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The group of units of a group algebra of characteristic p

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1. Introduction

In this paper we give a survey of results and unsolved problems concerning the group of units of a group algebra of characteristic p. The results presented here are chosen from among the most recent ones and are even partly unpublished, in order to describe the present state of research in this area.

The study of the group of units is one of the classical topics in group rings and started in the early 40s with the papers of Higman [1, 2] and Jennings [1]. The most fundamental characterization theorems were published during the 70s. Sehgal's book [4] gave an overview on this topic of the results known to us in the late 70s, and proposed 42 problems. In the next decade this book determined the directions of research to a great extent.

Research on the groups of units of group rings is a meeting point of concepts, techniques and problems of group theory and ring theory, since one can examine ring theoretical properties of the group ring and group theoretical properties of its group of units, and in both cases one encounters the need for tools from both theories. Lie properties of group algebras were described during the 70s. These results were applied first to the unit group by Bovdi and Khripta [3], who obtained the description of group algebras

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of torsion groups with solvable group of units. Clearly, Lie commutators are considerably easier to calculate than group commutators and one may think of obtaining information on the group of units by looking at the associated Lie algebra. It has turned out that, in most cases, the Lie structure reflects well the characteristics of the group of units and there is a close relationship between properties of these two.

Since then, only the surveys by A. Bovdi [10], Dennis [2], Giambruno [1], Polcino Milies [4] and Shalev [1, 2] have appeared on certain questions concerning the group of units of group algebras over fields of characteristic p, not counting the isomorphism problem, on which Sandling gave a detailed survey [3]. Since Giambruno's paper gives a good overview of Hartley's problem on the group identities of the group of units of group algebras, related questions will not be discussed here. Külshammer [1] gives group-theoretical descriptions of ring-theoretical invariants of group algebras of finite groups.

2. Basic notions

If $\mathbb{F}$ is a field of characteristic p and the group G contains an element of order p then $\mathbb{F}G$ is called a *modular group algebra*.

Clearly, all units of the group ring $\mathbb{F}G$ form a group called the *group of units* of $\mathbb{F}G$ and denoted by $U(\mathbb{F}G)$. The subgroup

$$V(\mathbb{F}G) = \Big\{ u = \sum_{g\in G} a_g g \in U(\mathbb{F}G) \;\Big|\; \sum_{g\in G} a_g = 1 \Big\}$$

is called the *group of normalized units* of $\mathbb{F}G$.

Obviously, G is a subgroup of $V(\mathbb{F}G)$ and $U(\mathbb{F}G) = V(\mathbb{F}G) \times U(\mathbb{F})$.

Often it is difficult to determine whether a given element in a group ring is a unit.

Suppose that $a, b \in \mathbb{F}G$. A group algebra $\mathbb{F}G$ is said to be von Neuman finite if $ab = 1$ implies $ba = 1$. Kaplansky proved that if $\mathbb{F}$ is a field of characteristic zero, then $\mathbb{F}G$ is von Neuman finite. The corresponding question in characteristic p is open.

One can easily see that elements of the set $U(\mathbb{F})G = \{cg \mid c \in U(\mathbb{F})$ $g \in G\}$ are units in $\mathbb{F}G$, called the *trivial units*.

It is possible that the group algebra $\mathbb{F}G$ has only trivial units. For instance, $V(\mathbb{F}G) = G$ if one of the following conditions holds:

1. $\mathbb{F}$ is the field of two elements and G is the group of order 2 or 3;
2. $\mathbb{F}$ is an arbitrary field and G is a u.p.-group (Strojnowski [1]).

Definition. If $V(\mathbb{F}G) = G$, then the group $V(\mathbb{F}G)$ is called *trivial*.

It is an old and difficult problem, due to Kaplansky [1, 2] and Mal'cev, to determine whether $V(\mathbb{F}G)$ is trivial for a torsion-free group G. This is true for torsion-free nilpotent groups, because they are u.p.-groups, but the problem is open for torsion-free supersolvable groups.

Clearly, if x is a nilpotent element and $x^n = 0$, then the element $1+x$ has the inverse $1 - x + x^2 - x^3 + \cdots + (-1)^{n-1}x^{n-1}$. Units of form $1 + x$ will be called *unipotent*.

Unipotent units of the following form play an important role in the investigation of $V(\mathbb{F}G)$. Let $a \in G$ be an element of order n and put $\bar{a} = 1 + a + a^2 + \cdots + a^{n-1}$. If $g \in G$ and $g \notin N_G(\langle a \rangle)$, then $(a-1)g\bar{a}$ is a nonzero nilpotent element of index 2. Hence, $u_{a,g} = 1 + (a-1)g\bar{a}$ is a unit called a *bicyclic unit*. The subgroup of $V(\mathbb{F}G)$ generated by all bicyclic units is denoted by $B_2(\mathbb{F}G)$.

Let $x = \sum_{g \in G} \alpha_g g \in \mathbb{F}G$. Then the subset $\mathrm{Supp}(x)=\{g \in G \mid \alpha_g \neq 0\}$ is called the support of x, and the subgroup $\langle \mathrm{Supp}(x) \rangle$ of G generated by $\mathrm{Supp}(x)$ is called the support subgroup of x.

Question (Sandling). Suppose that G is infinite and $x \in V(\mathbb{F}G)$. How does the support of x^{-1} depend on that for x?

Pappas [1] obtains a very interesting result on the support of $*$-symmetric units x (i.e. $x^* = x$, where $*$ is the classical involution). Friedman, Gustavson and Papas [1] shows that if G contains an abelian subgroup of finite index, then for each finite set $X \subset G$ there is a finite set $Y(X) \subset G$ with the following property: for each unit X such that $\mathrm{Supp}(x) \subseteq X$ the support of x^{-1} is a subset of $Y(X)$.

Wef finish this part with the following remark: several results in this surveys and on the isomorphism problem work only for a prime field $\mathbb{F}_p$. It would be nice to see everything extended to any field of characteristic p.

3. Units for locally finite groups

Let G be a locally finite p-group and let $\mathbb{F}$ be a field of characteristic p. Then the group of normalized units $V(\mathbb{F}G)$ is a locally finite p-group which coincides with $1 + \mathfrak{I}_{\mathbb{F}}(G)$, where $\mathfrak{I}_{\mathbb{F}}(G)$ is the augmentation ideal of $\mathbb{F}G$. Furthermore,

1. if G is finite then $V(\mathbb{F}G)$ is nilpotent;

2. if $\mathbb{F}G$ is finite, then $V(\mathbb{F}G)$ is of order $|\mathbb{F}|^{|G|-1}$.

In a natural way the following question arises: when do we have

$$V(\mathbb{F}G) = 1 + \mathfrak{I}_{\mathbb{F}}(G)$$

i.e., when does $\mathbb{F}G$ have "many" units? The answer to this question is known only in certain cases. For a p-group G of finite exponent $V(\mathbb{F}G) = 1 + \mathfrak{I}_{\mathbb{F}}(G)$ implies that G is locally finite. Indeed, in this case by one of its characterizations the Jacobson radical of $\mathbb{F}G$ is $\mathfrak{I}_{\mathbb{F}}(G)$. Using Zelmanov's theorem [1, 2] from Lichtman's result [1] we conclude that the group G is necessarily locally finite.

4. Units for abelian groups

Now let G be a finite abelian p-group and $\mathbb{F}_p$ the field of p elements. Then the structure theorem on $V(\mathbb{F}_pG)$ is well-known. In the direct decomposition of the group $V(\mathbb{F}_pG)$ the number of cyclic groups of order p^i is $f_i(V) = |G^{p^{i-1}}| - 2|G^{p^i}| + |G^{p^{i+1}}|$. Moreover, let $G = \langle a_1\rangle \times \langle a_2\rangle \times \cdots \times \langle a_t\rangle$ and let a_i be of order q_i. Denote by $L(G)$ the set of all t-tuples of integers $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_t)$ with components satisfying $0 \le \alpha_i < q_i$ and the property that at least one of them is not divisible by p. Define $u_\alpha = 1 + (a_1-1)^{\alpha_1}(a_2-1)^{\alpha_2}\dots(a_t-1)^{\alpha_t}$. By Sandling's result [2]

$$V(\mathbb{F}_pG) = \prod_{\alpha\in L(G)} \langle u_\alpha\rangle.$$

A. Bovdi and Szakács [2] extend this result of Sandling, and give a basis for $V(\mathbb{F}G)$ over an arbitary finite field $\mathbb{F}$.

