu : (Z, X) \in \bigcup_{x \in U} (\operatorname{Int} K(x) \times T(x)) \to \nu(Z, X) \in \mathbb{R}^2, \tag{13}$$

where

$$\nu(Z, X) = (\alpha(Z, X), \beta(Z, X)) \in \mathbb{R}^2,$$

and the real numbers are determined by

$$\begin{aligned} \alpha(Z, X) &= \min\{\lambda \in \mathbb{R} \mid X \geq \lambda Z\}, \\ \beta(Z, X) &= \max\{\lambda \in \mathbb{R} \mid \lambda Z \geq X\}, \end{aligned} \tag{14}$$

or equivalently

$$\begin{aligned} \alpha(Z, X) &= g(Z, X)(g(Z, Z))^{-1} + D(X), \\ \beta(Z, X) &= g(Z, X)(g(Z, Z))^{-1} - D(X). \end{aligned} \tag{14'}$$

e) A norm on T_xM. For every $x \in U$ the map $|\ .\ |_Z$: $X \in T_xM \to |X|_Z \in \mathbb{R}$, where $T_x(U) = T_xM$ and

$$|X|_Z = \min\{\lambda \in \mathbb{R} \mid \lambda \geq 0,\ -\lambda Z(x) \leq X \leq \lambda Z(x)\} \tag{15}$$

or equivalently

$$\begin{aligned} |X|_Z &= \max\{|\alpha(Z(x), X)|, |\beta(Z(x), X)|\} \\ &= |g(Z(x), X)|(g(Z(x), Z(x))^{-1} + D(X) \\ &= |g(Z(x), X)|(g(Z(x), Z(x))^{-1} + [(g(Z(x), X))^2 \\ &\quad - g(Z(x), Z(x))g(X, X)]^{1/2}(g(Z(x), Z(x))^{-1} \end{aligned} \tag{15'}$$

is a norm on the tangent vector space T_xM.

The norm $| \, . \, |_Z$ is invariant with respect to the replacement of the tensor g by the tensor $g^* = \rho g$, where $\rho \in \{\rho : U \to (0,\infty),\ \rho$ is C^∞-differentiable$\}$ and it is determined by $Z(x)$ and $K(x)$ or by $Z(x)$ and any tensor field g^*.

If one replaces Z by ρZ, (where $\rho : U \to (0,\infty), \rho$ is C^∞) then

$$\forall X \in T_x M \ \big| \ |X|_{\rho Z} = (\rho(x))^{-1} |X|_Z. \tag{16}$$

Therefore we can say that the vector $Z(x)$ serves as a unit of measure in $T_x M$.

In a local preferential chart (see (6)) one has

$$|X|_Z = |g^*(Z(x), X)| + [(g(Z(x), X))^2 - g^*(X, X)]^{1/2}.$$

Geometrical and physical interpretation of the norm: If for an arbitrary $X \in T_x M$ we put $X = \rho Z(x) + X_1, g(Z(x), X_1) = 0$ then

$$\begin{gathered} g(Z(x), X)(g(Z(x), Z(x))^{-1} = \rho \\ D(X) = (-g(X_1, X_1)(g(Z(x), Z(x))^{-1})^{1/2}. \end{gathered} \tag{17}$$

By means of (15') one obtains

$$|X|_Z = |\rho| + [-g(X_1, X_1)(g(Z(x), Z(x))^{-1}]^{1/2} \tag{18}$$

and thus the following geometrical and physical interpretations of the norm determined by (15) arise:

Geometrical interpretation: $|X|_Z$ is the sum of $|\rho|$ and of the quotient of two norms: the norm of X_1 determined by $-g$ and the Minkowski norm of $Z(x)$ determined by g (both norms are real positive numbers).

Physical interpretation: if for $n = 3$ we consider the Minkowski 4-dimensional space $[T_x M, g(x)]$ and in a local chart of M, normal in x (i.e. in this chart for the tensor $g(x)$ the relations $g_{ij} = -\delta_{ij}$, $g_{n+1\, i} = 0$, $g_{n+1\, n+1} = c^2$, $i, j = 1, 2, 3$ hold) we put $Z(x) = (0, 0, 0, 1/c)$, where c is the speed of light, then for an arbitrary vector $X(X^1, X^2, X^3, t) \in T_x M$ the norm $|X|_Z$ is the sum of euclidean distances $|ct|$ and $(\delta_{ij} X^i X^j)^{1/2}$.

