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Some remarks on central idempotents in group rings

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Abstract. Let G be a group, K a field of characteristic 0, and T the set of all elements of finite order in G. In this note we give necessary and sufficient conditions under which every idempotent of KT is central in KG.

1. Introduction

Let G be a group and K a field of characteristic 0. We denote by $U(KG)$ the group of units of the group ring of G over K. Also, if X is a group, we shall denote by $T(X)$ the *torsion* of X; i.e., the set of all elements of finite order in X. The study of group theoretical properties of $U(KG)$ has lead, on occasions, to the condition that $T = T(G)$ is a subgroup and that every idempotent of KT is central in KG. In what follows we study this condition and prove the following

Theorem 1. *Let K be a field of characteristic 0 and let T be the set of elements of finite order of a group G. Then, every idempotent of KG with support in T is central in KG if and only if the following conditions hold:*

(i) For every element $t \in T$ and every $x \in G$ there exists a positive integer j such that $xtx^{-1} = t^j$.

(ii) If $t \in T$ is an element of order n and ζ_n is a primitive nth root of unity over K, then, for each exponent j obtained as in (i), there exists a map $\sigma \in \mathrm{Gal}(K(\zeta_n) : K)$ such that $\sigma(\zeta_n) = \zeta_n^j$.

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(iii) *T is either an abelian group or a Hamiltonian group such that for each element $t \in T$ of odd order k, the field $K(\zeta_k)$ contains no non-trivial solution of the equation $x^2 + y^2 + z^2 = 0$.*

This problem was first studied in [1] where, due to an oversight, condition (ii) was stated in the following weaker form:

(ii') For every non-central element t of order n, K contains no root of unity of order n.

This condition is actually not sufficient, as is shown below. The authors are grateful to Prof. E. Jespers, who spotted the oversight and suggested this counterexample.

Example. Let $G = \langle x, y \mid yxy^{-1} = x^5,\ x^8 = 1\rangle$; let ζ be a primitive root of unity of order 8 and set $K = \mathbb{Q}(\zeta + \zeta^{-1})$.

Write:

$$e_1 = \frac{1}{8}\sum_{i=0}^{7}(\zeta x)^i \quad \text{and} \quad e_2 = \frac{1}{8}\sum_{i=0}^{7}(\zeta^{-1}x)^i.$$

Then, e_1, e_2 are orthogonal idempotents in $\mathbb{Q}(\zeta)\langle x\rangle$ and thus $e = e_1 + e_2 \in K\langle x\rangle$ is also an idempotent. Notice that the coefficient of x in e is equal to $\sqrt{2}/8$ and the coefficient of x^5 is equal to $-\sqrt{2}/8$. It follows that the coefficient of x in the element yey^{-1} is $-\sqrt{2}/8$ so $e \neq yey^{-1}$ and thus e is not central in KG. Since K is a real field, it contains no root of unity other than ± 1 and it is easily seen that conditions (i) and (iii) are also satisfied.

2. Proof of the Theorem

The fact that conditions (i) and (iii) are necessary follows as in [1]. To prove (ii), first notice that, given an element $t \in T$ we can write $K\langle t\rangle$ as a direct sum

$$K\langle t\rangle \cong K_1 \oplus \cdots \oplus K_s$$

with $K_i = (K\langle t\rangle)e_i \cong K(\zeta_i)$, where e_i, $1 \leq i \leq s$, is the set of principal idempotents of $K\langle t\rangle$, each ζ_i denotes a root of unity and at least one of them, ζ_1 say, is such that $o(\zeta_1) = o(t)$. Also notice that, in the isomorphism above, the element t corresponds to $(\zeta_1, \ldots, \zeta_s)$.

Since every idempotent of KT is central in KG, it follows that conjugation by an element $x \in G$ induces an automorphism $\theta : K\langle t\rangle \to K\langle t\rangle$ which, in turn, induces automorphisms θ_i on each simple component K_i, $1 \le i \le s$.

Each K_i contains $\overline{K}_i = Ke_i$, which is an isomorphic copy of K and, since $xtx^{-1} = x^j$ for some positive integer j, we see that, in particular, $\theta_1 : K(\zeta_1) \to K(\zeta_1)$ fixes K and is such that $\theta_1(\zeta_1) = \zeta_1^j$ so $\theta_1 \in \mathrm{Gal}(K(\zeta_1) : K)$, as desired.

To prove sufficiency we use the same methods as in [1] showing that they work also in the present case. Assume first that T is an abelian group and let $e \in KT$ be an idempotent. When considering $\mathrm{supp}(e)$ we may assume that T is finite. Furthermore, since every idempotent is a sum of primitive idempotents, we may restrict ourselves to the case where e is itself primitive. We wish to show that, for each fixed element $x \in G$, we have $xex^{-1} = e$.

Write $T = \langle t_1\rangle \times \cdots \times \langle t_s\rangle$, a direct product of cyclic groups and set $t_0 = t_1 \cdots t_s$. Then $xt_0x^{-1} = t_0^j$ for some positive integer j and thus also $xtx^{-1} = t^j$, for all $t \in T$. Notice that $o(t_0)$, the order of t_0, is equal to the exponent of T. So if ζ is a primitive root of unity whose order is equal to $o(t_0)$, then $K(\zeta)$ is a splitting field for T. Hence, $e \in KT \subset K(\zeta)T$ is a sum of primitive idempotents of $K(\zeta)T$. Let f be one of these idempotents.