The study the group of units $V(\mathbb{F}G)$ for infinite abelian p-group G was started by Berman and May. Let $\mathbb{F}$ be perfect field of characteristic p and let G be an arbitary abelian p-group. If $\mathbb{F}G^{p^\alpha}$ is infinite then it is easy to calculate the Ulm–Kaplansky invariant

$$f_\alpha(V(\mathbb{F}G)) = \begin{cases} \max\left\{|\mathbb{F}|, \left|G^{p^\alpha}\right|\right\} & \text{if } G^{p^\alpha} \ne G^{p^{\alpha+1}}, \\ 0 & \text{if } G^{p^\alpha} = G^{p^{\alpha+1}}. \end{cases}$$

It is an open question whether G is a direct factor of $V(\mathbb{F}_pG)$ and whether $V(\mathbb{F}_pG)/G$ is totally projective.

Hill and Ullery in [1] answer this conjecture affirmatively for a group G which is an isotype subgroup of a direct product of countable reduced p-groups.

5. Normal complement in the group of units

Definition. Let M be an ideal of the group ring $\mathbb{F}_pG$. Then the subgroup $M^+ = \{u \in V(\mathbb{F}_pG) \mid u - 1 \in M\}$ is called a *congruence subgroup* of $V(\mathbb{F}_pG)$.

The existence of a normal complement N to G in $V(\mathbb{F}_pG)$ is still an open problem, (see Dennis [2]). Moreover, one may ask: is there any congruence subgroup M^+ such that $V(\mathbb{F}_pG) = M^+ \rtimes G$? Ivory [1] gave the first example showing that a decomposition $V(\mathbb{F}_pG) = G \ltimes N$ need not exist. It happens, for example, when G is a dihedral group, a semidihedral group, or a generalized quaternion group of order greater than 8. We emphasize:

Question. Does the normal complement to G exist for $p \neq 2$?

It is easy to verify (see, for example A. Bovdi [3]) that if G is a circle group (i.e. G is isomorphic to the adjoint group of a certain Jacobson radical ring R of characteristic p with respect to the circle operation $a \circ b = a + b + ab$ with $a, b \in R$), then $V(\mathbb{F}_pG) = M^+ \rtimes G$.

Using this statement the following earlier results can easily be obtained, because all these groups are cicrle groups:

1. (Moran and Tench [1], Bovdi [2, 3]). *G is a nilpotent p-group of class* 2 *and exponent p*;
2. (A. Bovdi [2, 9]). *G is a nilpotent group of class* 2 *and exponent* 4.

The normal complement problem was discussed in detail in Roggenkamp and Scott's paper [2], especially for $p = 3$, which was an important motivation for Sandling's result [4]: *if the commutator subgroup of the finite p-group G is central and elementary abelian, then there exist a normal complement N for G in $V(\mathbb{F}_pG)$.* However, the question about the existence of a normal complement which is a congruence subgroup of some ideal is still open even in this case.

6. Conjugacy classes

For modular group algebras we know only few results on the conjugacy classes of the units of the group algebra $\mathbb{F}_pG$. Studies in this direction were initiated by Coleman [1].

Theorem. *Let H be a normal subgroup of the finite p-group G and let $\mathbb{F}$ be a field of characteristic p. Then*

$$N_{V(\mathbb{F}G)}(H) = G \cdot C_{V(\mathbb{F}G)}(H).$$

We obtain a corollary for a finite p-group G:

1. $N_{V(\mathbb{F}G)}(G) = G \cdot \zeta(V(\mathbb{F}G))$;
2. *if the conjugacy class C of $V(\mathbb{F}G)$ contains an element of G then $C \cap G$ is a conjugacy class in G;*
3. *the group $V(\mathbb{F}G)$ possesses an outer automorphism of order p.*

Let $a \in \mathbb{F}G$ and $C_a = \{x^{-1}ax \mid x \in V(\mathbb{F}G)\}$. Rao and Sandling [2] proved that p^2 divides the order $|C_a|$ for every noncentral element $a \in \mathbb{F}G$. We improve this result as follows.

Theorem. *Let G be a nonabelian finite p-group and let $\mathbb{F}$ be a field of characteristic p. If u is a noncentral element in $\mathbb{F}G$ then the conjugacy subset C_u has the following properties:*

1. *if $\mathbb{F}$ is an infinite field then C_u is infinite;*
2. *if the commutator subgroup of G is a subgroup of $C_G(u)$, $p \neq 2$ and $|\mathbb{F}| = p^t$, then $p^{t+1} \bigm| |C_u|$;*
3. *if $|\mathbb{F}| = 2^t$, the commutator subgroup of the 2-group G is a subgroup of $C_G(u)$ and $G/C_G(u)$ is a cyclic group then $2^{t+1} \bigm| |C_u|$;*
4. *if $p = 2$ and $2^r = \max\{4, |\mathbb{F}|\}$ then $2^r \bigm| |C_u|$.*

Clearly, if $|C_u| = p$ then the commutator subgroup of G is a subgroup of $C_G(u)$ and by it is imposible. It is reasonable to ask about the order of C_u for which the commutator subgroup of G is not a subgroup of $C_G(u)$.

If G is a finite p-group and $\mathbb{F}$ is a field of characteristic p then $V(\mathbb{F}G)$ has a conjugacy class C_u which contains no element of G. Therefore, the analogue of the first conjecture of Zassenhaus for modular group algebras is not true.

The study the group of units of group rings with finite conjugacy classes was started by Sehgal and Zassenhaus [1]. They gave answers for group algebras when $\mathbb{F}$ is a field of characteristic 0 and G has no subgroups of type p^∞. The case when $\mathbb{F}$ is an infinite field of characteristic $p \neq 2$ was studied by Polcino Milies [1, 2] and the problem was solved by Cliff and Sehgal [1]. A more detailed characterization is given Coelho and Polcino Milies [1].

Theorem. *Let $\mathbb{F}G$ be a noncommutative infinite modular group algebra of characteristic p. The group of units $V(\mathbb{F}G)$ is an FC-group (a group with all conjugacy classes finite) if and only if $p = 2$, $|G'| = 2$, $T(G) = G' \times A$, where A is a finite central subgroup of odd order.*

For nonmodular group algebras the result is more interesting.

Let G be an arbitary group. We define the subgroup

$$\Delta(G) = \{u \in G \mid |G : C_G(u)| < \infty\},$$

called the FC-center of G. Clearly, if $G = \Delta(G)$ then G is an FC-group.

The subgroup $S_{\mathbb{F}}(G) = \Delta(U(\mathbb{F}G)) \cap G$ is called the $\mathbb{F}$-supercenter of G. This concept was introduced by Sehgal and Zassenhaus for integral group rings. Clearly, the center of G is a subgroup of $S_{\mathbb{F}}(G)$ and for an infinite group algebra $\mathbb{F}G$ of a torsion group G they coincide.

Polcino Milies and Sehgal [1] studied the $\mathbb{F}$-supercenter and this investigation was completed in a paper of Coelho and Polcino Milies [1]. Considering the case $S_{\mathbb{F}}(G) = G$ we are able to describe fully those $\mathbb{F}$ an G for which $U(\mathbb{F}G)$ is an FC-group.

V. Bovdi [1] describes the FC-center $\Delta(U)$ of the unit group $U(\mathbb{F}G)$ for arbitary infinite group algebras $\mathbb{F}G$. Remember that by Neumann's theorem the set of all elements of finite order $T(\Delta(U))$ is a subgroup of $\Delta(U)$ containing all elements of finite order of $\Delta(U)$. By V. Bovdi's result the elements of the commutator subgroup of $T(\Delta(U))$ are unipotent and central in $\Delta(U)$. Therefore, $\Delta(U)$ is a solvable group of length at most 3, and the subgroup $T(\Delta(U))$ is nilpotent of class at most 2.

7. The center

Let G be a finite p-group and let C_g be the conjugacy class of G containing $g \in G$. The element $\widehat{C_g} = \sum_{g \in C_g} g$ of the group algebra $\mathbb{F}G$ is called a *class sum.* Clearly, the class sums form a basis for the center of $\mathbb{F}G$.

Definition. Let G be a finite p-group and $|C_g| > 2$. The *depth of the conjugacy class* C_g is the nonnegative integer d which satisfies the conditions that $C_G(g) = C_G(g^{p^d})$ and $C_G(g^{p^d})$ is a proper subgroup of $C_G(g^{p^{d+1}})$.

Recall that the centralizer of any element of a non-singleton conjugacy class C_g is called the *defect group of the conjugacy class*.

Clearly, the defect group of a conjugacy class is unique up to conjugation as centralizers of conjugate elements are conjugate subgroups.

A. Bovdi and Patay [2] describe the center $\zeta(V)$ of $V(\mathbb{F}_pG)$ for a finite p-group G. For groups G with zero depth of any of the conjugacy classes, this is Sehgal's result [1].