f) Linear connections associated to a pair (U, Z). A linear connection defined on U, for which the cone field K and the vector field Z are invariant in the parallel displacement along any curve in U, will have to satisfy the relations

$$g_{ab|c} + (\partial(\log\rho)/\partial x^c)g_{ab},\ Z^a_{|b} = 0,\ \rho : U \to (0,\infty)$$
$$a, b, c = 1, \dots, n+1. \tag{19}$$

Using any of the preferential local charts defined in the Remark 3 (1.I), one gets $\Gamma^a_{n+1c} = 0$ and $\rho = (g_{n+1\,n+1})^{-1}$. Therefore in these preferential local charts, the conditions (19) are equivalent to

$$\begin{gathered}\rho = (g_{n+1\,n+1})^{-1}\\ \Gamma^a_{n+1\,b} = 0\\ \partial g_{i\,n+1}/\partial x^a + (\partial \log\rho/\partial x^a)g_{i\,n+1} = \Gamma^b_{ia}g_{b\,n+1},\\ \partial g_{ij}/\partial x^a + (\partial \log\rho/\partial x^a)g_{ij} = \Gamma^b_{ia}g_{bj} + \Gamma^b_{ja}g_{ib},\\ i, j = 1, \dots, n;\ a, b = 1 \dots, n+1.\end{gathered} \tag{20}$$

This is a linear system of $n(n+1)(n+3)/2$ equations with $n(n+1)^2$ unknown functions $\{\Gamma^a_{ib}\}$. At least $(n^2-1)/2$ functions will remain undetermined. Consequently there are linear connections associated to an arbitrary pair (U, Z).

Remark that if a tangent vector field X defined on U is parallel in the parallel transport determined by a linear connection associated to the pair (U, Z) then the functions $\rho g(Z, X)$ and $\rho g(X, X)$ are constant in this parallel transport. Therefore all geometric relations and concepts defined above (the relation $X \le Y$, see (8), the pseudonorm D, see (10) and (10'), the function ν, see (14) and (14'), and the norm $|\,.\,|_Z$, see (15) and (15')) are invariant in the parallel transport determined by these linear connections.

If we try to find a symmetric linear connection which satisfies (19) then the following conditions must be verified by the tensor field g : $\partial(g_{ia}/g_{n+1\,n+1})/\partial x^{n+1} = 0$. For the symmetric linear connection Γ also $\Gamma^a_{b\,n+1} = 0$ holds true. The linear system (20) supplies a linear system for Γ of $n(n+1)(n+2)/2$ equations with $n(n+1)^2/2$ unknown functions. The number of equations is greater than the number of functions by $n(n+1)/2$. Obviously, a symmetric linear connection for which the field of cones K and the vector field Z are invariant in the parallel displacement along any curve of M, generally does not exist.

2. Some particular Lorentzian manifolds

1. Time-normalized space–time manifolds

Definition 3. A space–time manifold is a Lorentzian manifold M for which there is a global C^∞-differentiable tangent vector field Z of timelikes vectors, i.e. a global tangent vector field Z for which

$$\forall x \in M \;\Big|\; g(x)(Z(x), Z(x)) > 0. \tag{21}$$

A time-normalized space–time manifold is a space–time manifold for which a global C^∞-differentiable tangent vector field Z of timelike vectors is fixed.

A time-normalized space–time manifold will be noted by $[M, g; Z]$.

If for a structure $[M, g; Z]$ the pair $(U = M, Z)$ is considered, then the geometry of this pair (U, Z) developed in the first part will be the geometry of the structure $[M, g; Z]$. In this manner the fundamental elements of a new geometry of a time-normalized space–time manifold are obtained. Note that in this case for the manifold M one may use a preferential atlas so that in the local charts of this atlas the tangent vector field Z is represented by $Z = \partial/\partial x^{n+1}$. The coordinate transformations for the local charts of this atlas are (5).

2. Almost Minkowskian manifolds

Definition 4. An almost Minkowskian manifold is a time-normalized space–time manifold $[M, g; Z]$ for which the differential system ("distribution" in Chevalley sense) $g(Z, dx) = 0$ defined on M is totally integrable (involutive).

A geometrical interpretation of the condition for a time-normalized space–time manifold to be an almost Minkowskian manifold is the following: for every $x \in M$ the linear n-dimensional subspace of T_xM determined by the differential system $g(Z, dx) = 0$ is the correspondent of $Z(x)$ in the polarity of T_xM determined by the hyperquadric $g(X, X) = 0$. The n-dimensional distribution determined this way must be totally integrable.