Every K-automorphism of $K(\zeta)$ extends in a natural way to an automorphism of $K(\zeta)T$. We define:

$$H = \{\phi \in \mathrm{Gal}(K(\zeta) : K) \mid \phi(f) = f\}$$

and take $\phi_1 = I$, $\phi_2, \dots, \phi_r$ a transversal of H in $\mathrm{Gal}(K(\zeta) : K)$.

Set $e* = \phi_1(f) + \cdots + \phi_r(f)$. Exactly as in [1], it can be shown that $e* = e$.

According to [5, Theorem 2.12], we can write f in the form:

$$f = \frac{1}{|T|}\sum_{t \in T} \chi(t^{-1})t$$

where χ is an irreducible character of T with values in $K(\zeta)$.

Then,

$$e = \frac{1}{|T|} \sum_{t \in T} \left(\sum_i \phi_i \left(\chi(t^{-1}) \right) \right) t$$

so

$$xex^{-1} = \frac{1}{|T|} \sum_{t \in T} \left(\sum_i \phi_i \left(\chi(t^{-1}) \right) \right) t^{-j}.$$

Thus, our result will follow if we show that $\sum_i \phi_i \left(\chi(t^{-1}) \right) = \sum_i \phi_i \left(\chi(t^{-j}) \right)$. Since $K(\zeta)$ is a splitting field for T, we have that $\chi(t)$ is a root of unity of order dividing $\exp(T)$, so we may assume that $\chi(t) = \xi$, where ξ is some power of ζ.

By (ii), there exists a map $\sigma \in \mathrm{Gal}(K(\zeta) : K)$ such that $\sigma(\zeta) = \zeta^j$, which we extend in the natural way to an automorphism of $K(\zeta)T$ and denote also by σ. Since $\{\phi_i\}_{1 \le i \le r} = \{\phi_i \circ \phi\}_{1 \le i \le r}$, we have that:

$$\sum_i \phi_i \left(\chi(t^{-1}) \right) = \sum_i \phi_i(\bar{\xi}) = \sum_i \phi_i \circ \phi(\bar{\xi}) = \sum_i \phi_i(\xi^{-j}) = \sum_i \phi_i \left(\chi(t^{-j}) \right)$$

as we intended to prove.

The case where T is Hamiltonian now follows as in [1].

3. Final comments

We recall that the supercenter of a group G over a field K is defined as the set $S = S_K(G)$ of all elements in G having a finite number of conjugates in $U(KG)$, the group of units of KG. This subgroup was studied in [2] and its description, in the case where $\mathrm{char}(K) = 0$, was obtained using the theorem on central idempotents given above. Though the statement of [2, Theorem C] is correct, it can now be stated in a more precise form:

Theorem 2. *Let K be a field of characteristic 0 and let G be a non-torsion group. Then one of the following statements holds:*

(i) $S = \mathcal{Z}(G)$, the centre of G.

(ii) $T(S)$ is an abelian group such that for all $t \in T(S)$ and all $x \in G$ we have that $xtx^{-1} = t^j$ for some positive integer j and, if ζ is a

primitive root of unity of order $o(t)$*, then there exists an element* $\sigma \in \mathrm{Gal}(K(\zeta) : K)$ *such that* $\sigma(\zeta) = \zeta^j$*. Furthermore, if* $T(S)$ *is infinite, then* $T(S) = \mathbb{Z}(q^\infty) \times B$ *where* q *is a prime rational integer,* B *is finite central in* G*,* $\mathbb{Z}(q^\infty)$ *is central in* S*,* $(G, S) \subset \mathbb{Z}(q^\infty)$ *and there exists a positive integer* k *such that* K *does not contain roots of unity of order* q^k*.*

Also, [3, Theorem 3.2] can now be stated as follows.

Theorem 3. *Let* G *be a nilpotent or FC group and let* K *be a field of characteristic* 0*. Then* $TU(KG)$ *is a subgroup if and only if the following conditions hold:*

(i) T *is abelian.*

(ii) For each $t \in T$ *and each* $x \in G$*, there exists a positive integer* j *such that* $xtx^{-1} = t^j$ *and, if* ζ *is a primitive root of unity of order* $o(t)$*, then there exists a map* $\sigma \in \mathrm{Gal}(K(\zeta) : K)$ *such that* $\sigma(\zeta) = \zeta^k$*.*

Theorem 1 was used in [4] to obtain a similar result in a nonassociative context, which should now be stated as follows:

Theorem 4. *Let* L *be an RA loop with torsion subloop* T *and let* K *be a field of characteristic* 0*. Then* $TU(KL)$ *is a subloop if and only if the following conditions hold:*

(i) T *is an abelian subgroup.*

(ii) For each $t \in T$ *and each* $x \in L$*, there exists a positive integer* i *such that* $xtx^{-1} = t^i$*.*

(iii) For each noncentral element $t \in T$ *and each* $x \in L$*, there exists a positive integer* j *such that* $xtx^{-1} = t^j$ *and, if* ζ *is a primitive root of unity of order* $o(t)$*, then there exists a map* $\sigma \in \mathrm{Gal}(K(\zeta) : K)$ *such that* $\sigma(\zeta) = \zeta^j$*.*

The proofs are essentially the same as the original ones, requiring only minor changes.

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