Theorem. *Let $\mathbb{F}_pG$ be the group algebra of a finite p-group G. Denote by P_i $(i = 1, 2, \ldots, s)$ the defect groups of conjugacy classes of G. For an arbitrary integer l denote by $R_l(P_i)$ the set of conjugacy classes of the form $C_{g^{p^l}}$ with defect group P_i for which the depth of the conjugacy class C_g is not less than l. Furthermore, let $C = \zeta(G)$ be the center of G, and $q_{i,l}$ the order of $\zeta(P_i)^{p^l}[p]$.*

In the direct decomposition of the center $\zeta(V)$ of $V(\mathbb{F}_pG)$ the number of the cyclic groups of order p^l is

$$f_l(\zeta(V)) = \left|C^{p^{l-1}}\right| - 2\left|C^{p^l}\right| + \left|C^{p^{l+1}}\right|$$
$$+\sum_{i=1}^{s}\left(|R_{l-1}(P_i)| - |R_l(P_i)|\frac{1+q_{i,l-1}}{q_{i,l-1}} + \frac{|R_{l+1}(P_i)|}{q_{i,l}}\right).$$

Therefore, we know the invariants of $\zeta(V)$ but it is very hard to find a basis for this subgroup. How can we supplement the Sandling basis of $V(\mathbb{F}_p\zeta(G))$ to a basis of $\zeta(V)$?

The Ulm–Kaplansky invariants of the center of $V(\mathbb{F}G)$ for some infinite p-group G over a field of characteristic p have been determined in A. Bovdi and Patay [2], Nachev and Mollov [1]. Let G be a p-group and denote by $E(G)$ the set of all elements $g \in \Delta(G)$ such that g has an infinite height in the center of the centralizer $C_G(g)$ of g. Under the assumption that $E(G)$ is a subgroup, A. Bovdi and Patay [3] describe the first Ulm-subgroup and the maximal divisible subgroup of the center of the unit group $V(\mathbb{F}G)$. Every nilpotent and FC-group satisfies these properties.

8. Sections of the group of units

Suppose that R is a commutative ring of characteristic p, H is a finite normal subgroup of the group G and p divides $|H|$. If $\widehat{H} = \sum_{h\in H} h$ then $\widehat{H}$

is central in RG, $\widehat{H}^2 = 0$ and $1+x\widehat{H} \in V(RG)$ for any $x \in RG$. Obviously, $A_H = \{1+x\widehat{H} \mid x \in RG\}$ is a commutative subgroup of $V(RG)$ and since $y^{-1}(1+x\widehat{H})y = 1+y^{-1}xy\widehat{H} \in A_H$ for any $y \in V(RG)$, A_H is a normal abelian subgroup.

Let $T_l(G/H)$ be a complete system of coset representatives of the subgroup H in G, and $\chi(x)$ the augmentation of x. Any $x \in RG$ can be written as $x = \sum_i u_i x_i$, where $x_i \in RH$ and $u_i \in T_l(G/H)$. By $x_i\widehat{H} = \chi(x_i)\widehat{H}$ we have

$$1+x\widehat{H} = 1+\sum_i \chi(x_i)u_i\widehat{H} = \prod_i \left(1+\chi(x_i)u_i\widehat{H}\right).$$

From this we obtain the direct sum decomposition

$$A_H = \prod_{u_i \in T_l(G/H)} \langle 1+\lambda u_i\widehat{H} \mid \lambda \in R\rangle,$$

and it is easy to see that the group $\langle 1+\lambda u_i\widehat{H} \mid \lambda \in R\rangle$ is isomorphic to the additive group of the ring R.

Evidently, the abelian group A_H is of exponent p, and A_H is central if and only if G' is a subgroup of H.

From this direct sum decomposition of A_H we obtain the next useful result.

Lemma. *Let R be a commutative ring of characteristic p, and let H be a finite normal subgroup of the group G of order divisible by p. Fix an element $g \in G$ such that the centralizer $C = C_G(g\widehat{H})$ is a proper subgroup of G and let $L = N_G(C)$ be the normalizer of C in G. If*

$$B = \langle 1+\lambda u_i^{-1} g u_i\widehat{H} \mid \lambda \in R,\, u_i \in T_l(L/C)\rangle,$$

then

$$B = \prod_{u_i \in T_l(L/C)} \langle 1+\lambda u_i^{-1} g u_i\widehat{H} \mid \lambda \in R\rangle,$$

C is a normal subgroup of $B \cdot L$, and BL/C is isomorphic to the wreath product of the additive groups of the ring R and L/C.

Coleman and Passman [1] proved that a wreath product of two groups of order p is involved in $V(\mathbb{F}_pG)$ (i.e. there is a subgroup of $V(\mathbb{F}_pG)$ which can be mapped homomorphically onto the wreath product $C_p \wr C_p$). If the commutator subgroup of finite p-groups is not cyclic, the next assertion is just the theorem of Coleman and Passman.

Corollary. *Let G be a nilpotent group which has a finite normal subgroup H with the characteristic of the field $\mathbb{F}$ dividing $|H|$ and assume that H does not contain the commutator subgroup of G. Then $R \wr (L/C)$ is involved in $V(\mathbb{F}G)$.*

Obviously, a nonabelian 2-group H does not contain a wreath product of two groups of order 2 if and only if the set $\Omega(H) = \{x \in H \mid x^2 = 1\}$ is an abelian subgroup. In the next theorem V. Bovdi and Dokuchaev describe all 2-groups G for which $V(\mathbb{F}G)$ does not contain a wreath product of two groups of order 2.

Theorem. *Let $\mathbb{F}$ be a field of characteristic 2, and let G be a finite nonabelian 2-group. Then all elements of order two of $V(\mathbb{F}G)$ generate an abelian subgroup $\Omega(V(\mathbb{F}G))$ if and only if one of the following holds:*

1. *$G = S_{n,m} = \langle a, b \mid a^{2^n} = b^{2^m} = 1,\ a^b = a^{1+2^{n-1}},\ n,m \geq 2\rangle$, or $G = Q_8$;*
2. *G is a direct product of the quaternion group of order 8 and a cyclic group of order 2^n, or the direct product of two quaternion groups of order 8;*
3. *G is the semidirect product of the cyclic group $\langle d \mid d^{2^n} = 1\rangle$ with the quaternion group $\langle\, a, b \mid a^4 = 1,\ a^2 = b^2,\ a^b = a^{-1}\rangle$ such that $(a,d) = d^{1+2^{n-1}}$ and $(b,d) = 1$;*
4. *G is the central product of the group $S_{2,2} = \langle\, a, b \mid a^4 = b^4 = 1,\ a^2 = (b,a)\,\rangle$ with the quaternion group of order 8, the nontrivial element common to the two central factors being a^2b^2;*
5. *G is isomorphic to*

$$H_{32} = \Big\langle x, y, u \mid x^4 = y^4 = x^{-2}(y,x) = 1,\ y^2 = u^2 = (u,x),\ x^2y^2 = (u,y)\Big\rangle\,.$$

Let G be a finite 2-group and let a, b be elements of order two of $V(\mathbb{F}G)$. Then the subgroup generated by a and b in $V(\mathbb{F}G)$ is a dihedral subgroup of order 2^{n+1}. Clearly, there exists a dihedral subgroup of maximal order 2^{t+1} in $V(\mathbb{F}G)$ and this t we call *the dihedral depth* of $V(\mathbb{F}G)$.

It would be of interest to investigate the dihedral depth of $V(\mathbb{F}G)$. Above conditionons were established for the dihedral depth of $V(\mathbb{F}G)$ to be equal to 1.

The description of nonabelian p-groups G for $p \neq 2$ with the following property is given by Baginski [1]: $V(\mathbb{F}_pG)$ does not contain a wreath product of two groups of order p if and only if

$$G = \langle a,\, b \mid a^{p^m} = b^{p^n} = 1,\ b^{-1}ab = a^{p^{n-1}+1},\ n > 1,\ m > 1\rangle.$$

Several extensions of the Coleman–Passman theorem have been obtained recently. They all show that, under suitable conditions on G certain larger wreath products are involved in $V(\mathbb{F}G)$. Shalev [4] has developed methods of constructing larger wreath products inside the group of units. A. Mann [1] applies these methods to prove that for finite p-groups with $|G'| > p^2$ and $p \neq 2$, the group $U(\mathbb{F}_pG)$ involves a wreath product $C_p \wr A$, where $|A| = p^2$. We do not know whether it suffices to assume $|G'| \geq p^2$.