An almost Minkowskian manifold is characterised as follows:

Proposition. *The necessary and sufficient condition that a time-normalized space–time manifold $[M, g; Z]$ be an almost Minkowskian manifold is: for every point $x \in M$ there exists an admissible local chart of M in which for the fundamental tensor $g^* = g(g_{n+1\,n+1})^{-1}$ and the vector field Z the conditions*

$$g^*_{n+1\,n+1} = 1,\ g^*_{n+1\,a} = 0, \quad a = 1, \dots, n,\ Z = \partial/\partial x^{n+1} \tag{22}$$

hold true.

Proof. Conditions (22) are necessary. Indeed, let $[M, g; Z]$ be an almost Minkowskian manifold. For an arbitrary $x \in M$ there is an admissible local chart of M such that x belongs to the geometric zone of this chart and the vector field Z will be represented in this chart by $Z = \partial/\partial x^{n+1}$ (local Frobenius theorem). We also replace the fields of tensors g by $g^* = g(g_{n+1\,n+1})^{-1}$. Obviously $g^*_{n+1\,n+1} = 1$. Also, in this local chart the differential system from the Definition 4 will be given by the Pfaff's equation $g^*(Z, dx) = g^*_{n+1\,i}dx^i = g^*_{n+1\,a}dx^a + dx^{n+1} = 0$. This differential system is integrable if and only if the following conditions are satisfied:

$$\begin{aligned} &g^*_{n+1\,i}(\partial g^*_{n+1\,j}/\partial x^{n+1}) - g^*_{n+1\,j}(\partial g^*_{n+1\,i}/\partial x^{n+1}) \\ &+\partial g^*_{n+1\,i}/\partial x^j - \partial g^*_{n+1\,j}/\partial x^i = 0, \quad i = 1, \dots, n. \end{aligned} \tag{23}$$

The differential system being integrable, it admits n-dimensional integral manifold which, in the considered admissible local chart, is given by:

$$x^a = u^a, \quad x^{n+1} = f(u^1, \dots, u^n), \quad a = 1, \dots, n$$

where $(x^1, \dots, x^n, x^{n+1})$ belongs to the arithmetical zone of this local chart. The images in T_xM of vectors $\{\partial/\partial u^i; i = 1, \dots, n\}$ are vectors $X_i = \partial/\partial x^i + (\partial f/\partial x^i)\partial/\partial x^{n+1}$, $i = 1, \dots, n$. For these vectors $g^*(Z, dx)(X_i) = g^*_{n+1\,i} + \partial f/\partial x^i = 0$ holds. Hence: $g^*(Z, dx) = d(x^{n+1} - f(x^1, \dots, x^n))$. If one proceeds to the change of local coordinates $x'^a = x^a$, $x'^{n+1} = -f(x^1, \dots, x^n) + x^{n+1}$, one obtains a new preferential local chart, with the same geometrical zone, in which $g^*(Z, dx) = dx'^{n+1}$ and consequently (22) will be satisfied.

Conditions (22) are sufficient. Indeed, let $[M, g; Z]$ be a time-normalized space–time manifold. We assume that for every point x of M there is an admissible local chart of M in which, for the tensorial field g^* and Z, the conditions (22) are satisfied. By virtue of the second condition (22), the differential system determined by the Pfaff's equation $g^*(z,dx) = g^*_{n+1\,i}dx^i = dx^{n+1} = 0$ is integrable and thus the condition from Definition 4 is satisfied.

Corollary. *Let $[M,g;Z]$ be an almost Minkowskian manifold. A local chart of the manifold M in which the relations* (22) *are satisfied is called a preferential local of M. A transformation of coordinates between two preferential charts is given by*

$$\begin{aligned} x'^{i} &= x'^{i}(x^1, x^2, \dots, x^n), \quad i = 1, \dots, n \\ x'^{n+1} &= x^{n+1} + c. \end{aligned} \tag{24}$$

For the almost Minkowskian manifolds, in preferential local charts, the cones $K(x)$ are determined by:

$$\forall x \in M \mid K(x) = \{X \in T_xM \mid X^{n+1} \geq 0, (X^{n+1})^2 - g^*_{ab}X^aX^b \geq 0\} \tag{25}$$

and the fundamental functions α and β by:

$$\forall (Z,X) \in \bigcup (\mathrm{Int}\, K(x) \times T_xM) \;\Big|\; \begin{aligned} \alpha(Z,X) &= X^{n+1} + (-g^*_{ab}X^aX^b)^{1/2} \\ \beta(Z,X) &= X^{n+1} - (-g^*_{ab}X^aX^b)^{1/2} \\ a,b &= 1, \dots, n. \end{aligned} \tag{26}$$