The following conjecture of Shalev seems very hard: *Does $V(\mathbb{F}G)$ always possess a section isomorphic to a wreath product of a cyclic group of order p and the commutator subgroup G'?*

Shalev [4] establishes this for a finite p-group ($p > 2$) with a cyclic commutator subgroup, and Konovalov [1] for finite 2-groups of maximal class. Shalev [7] establishes the following result which may be considered as an asymptotic generalization of the Coleman–Passman theorem:

Theorem. *For any prime p there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that if G is a finite p-group with the order of the commutator subgroup greater than or equal to $p^{f(k)}$, then $V(\mathbb{F}_pG)$ involves a wreath product of a cyclic subgroup of order p and an abelian group of order p^k.*

9. Normal and subnormal subgroups

We start by considering the question: Determine all subgroups H of G such that H is normal in $U(\mathbb{F}G)$. This question has been considered by Pearson [1, 2] for finite groups G and in a more general situation by Bovdi and Khripta [1].

Theorem. *Let R be an algebra over a field $\mathbb{F}$ of characteristic p and suppose that the elements of a normal subgroup H of $U(R)$ are linearly independent over $\mathbb{F}$. Then all torsion elements form a subgroup $T(H)$. Suppose that H is noncentral. Then:*

1. *if $T(H)$ is a nontrivial subgroup and it contains no element of order p then $T(H)$ is a subgroup of type q^∞ and $H/\langle b\rangle$ is a central subgroup of $U(R)/\langle b\rangle$, where b is an element of prime order q;*
2. *if $T(H)$ is nonabelian then H is a dihedral group of order 6 and $\mathbb{F}$ is the field of two elements;*
3. *if $T(H)$ is an abelian subgroup and contains an element of order p then $T(H)$ is a subgroup of order 2, the factor group $H/T(H)$ is central in $U(R)/T(H))$, and $\mathbb{F}$ is the field of two elements.*

The final answer describing when G is normal in $U(\mathbb{F}G)$ was given by Cliff and Sehgal [2].

Theorem. *Let $\mathbb{F}$ be a field of characteristic p and let G be a nonabelian group with torsion elements. Then, G is normal in $U(\mathbb{F}G)$ if and only if one of the following conditions holds:*

1. *G is the dihedral group of order 6 and $\mathbb{F}$ is the field of two elements;*
2. *$T(G)$ is a subgroup of order 2, $G' = T(G)$ and $\mathbb{F}$ is the field of two elements;*
3. *$T(G)$ is a subgroup of type q^∞, $p \neq q$ and G' is of prime order q and the following conditions is satisfied:*
   a) *whenever $T(G)$ has an element of order q^n the q^nth cyclotomic polynomial $\Phi_{q^n}(x)$ is irreducible over $\mathbb{F}$;*
   b) *either $T(G)$ is central in G or $\mathbb{F}$ is the field of two elements, $T(G)$ is a subgroup of order 3 or 5, and if $g \in G$ does not centralize T then $g^{-1}tg = t^{-1}$ for all $t \in T(G)$.*

The next question of determining subgroups H of G which are subnormal in $U(\mathbb{F}G)$ has been considered by Pearson and Taylor [1] for a finite group G, and Gonçalves [4] for a torsion group G over an infinite field $\mathbb{F}$ assuming that the subgroup H is finite or nilpotent.

The next theorem was proved by A. Bovdi and Khripta [1] for abelian normal subgroups. This is true also in a more general situation.

Theorem. *Let RG be the group ring of a p-group G over the integral domain R of characteristic p with a nontrivial group of units $U(R)$. Let A be a normal subgroup of $V(RG)$ such that any two elements of A, which are conjugate in $V(RG)$, commute. Then*

1. *if $p \neq 2$ then A^p and $A \cap G$ are central subgroups in $V(RG)$;*
2. *if $p = 2$ then A^2 and $(A \cap G)^2$ are central subgroups in $V(RG)$;*
3. *the cardinality of the maximal elementary subgroup $A^p[p]$ in A^p is not less then the cardinality of $U(R)$.*

As it is well-known, in a nilpotent group A of class 2 any conjugates commute.

Corollary. *Let RG be the group ring of a p-group G over the integral domain R of characteristic p with $U(R)$ nontrivial. If $\zeta(V)$ is the center of $V(RG)$ and A is a normal subgroup of $V(RG)$ such that $A/A \cap \zeta(V)$ is abelian then A^p and $(A \cap G)^2$ are central in $V(RG)$.*

Corollary. *Let G be a p-group with finite commutator subgroup and let $\mathbb{F}$ be a field of characteristic p. Then the factors $\zeta_{i+1}(V(\mathbb{F}G))/\zeta_i(V(\mathbb{F}G))$ of the upper central series*

$$1 \subset \zeta_1(V(\mathbb{F}G)) \subset \zeta_2(V(\mathbb{F}G)) \subset \cdots \subset \zeta_s(V(\mathbb{F}G)) = V(\mathbb{F}G)$$

of the nilpotent group $V(\mathbb{F}G)$ are elementary abelian p-groups for all $i \geq 1$.

The previous results suggest the following conjecture: *If G is a finite p-group, $\mathbb{F}$ is a field of characteristic p and N is a normal subgroup of $V(\mathbb{F}G)$ then $N^p \subseteq N' \cdot \zeta(V(\mathbb{F}G))$ where N' is the commutator subgroup and $\zeta(V(\mathbb{F}G))$ is the center.*

The next theorem may be true for groups G generated by torsion elements.

Theorem. *Let R be an infinite commutative local ring which has no zero divisors and let G be a torsion group. If $\Delta(G)$ is a FC-center of G and the characteristic of R does not divide the order of any element of the subgroup $\Delta(G)$ then*

1. *if the characteristic of R does not equal two then every abelian normal subgroup of $U(RG)$ is central;*
2. *if the characteristic of R does not divide 6 then every solvable normal subgroup of $U(RG)$ is central.*

10. Generators of the group of units

Let G be a finite p-group. We gave an algorithm describing a construction of a generating system for the group $V(\mathbb{F}_pG)$ which coincides with Sandling's basis when the p-group G is abelian. Until now, the non-abelian case has not been studied, with the exception of the 2-groups of order at most 32 (Sandling [5] and Rao [1]).

Recall that the dimension subgroups D_n of $\mathbb{F}_pG$ form a central series. Let

$$D_i/D_{i+1} = \prod_{j=1}^{d_i} \langle u_{ij}D_{i+1}\rangle$$

be the direct decomposition into a product of cyclic groups of order p.

Now we begin the construction of a generating system for the group $V(\mathbb{F}_pG)$. For this, we will choose a special dimension basis $\{u_{ij} \mid j = 1,\dots,d_i,\ i = 1,2,\dots,s\}$ of the group G. Simultaneously, we define a sequence of integers $\ell_1,\ell_2,\dots,\ell_s$ with the property that for $j \le \ell_i$ the element u_{ij} is a p-th power of an element u_{km} for some $k < i$. We do this in the following way:

Step 1: Choose the elements $u_{11},u_{12},\dots,u_{1d_1} \in D_1$ so that their cosets form a basis for D_1/D_2 and set $\ell_1 = 0$.

Step 2: By a property of dimension subgroups we know that the elements u_{1i}^p belong to D_2 $(i = 1,\dots,d_1)$, but some of them can belong to a smaller group D_k. For $k \ge 2$, let us choose the elements $\{u_{k1},u_{k2},\dots\}$ so that $\{u_{k1}D_{k+1},u_{k2}D_{k+1},\dots\}$ is a maximal linearly independent subset of $\{u_{1i}^pD_{k+1} \mid u_{1i}^p \in D_k\}$. We denote by ℓ_2 the number of elements in the set $\{u_{21},u_{22},\dots\}$. Usually these elements form only a part of a basis for D_2/D_3. We supplement $\{u_{21},u_{22},\dots,u_{2\ell_2}\}$ arbitrarily with elements $u_{2(\ell_2+1)},u_{2(\ell_2+2)},\dots,u_{2d_2}$ such that the corresponding cosets form a basis for D_2/D_3.

Step 3: We know that

$$\{u_{1i}^{p^2} \mid i = 1,\dots,d_1\} \cup \{u_{2i}^p \mid i = 1,2,\dots,d_2\} \subseteq D_3,$$

but some of these elements can belong to a smaller subgroup D_k $(k \ge 3)$. For $k \ge 3$ we complement the subset $\{u_{k1},u_{k2},\dots\}$ constructed in Step 2 with some elements from

$$\left(\left\{u_{1i}^{p^2} \mid i = 1,\dots,d_1\right\} \cup \{u_{2i}^p \mid i = 1,2,\dots,d_2\}\right) \cap D_k$$

so that their cosets form a maximal independent subset of D_k/D_{k+1}. In particular, for $k = 3$ we obtain the elements $\{u_{31}, u_{32}, \dots, u_{3\ell_3}\}$. If this set happens to be empty we let $\ell_3 = 0$. Again, we complement this set with $u_{3(\ell_3+1)}, u_{3(\ell_3+2)}, \dots, u_{3d_3}$ so that the cosets obtained form a basis for D_3/D_4.

In general, in Step t we do the following. Consider all the elements

$$B_t = \left\{ u_{ij}^{p^{t-i}} \mid 1 \le j \le d_i,\ 1 \le i \le t-1 \right\}.$$

For $k \ge t$ we complement the subset $\{u_{k1}, u_{k2}, \dots\}$ constructed in Step $(t-1)$ with some elements from $B_t \cap D_k$ so that their cosets form a maximal linearly independent system.

Again, we complement $\{u_{t1}, \dots, u_{t\ell_t}\}$ with $u_{t,\ell_t+1}, \dots, u_{t,d_t}$ so that the obtained cosets form a basis for D_t/D_{t+1}.

The basis $\{u_{i,j}\}$ so constructed will be called a dimension p-basis.

If the element

$$w = \prod_{k=1}^{s} \prod_{j=1}^{d_k} (u_{kj} - 1)^{y_{kj}} \ (0 \le y_{kj} < p) \tag{1}$$

has indices of its factors in the lexicographic order then we call it regular.

Let w be a regular element of the form (1). Suppose w is expressed in terms of a dimension p-basis of a finite group G. If w contains at least one nonzero exponent y_{kj} with index j greater than the number ℓ_k determined in the p-basis then we call w an α-regular element.

Let us compare our construction with that of Sandling, when G is an abelian p-group. Then $G = \langle u_{11} \rangle \times \langle u_{12} \rangle \times \cdots \times \langle u_{1d_1} \rangle$ and $G = D_1$, $G^p = D_2 = \ldots = D_p$, $G^{p^2} = D_{p+1} = \ldots = D_{p^2}, \ldots$. Let $\{u_{ij}\}$ be a dimension p-basis of the group G. If $i > 1$, then $\ell_i = d_i$ and $u_{ij} = u_{1k}^{p^{r(i,j)}}$ for some k. Therefore if w is of the form (1) and α-regular then $y_{1j} \ne 0$ for some j. Thus we may write it in the form $(u_{11} - 1)^{\beta_1} (u_{12} - 1)^{\beta_2} \dots (u_{1d_1} - 1)^{\beta_{d_1}}$ which has the following properties: $0 \le \beta_i < o(u_{1i})$ and p does not divide β_j. These are precisely the elements from Sandling's basis for $V(\mathbb{F}_p G)$ which we have defined before.

Theorem. *Let G be a finite p-group and let $v_1, v_2, \dots, v_n$ be all α-regular elements contained in a dimension p-basis, constructed as above, for* *a group G. Then $\{1+v_i \mid i=1,2,\dots,n\}$ is a generating system for the group $V(\mathbb{F}_pG)$. Moreover, if G is abelian then this system is minimal.*

In the next step, using commutator calculations, we give a new algorithm to exclude some elements from this generating system.

We apply this algorithm to some 2-groups and metacyclic p-groups.

Theorem. *Let G be one of the following noncommutative groups:*

1. $D_{2^n} = \langle a, b \mid a^{2^n} = b^2 = 1,\ b^{-1}ab = a^{-1}\rangle$;
2. $Q_{2^n} = \langle a, b \mid a^{2^n} = 1,\ b^2 = a^{2^{n-1}},\ b^{-1}ab = a^{-1}\rangle$;
3. $D^{-}_{2^n} = \langle a, b \mid a^{2^n} = b^2 = 1,\ b^{-1}ab = a^{2^{n-1}-1}\rangle$.

Then $V(\mathbb{F}_2G)$ is generated by the units

$$\left\{a,\ b,\ 1+(a+1)^{4k+1}(b+1) \;\middle|\; 0 \le k \le 2^{n-2}-1\right\}.$$

It is a very difficult question to determine when the group of units $V(\mathbb{F}G)$ is finitely generated. Krempa in [1] gives an overview of this question. We recall only Mirowicz's [1] results: Let D_∞ be an infinite dihedral group. Then the groups of units $U(\mathbb{F}_2D_\infty)$ and $U(\mathbb{F}_3D_\infty)$ are not finitely generated.

11. Involutions in group rings

Let R be a commutative ring and the mapping $x \mapsto x^*$ an anti-isomorphism of RG of order 2, i.e. a bijective mapping with

$$(x+y)^* = x^* + y^*, \quad (xy)^* = y^*x^* \quad \text{and} \quad (x^*)^* = x$$

for any $x, y \in RG$, called an *involution* of RG.

Examples of involutions:

1. Let f be a homomorphism of G into the unit group $U(R)$ of the commutative ring R with identity. Then the mapping defined by

   $$x = \sum_{g\in G} \alpha_g g \mapsto x^f = \sum_{g\in G} \alpha_g f(g) g^{-1}$$

   is an involution of RG. In particular, if $f(g)=1$ for all $g \in G$, then the mapping $x = \sum_{g\in G}\alpha_g g \mapsto x^* = \sum_{g\in G}\alpha_g g^{-1}$ is an involution of RG.

2. Let G be a finite 2-group and let C be the center of G such that G/C is a direct product of two groups of order two and the commutator subgroup $G' = \langle e \mid e^2 = 1\rangle$ is of order two. Then the mapping $\odot : G \to G$, defined by

$$g^{\odot} = \begin{cases} g & \text{if } g \in C, \\ ge & \text{if } g \notin C \end{cases}$$

is an anti-automorphism of order two. If $x = \sum_{g \in G} \alpha_g g \in \mathbb{F}_2 G$, then $u \mapsto u^{\odot} = \sum_{g \in G} \alpha_g g^{\odot}$ is an involution.

Description of involutions of group rings is a very interesting problem. In their papers Rosenbaum [1] and Mahrhold–Rosenbaum [1] investigated involutions of group algebras of finite groups in the case when the group algebra is semisimple. In the modular case this problem is still open.

12. Unitary units and subgroups

The element $u \in U(RG)$ is called *unitary*, if $u^{-1} = u^* \varepsilon$, where ε is a unit of R. Evidently, unitary elements form a subgroup, denoted by $U_*(RG)$ and called the unitary subgroup of $U(RG)$. In case $U_*(RG) = U(RG)$ the group $U(RG)$ is called *unitary*.

In the case of the first involution of the above examples the unitary subgroup is denoted by $U_f(\mathbb{Z}G)$.

In the study of $U(RG)$ the set of symmetric units

$$S_*(RG) = \{x \in V(RG) \mid x^* = x\}$$

plays an important role. Notice that this set does not always form a subgroup.

The interest in $U_f(\mathbb{Z}G)$ arouse from algebraic topology and K-theory. Its significance was noticed by S. Novikov. The study and description of $U_f(\mathbb{Z}G)$ in certain cases is known as Novikov's problem. Results on this topic are treated in Bovdi's book [6] *"The multiplicative group of an integral group ring"*.

Below we shall study the unitary subgroup for finite modular commutative group algebras when the involution is of the form

$$x = \sum_{g \in G} \alpha_g g \mapsto x^* = \sum_{g \in G} \alpha_g g^{-1}.$$

The problem of the description of invariants and a basis of $V_*(\mathbb{F}_pG)$ was raised by S. Novikov, its solution will be discussed below. Note that this problem was solved also for other involutions by Szakács, V. Bovdi and Rozgonyi [1].

Theorem (A. Bovdi, Szakács [1]). *Let G be a finite abelian p-group and let $\mathbb{F}_p$ be the field of p ($p>2$) elements. Then in the direct decomposition of $V_*(\mathbb{F}_pG)$ into a direct product of cyclic groups the number of cyclic factors of order p^i ($i=1,2,\dots$) is*

$$f_i(V_*)=\frac{1}{2}\left(\left|G^{p^{i-1}}\right|-2\left|G^{p^i}\right|+\left|G^{p^{i+1}}\right|\right).$$

Let $G=\langle a_1\rangle\times\langle a_2\rangle\times\cdots\times\langle a_n\rangle$ be the decomposition of the finite abelian p-group into a direct product of cyclic groups, and let a_i be of order q_i. Above we have already defined the set $L(G)$ as the set of all the vectors $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)$ with $\alpha_i\in\{0,1,\dots,q_i-1\}$, and with at least one of the components α_j not divisible by p.

Theorem (A. Bovdi, Szakács [1]). *Let G be a finite abelian p-group and let $\mathbb{F}_p$ be the field of p ($p>2$) elements.*

Let $u_\alpha=1+(a_1-1)^{\alpha_1}(a_2-1)^{\alpha_2}\dots(a_n-1)^{\alpha_n}$ and $L_1(G)=\{\alpha\in L(G)\mid \alpha_1+\alpha_2+\cdots+\alpha_n$ is odd $\}$. Then $B(V_*)=\{v_\alpha={u_\alpha}^*{u_\alpha}^{-1}\mid\alpha\in L_1(G)\}$ is a basis for $V_*(\mathbb{F}_pG)$, i.e. $V_*(\mathbb{F}_pG)=\prod_{\alpha\in L_1(G)}\langle {u_\alpha}^*{u_\alpha}^{-1}\rangle$.