The norm determined by the global vector field Z of an arbitrary tangent vector $X \in T_xM$ is supplied by:

$$\begin{aligned} |X|_Z &= \max\{|X^{n+1} + (-g^*_{ab}X^aX^b)^{1/2}|, |X^{n+1} - (-g^*_{ab}X^aX^b)^{1/2}|\} \\ &= |X^{n+1}| + (-g^*_{ab}X^aX^b)^{1/2}. \end{aligned} \tag{27}$$

3. Local Minkowskian manifolds

Definition 5. A local Minkowskian manifold is a Lorentzian manifold $[M,g]$ for which there is an admissible atlas A of M such that in any local chart of A the tensor field g satisfies the conditions:

$$g_{ij} = -\delta_{ij}, \; g_{n+1\,i} = 0, \; g_{n+1\,n+1} = 1, \; i,j = 1, \dots, n. \tag{28}$$

Every local Minkowskian manifold is an almost Minkowskian manifold (the conditions (22) are satisfied, the global vector field Z being given in a local chart which belongs to A by $\partial/\partial x^{n+1}$).

The local charts of the atlas A from Definition 5 are preferential local charts of the considered local Minkowskian manifold. The coordinate transformation determined by two preferential local charts is:

$$\begin{aligned} x'^{i} &= a^i_j x^j + a^i, \quad \delta_{ij} a^i_h a^j_k = \delta_{hk}, \quad i,j,h,k = 1,\dots,n, \\ x'^{n+1} &= x^{n+1} + a. \end{aligned} \tag{29}$$

A local Minkowskian manifold M for which there exists a preferential chart whose geometrical zone is M and whose arithmetical zone is $\mathbb{R}^{n+1}$ is a Minkowskian manifold (a homogeneous Minkowskian space).

3. Example

We consider a Lorentzian manifold $[M,g]$ for which in a local chart $h(U,\chi)$ the tensor field g is determined by:

$$\begin{gathered} g_{ij} = -(1+W)\delta_{ij},\ g_{i4} = 0,\ g_{44} = (1-W)c^2;\ i,j = 1,2,3; \\ W = (k/4\pi)\int \sigma r^{-1} dV_0 \end{gathered} \tag{30}$$

where c is the speed of light ([2]; A. Einstein, 1955).

If one considers on U the vector field $Z(0,0,0,c^{-1})$, then for an arbitrary vector field $X(X^1,X^2,X^3,X^4)$ defined on U, one has:

$$\begin{aligned} &g(Z,Z) = 1-W,\ g(Z,X) = c(1-W)X^4, \\ &g(X,X) = -(1+W)((X^1)^2+(X^2)^2+(X^3)^2)+c^2(1-W)(X^4)^2. \end{aligned} \tag{31}$$

Hence by virtue of (12),

$$\begin{aligned} \alpha(Z,X) &= cX^4 + \Delta((X^1)^2+(X^2)^2+(X^3)^2)^{1/2}, \\ \beta(Z,X) &= cX^4 - \Delta((X^1)^2+(X^2)^2+(X^3)^2)^{1/2} \end{aligned} \tag{32}$$

holds, where $\Delta = (1+W)^{1/2}(1-W)^{-1/2}$. Also, (15) implies:

$$|X|_Z = c|X^4| + \Delta((X^1)^2+(X^2)^2+(X^3)^2)^{1/2}. \tag{33}$$

An arbitrary linear connection, for which the field of cones K and the vector field Z are invariant in the parallel transport determined by this connection, depends on 12 unknown real functions defined on U. A necessary condition for the existence of such a linear connection which is symmetric is that W does not depend on $x^4 = t$. The number of linear conditions which have to be satisfied by the coefficients of this symmetric connection (in the considered local chart) is greater by 6 than the number of coefficients.

Conclusions

In the study of an arbitrary time-normalized space–time manifold $[M, g; Z]$, or more generally, in the study of a locally time-normalized space–time manifold one can use the norm proposed by (3) and (4). This norm allows us to considerv canonical parameters on the curves in M and, subsequently, a length of a curve and a distance on the manifold M. We obtain a metric geometry of (locally) time-normalized space–time manifold $[M, g; Z]$ which appears more natural than the "metric" geometry determined by means of Minkowsky metric. If for two Observers the fundamental fields Z's are different (for a locally time-normalized space–time manifold the vector fields Z's are defined only locally!) the General Relativity Theory with all its concepts will remain the same, the Lorentzian transformations being involved. If Z is globally defined, then those Lorentzian transformations (transformations which conserve the fundamental cones) which conserve also the fundamental field Z, are Galilean transformations.

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