For a finite abelian 2-group G the description of $V_*(\mathbb{F}_2G)$ is more difficult. This question was solved by A. Bovdi and Szakács in [1, 2]. For a 2-group G we shall use the following notations: let

$$H=\langle a_1\rangle\times\cdots\times\langle a_s\rangle\quad\text{and}\quad C=\langle a_{s+1}\rangle\times\cdots\times\langle a_n\rangle$$

and let q_i be the order of the element a_i ($i=1,\dots,n$).

Suppose that $q_i\geq 4$ for $i=1,\dots s$, $q_{s+1}=\cdots=q_n=2$ (if $s<n$) and $G=H\times C$. Denote

$$N(H)=\{\alpha=(\alpha_1,\dots,\alpha_s)\mid\alpha_i\in\{0,q_i-1\}\}.$$

Let $G[2]$ be a subgroup of G generated by all elements of order 2. The subgroup $W(\mathbb{F}_2G)=\{x^*x^{-1}\mid x\in V(\mathbb{F}_2G)\}$ can be presented as a direct product $H^2 \times B(\mathbb{F}_2G)$, and $f_j(B) = d_{j-1} - 2d_j + d_{j+1} - f_{j+1}(H)$, where $d_j = \frac{1}{2}\big(|G^{2^j}| - |G^{2^j}[2]|\big)$. Then

$$D(\mathbb{F}_2G) = \Bigg\{1 + \sum_{\substack{\gamma\in N(H)\\ \gamma\neq(0,\dots,0)}} t_\gamma(1+a_1)^{\gamma_1}\cdots(1+a_s)^{\gamma_s} \;\Bigg|\; t_\gamma \in \mathbb{F}_2C\Bigg\}$$

is a group of exponent 2, $|D(\mathbb{F}_2G)| = 2^{|C|(|H[2]|-1)}$ and

$$V_*(\mathbb{F}_2G) = H \times V(\mathbb{F}_2C) \times B(\mathbb{F}_2G) \times D(\mathbb{F}_2G).$$

Therefore the invariants of $V_*(\mathbb{F}_2G)$ can be found easily from the above decomposition.

By Sandling's result we can find the basis of $V(\mathbb{F}_2C)$. It is easy to prove that if $\alpha = (\alpha_1,\dots,\alpha_s,\alpha_{s+1},\dots,\alpha_n)$ then the elements of the set $\{x_\alpha \mid \alpha \in L(G)$ and $(\alpha_1,\dots,\alpha_s) \in \{N(H)\setminus(0,\dots,0)\}\}$ form a basis for $D(\mathbb{F}_2G)$.

An explicit basis for $B(\mathbb{F}_2G)$ has not been found but Bovdi and Szakács [2] were able to give an algorithm for constructing a basis, which has been improvement by Szakács recently. By induction on s Szakács has constructed the following subset $L_*(G) \subset L(G)$.

In case $s = 1$, let

$$\begin{aligned} L_*(G) = \big\{\alpha = (\alpha_1,\widetilde{\alpha}) \mid \alpha_1 \equiv 1 \pmod 4,\ \alpha \neq (1,0,\dots,0)\big\}\\ \cup \big\{\alpha = (\alpha_1,\widetilde{\alpha}) \mid \alpha_1 \equiv 3 \pmod 4,\ \alpha_1 \neq q_1 - 1,\ \widetilde{\alpha} \in L(C)\big\}.\end{aligned}$$

(Here $(\alpha_1,\widetilde{\alpha})$ we mean the vector with first component α_1, and with the other component forming the vector $\widetilde{\alpha}$.)

Suppose that $s > 1$ and $\widetilde{H} = \langle a_2,\dots,a_s\rangle$, $\widetilde{G} = \widetilde{H} \times C$ and $q_1 = 2^t$. Let $L_*(G)$ be the set of all elements $\alpha = (\alpha_1,\widetilde{\alpha}) \in L(G)$ for which one of the conditions (1)–(5) holds:

(1) $\alpha_1 \in \{0, 2^i - 1, 2^i \mid i > 1\}$ and $\widetilde{\alpha} \in L_*(\widetilde{G})$;
(2) $\alpha_1 \equiv 1 \pmod 4$, $\alpha \neq (1,0,\dots,0)$;
(3) $\alpha_1 \equiv 3 \pmod 4$, $\alpha_1 \neq 2^i - 1$ $(i > 1)$ and $\widetilde{\alpha} \in L(\widetilde{G})$;
(4) $\alpha_1 = 2^i - 1$ $(1 < i < t)$, $C \neq 1$, $\widetilde{\alpha} \in N(\widetilde{H}) \times L(C)$ (a cartesian product of sets);
(5) α has the form $(\alpha_1, 0,\dots,0,1,0,\dots,0)$, where there is 1 in the j-th $(j = 2,\dots,s)$ position and $\alpha_1 \in \{2^i - 1, 2^i \mid i > 1\}$.

Then the set

$$\big\{(x_\alpha)^*(x_\alpha)^{-1} \mid \alpha \in L_*(G)\big\}$$

is a basis of $V_*(\mathbb{F}_2G)$, and therefore a basis of $V_*(KG)$ is found for $p = 2$.

13. Complement in the unitary subgroup

We know few facts on $V_*(\mathbb{F}_pG)$ for noncommutative G. A. Bovdi [7] describing when the group $V(\mathbb{F}G)$ is unitary, i.e. $V(\mathbb{F}G) = V_f(\mathbb{F}G)$.

The next very interesting problem arises: *when does G have a normal complement in $V_*(\mathbb{F}_pG)$?* Since G is a subgroup of $V_*(\mathbb{F}_pG)$, if G has a normal complement in $V(\mathbb{F}_pG)$ it clearly has one also in $V_*(\mathbb{F}_pG)$.

A. Bovdi and Erdei show that if G is a 2-group of maximal class, then G has no normal complement in $V_*(\mathbb{F}_2G)$ if and only if G is the dihedral group D_{2^n} $(n \geq 4)$ or the semidihedral group $D^-_{2^n}$ $(n \geq 4)$. They [1] describe $V_*(\mathbb{F}_2G)$ for groups G of order 16 and they give a presentation for those unitary subgroups. Those groups G have a normal complement in $V_*(\mathbb{F}_2G)$.

V. Bovdi and Rozgonyi [1] give a complete characterization of the unitary subgroup for the following group algebras: let G be a finite 2-group which contains an abelian normal subgroup A of index two. Suppose that there exists an element $b \in G \setminus A$ of order 4 such that $b^{-1}ab = a^{-1}$ for all $a \in A$. Then the unitary subgroup $V_*(\mathbb{F}_2G)$ is the semidirect product of G and a normal subgroup H. The subgroup H is the semidirect product of the normal elementary abelian 2-group $W = \{1 + (1 + b^2)zb \mid z \in \mathbb{F}_2A\}$ and the abelian subgroup L, where $V_*(\mathbb{F}_2A) = A \times L$. The abelian group W is the direct product of $\frac{1}{2}|A|$ copies of the additive group of the field $\mathbb{F}_2$.

It is easy to see that the structure of normalizer $N_{V(\mathbb{F}G)}(V_*(\mathbb{F}G))$ of the unitary subgroup is given by

$$\{y \in V(\mathbb{F}G) \mid yy^* \text{ is a central element in } V(\mathbb{F}G)\}.$$

V. Bovdi, Kovács [1] describing when the subgroup $V_*(\mathbb{F}G)$ is normal in $V(\mathbb{F}G)$:

Theorem. *Let $\mathbb{F}$ be a field of characteristic p and let G be a non-abelian locally finite p-group. The subgroup $V_*(\mathbb{F}G)$ is normal in $V(\mathbb{F}G)$ if and only if $p = 2$ and G is the direct product of an elementary abelian group with a group H for which one of the following holds:*

1. *H has no direct factor of order 2, but it is a semidirect product of a group $\langle h\rangle$ of order 2 and an abelian 2-group A with $h^{-1}ah = a^{-1}$ for all a in A;*
2. *H is an extraspecial 2-group or the central product of such a group with a cyclic group of order 4.*

Recall that a p-group is extraspecial if its center, commutator subgroup and Frattini subgroup are all equal and have order p (we do not require the group itself to be finite).

14. Exponent of the group of units

Let G be an abelian p-group and let $G[p^n] = \{g \in G \mid g^{p^n} = 1\}$. Denote by $\mathfrak{I}(G[p^n])$ be the ideal in $\mathbb{F}G$ generated by elements of the form $g-1$ with $g \in G[p^n]$. Then $1+\mathfrak{I}(G[p^n])$ consists of all units of $V(\mathbb{F}G)$ of order dividing p^n.

Now let G be a nonabelian finite p-group and $u \in V(\mathbb{F}G)$ an element of order p^n.

Question. Give a necessary condition that u be of order p^n, using the support and the support subgroup of u.

In the dissertation of Patay the study of the exponent of $V(\mathbb{F}_pG)$ was initiated. For a finite nonabelian p-group G, by a result of Coleman and Passman $\exp(V(\mathbb{F}_pG)) \geq p^2$. By the theorem of Passman on polynomial identities for every $p \neq 2$ there exists a sequence $G_1, G_2, \dots, G_m, \dots$ of finite p-groups, each of exponent p, such that $\exp(V(\mathbb{F}_pG_m)) \to \infty$ as $m \to \infty$. For example, we may choose G_m as the free nilpotent group of class 2 and exponent p with n generators. Therefore, one cannot expect general inequalities of the form $\exp(V(\mathbb{F}_pG)) \leq f(\exp(G))$ for a fixed function $f : \mathbb{N} \to \mathbb{N}$.

Let R be a ring of characteristic p and let the m-th Lie power $R^{(m)}$ be zero. If $m = 1+(p-1)p^{e-1}$, then R satisfies the indentity

$$(x+y)^{p^e} = x^{p^e} + y^{p^e}. \tag{1}$$

As a corollary, Shalev [5] obtains that if $\exp(G) = p^e$ and $(FG)^{(m)} = 0$ then $\exp(V(\mathbb{F}_pG)) = \exp(G)$. For finite p-groups G with $p > 5$ and $\exp(G)^3 > |G|$ this is true as well. Therefore, for those groups the exponent of the unitary subgroup $V_*(\mathbb{F}G)$ equals $\exp(G)$.

The exponent of $V(\mathbb{F}_pG)$ is determined by Shalev [5] for every p-group of order p^n, where $n \leq 5$ and $p > 5$ and by A. Bovdi and Lakatos [1] for a 2-group G which contains an abelian subgroup of index 2.

A. Bovdi and Lakatos [1] proved that if G is a finite p-group, $p \neq 2$ and the commutator subgroup of G is cyclic then the exponents of G and $V(\mathbb{F}_pG)$ are distinct if and only if the exponents of G and of the commutator subgroup coincide. Then the exponent of $V(\mathbb{F}_pG)$ is $p{\cdot}\exp(G)$.

It is likely that the exponents of G and $V_*(\mathbb{F}_pG)$ always coincide.

Question. Describe finite p-groups G for which (1) is an identity in the group algebra $\mathbb{F}G$.

15. Exponent units over the center

Sehgal [4] proposed the following problem: *When every* $u \in V(\mathbb{F}G)$ *satisfies* $u^n \in \zeta(V)$ *for some (or fixed)* n*, where* $\zeta(V)$ *is the center of* $V(\mathbb{F}G)$*. For a fixed* n *we write this as* $V(\mathbb{F}G)^n \subseteq \zeta(V)$*.*

This problem was considered by Cliff and Sehgal [3] for solvable groups, and they obtained an answer with certain restrictions on the exponent n. Coelho [1] continued this investigation in the cases when G is a locally finite group, and when G is either a solvable or an FC non-torsion group.

Theorem. *Let* $\mathbb{F}$ *be a field of characteristic* p *and let* G *be a locally finite group. Then* $V(\mathbb{F}G)^n \subseteq \zeta(V)$ *for some* n *if and only if* $G^l \subseteq \zeta(G)$ *for some* l *and the following conditions hold:*

1. G *contains a normal* p*-abelian subgroup of finite index;*
2. *either every* p'*-element of* G *is central, or* G *is of bounded exponent and* $\mathbb{F}$ *is finite.*

Theorem. *Let* $\mathbb{F}$ *be a field of characteristic* p *and let* G *be either a solvable or an* FC *non-torsion group. Then* $V(\mathbb{F}G)^n \subseteq \zeta(V)$ *for some* n *if and only either* $\mathbb{F}G$ *is Lie* m*-Engel for some* m *or if* $G^l \subseteq \zeta(G)$ *for some* l *and the following conditions hold:*

1. *if* A *is a non-central abelian subgroup of* G*, then* $\mathbb{F}$ *is finite,* A *is of bounded exponent and for every* $g \in G$ *and every* $a \in A$ *there exists an integer* r *such that* $g^{-1}ag = a^{p^r}$*, where* $[\mathbb{F} : \mathbb{F}_p] \mid r$*;*
2. *if there exist a non-finite subgroup* P *of* G *of bounded exponent centralizing* A*, then* G *contains a normal* p*-abelian subgroup of finite index.*

16. The torsion part of units

In this section we consider when the set of all units of finite order $T(V(\mathbb{F}G))$ in $V(\mathbb{F}G)$ is a subgroup.

First this question was investigated by Polcino Milies [3], Coelho and Polcino Milies [2] and later A. Bovdi [5] generalized this result the following way.

We now define for an arbitary group G the normal subgroup $\Lambda(G) = \{g \in G \mid \big[H : C_H(g)\big] < \infty\}$ for every finitely generated subgroup H of G. Of course, the torsion part $\Lambda^+(G)$ of $\Lambda(G)$ is a normal subgroup and the factor group $\Lambda(G)/\Lambda^+(G)$ is torsion free and abelian (Passman [2]).

Theorem. *Suppose that either the group of units of the field $\mathbb{F}$ of characteristic p or the group G has an element of infinite order and $T(V(\mathbb{F}G))$ is a subgroup. Then $T(G)$ is a subgroup and the p-Sylow subgroup P of $\Lambda^+(G)$ is normal in G and $A = \Lambda^+(G)/P$ is an abelian group. Moreover, if A is noncentral in G/P and G/P is non-torsion then the algebraic closure L of $\mathbb{F}_p$ in $\mathbb{F}$ is finite, and for all $g \in G/P$ and $a \in A$ there exists a natural number r such that $gag^{-1} = a^{p^r}$ and every such r is divisible by $[L : \mathbb{F}_p]$.*

If for every finitely generated subgroup H of G the subgroup $T(H)$ is finite then $T(G) = \Lambda^+(G)$.

Conversely, if $T(G) = \Lambda^+(G)$, $G)/\Lambda^+(G)$ is a right ordered group, $T(G) = \Lambda^+(G)$ and G satisfies the conditions described before then $T(V(\mathbb{F}G))$ is a subgroup.

17. Noetherian group of units

We consider the conditions under which the unit group of a group ring is Noetherian. Krempa [2] studies rings R with Noetherian group of units $U(R)$. Let S be a radical subring of the ring R. Then the following conditions are equivalent:

1. the subgroup of units $1 + S$ is a Noetherian group;
2. the subgroup $1 + S$ is a finitely generated nilpotent group;
3. the additive group S^+ is finitely generated.

As a corollary it is obtained that if I is a radical ideal of R, then $U(R)$ is Noetherian if and only if the additive group I^+ is finitely generated and $U(R/I)$ is Noetherian.

Let R be a semiprime ring with the group of units $U(R)$ Noetherian. Then there exists a decomposition of R as a direct sum:

$$R = I \oplus S_1 \oplus \ldots \oplus S_m,$$

where the ideal I is a reduced ring (i.e. has no nontrivial nilpotent elements) and all the ideals S_k are simple finite noncommutative rings.

Krempa [2] proved in another way A. Bovdi's results [4]: Let G have a finite subgroup which is not normal. Then $U(\mathbb{F}G)$ is a Noetherian group if and only if G and $\mathbb{F}$ are finite.

Let G be a finite nonabelian group and let $\mathbb{F}$ be infinite. Then $U(\mathbb{F}G)$ is Noetherian if and only if G is a Hamiltonian group and the subgroup $U(\mathbb{F}Q)$ is a Noetherian group, where Q is the quaternion subgroup of order 8 of G.

18. The unitary subgroup and $B_2(\mathbb{F}G)$

Above we have discussed the problem of constructing a basis for the commutative unitary subgroup $V_*(\mathbb{F}_pG)$. For noncommutative groups the determination of a generator system for $V_*(\mathbb{F}_2G)$ is an open problem. Earlier we have defined the bicyclic units $u_{a,g} = 1+(a-1)g\bar{a}$ and the subgroup $B_2(\mathbb{F}G)$ they generate. The following result describes the cases when the unitary subgroup contains $B_2(\mathbb{F}G)$. This result can be considered as a first step towards determining a generator system for $V_*(\mathbb{F}_2G)$.

Theorem (V. Bovdi, Kovács [1]). *Let G be a noncommutative p-group. All bicyclic units of $V(\mathbb{F}_pG)$ are unitary if and only if $p = 2$ and G is a direct product of an elementary abelian 2-group and a subgroup H which satisfies one of the following conditions:*

1. *H contains an abelian subgroup A of index 2 and an element b inverting elements of A;*
2. *H is an extraspecial 2-group or a central product of an extraspecial 2-group with a cyclic group of order 4;*
3. *H is a direct product of the quaternion group of order 8 and the cyclic group of order 4 or a direct product of two quaternion groups of order 8;*
4. *H is a central product of the group $\langle x, y \mid x^4 = y^4 = 1,\ x^2 = [y, x]\rangle$ and a quaternion group of order 8 with a nontrivial common element x^2y^2;*
5. *H is isomorphic to H_{32} or*

$$H_{245} = \langle x, y, u, v \mid x^4 = y^4 = (v, u) = 1,\ x^2 = v^2 = (y, x) = (v, y),\\ y^2 = u^2 = (u, x),\ x^2y^2 = (u, y) = (v, x)\rangle.$$

19. Symmetric elements and units

An element $s \in \mathbb{F}G$ is called symmetric if $x^* = x$. Ring theoretical properties of the group ring is determined to a great extent by the behaviour of the symmetric elements. For instance, in Giambruno and Sehgal's [2] those group algebras are described in which the subset of symmetric element is Lie nilpotent.

Above we have seen that the subset $S_*(\mathbb{F}G) = \{x \in V(\mathbb{F}G) \mid x^* = x\}$ of the symmetric units plays an important role in investigating $V(\mathbb{F}_pG)$. Clearly, if $V(\mathbb{F}_pG)$ contains a non-unitary element x then xx^* is a nontrivial symmetric element.

Lemma. *If $S_*(\mathbb{F}G)$ is a subgroup of $U(\mathbb{F}G)$ then it is a commutative normal subgroup.*

It is natural to ask when $S_*(\mathbb{F}_pG)$ is a subgroup. To this question only partial answers are known, but recently V. Bovdi has extended this result for an arbitary torsion group G.

Theorem (V. Bovdi, Kovács, Sehgal [1])**.** *Let G be a noncommutative p-group. Then $S_*(\mathbb{F}_pG)$ is a subgroup of $U(\mathbb{F}_pG)$ if and only if $p = 2$ and G is a direct product of an elementary abelian 2-group and a subgroup H, which satisfies one of the following conditions:*

1. *H contains an abelian subgroup of index 2 and an element b of order 4 inverting the elements of A;*
2. *H is a direct product of a quaternion group of order 8 and a cyclic group of order 4, or a direct product of two quaternion groups;*
3. *H is a central product of the group $\langle x, y \mid x^4 = y^4 = 1,\ x^2 = [y, x]\rangle$ and a quaternion group of order 8, with nontrivial common element x^2y^2;*
4. *H is isomorphic to H_{32} or H_{245}.*

20. Free subgroup and free product

Let R be a commutive ring. The following problem due to Hartley is a very difficult and interesting one:

When does the group of units $U(RG)$ contain no free group of rank 2?

We would like to deal with this problem only for the group of units $U(\mathbb{F}G)$ of a group algebra $\mathbb{F}G$ of characteristic p. Gonçalves [1, 3] gave necessary and sufficient conditions for this problem in case G is finite or some infinite solvable group. A. Bovdi [11] extended and generalized Gonçalves' theorems. Remember that earlier we defined the subgroup $\Lambda(G)$ of G.

Theorem. *Let $\mathbb{F}$ be a field of characteristic p and suppose that the nonabelian group $U(\mathbb{F}G)$ does not contain a free group of rank two. Then one of following conditions hold:*

1. *G is a torsion group and $\mathbb{F}$ is algebraic over its prime field $\mathbb{F}_p$;*
2. *The p-Sylow subgroup P of $\Lambda^+(G)$ is normal in G and $A = \Lambda^+(G)/P$ is an abelian group. Moreover,*

a. *if $\mathbb{F}$ is not algebraic over its prime field $\mathbb{F}_p$ then the centralizer $C_G(A)$ contains all elements of finite order of G/P;*

b. *if A is noncentral in G/P and G/P is non-torsion then the algebraic closure L of $\mathbb{F}_p$ in $\mathbb{F}$ is finite, and for all $g \in G/P$ and $a \in A$ there exists a natural number r such that $gag^{-1} = a^{p^r}$. Furthermore, each such r is divisible by $[L : \mathbb{F}_p]$.*

Corollary 1. *Let $\mathbb{F}$ be a field of characteristic p and let G be a group such that $T(G) = \Lambda^+(G)$ and $G/T(G)$ is a unique product group. Then $U(\mathbb{F}G)$ contains no free group of rank two if and only if G contains no free group of rank two and one of the statements 1–2 of the theorem holds.*

Corollary 2. *Let $\mathbb{F}$ be a field of characteristic 0 or p and let G be a solvable group such that $T(G) = \Lambda^+(G)$ and $G/T(G)$ are unique product groups. Then either $U(\mathbb{F}G)$ contains a free group of rank two or $U(\mathbb{F}G)$ has a normal p-subgroup N such the factor group $U(\mathbb{F}G)/N$ is solvable.*

Clearly, if G is a locally nilpotent group then $T(G) = \Lambda^+(G)$. If $U(\mathbb{F}G)$ does not contain a free group of rank two and $U(\mathbb{F})$ has an element of infinite order we assume that $T(G) = \Lambda^+(G)$. The last question is very difficult and was answered affirmatively by Gonçalves [3] in the following case: G is a solvable-by-finite group without p-elements and $\mathbb{F}$ is not algebraic over its prime subfield $\mathbb{F}_p$, and if $p = 2$ then the degree of transcendence of $\mathbb{F}$ over $\mathbb{F}_2$ is at least 2.

Vikas Bist [1] obtains necessary and sufficient conditions for the commutator subgroup of the group of units $U(\mathbb{F}G)$ of the group algebra $\mathbb{F}G$ to be torsion, if G is a locally finite or a locally FC-group. As a consequence of the theorem, we have also the following result.

Corollary 3. *Let $\mathbb{F}$ be a field of characteristic p and let G be a group such that $T(G) = \Lambda^+(G)$. Then the commutator subgroup of the non-abelian group $U(\mathbb{F}G)$ is torsion if and only if one of the following conditions holds:*

1. *G is a torsion group and $\mathbb{F}$ is algebraic over its prime subfield $\mathbb{F}_p$;*
2. *the p-Sylow subgroup P of $T(G)$ is normal in G and $A = T(G)/P$ is an abelian group;*

   a. *if $\mathbb{F}$ is not algebraic over its prime field $\mathbb{F}_p$, then A is a central subgroup of G/P;*

   b. *if A is noncentral in G/P and G/P is non-torsion, then the algebraic closure L of $\mathbb{F}_p$ in $\mathbb{F}$ is finite and for all $g \in G/P$ and*

$a \in A$ there exists a natural number r such that $gag^{-1} = a^{p^r}$. Furthermore, each such r is divisible by $[L : \mathbb{F}_p]$.

An answer to the next question would be very useful for applications: *Let $\mathbb{F}$ be a field of characteristic p. When does $U(\mathbb{F}G)$ contain a free product of two cyclic subgroups of order p?*

For example in A. Bovdi's paper [5] the following assertion is proved. Let H be a finite group and assume that its commutator subgroup is not a p-group. If G is a direct product of H and an infinite cyclic group then $U(\mathbb{F}_pG)$ contains a free product of two cyclic subgroups of order p.

The following results of Gonçalves and Passman [1] are very interesting: Let $\mathbb{F}$ be nonalgebraic over its prime field $\mathbb{F}_p$ and let the element λ be transcendental over $\mathbb{F}_p$. Suppose that G has an element a of order n and the element b which does not normalize $\langle a \rangle$, and the subgroup $\langle a, b^{-1}ab \rangle$ has no p-elements. We define the following elements: $\bar{a} = (1 + a + a^2 + \dots a^{n-1})$, $u = (1 - a)b\bar{a}$ and $v = \bar{a}b^{-1}(1 - a^{(-1)^p})$. Then the subgroup $\langle 1 + \lambda u, 1 + \lambda bab, 1 + \lambda(1 - b)aba(1 + b) \rangle$ of $U(\mathbb{F}G)$ is a free product of three cyclic subgroups of order p.

Note that the assumption that $\langle a, b^{-1}ab \rangle$ has no p-elements cannot be removed.

21. Solvability of the group of units

The study of finite groups G with solvable group of units $U(\mathbb{F}G)$ was initiated independently by Motose and Tominaga [2] and Bateman [1]. Some oversights of Bateman's paper were corrected by Motose and Ninomiya [1] and an alternative characterization was given by Bovdi and Khripta [2]. A nice exposition of these results was later given by Bovdi and Khripta [4], Passman [1] and Taylor [1].

After these results for a long time there was no progress in the study of group algebras with solvable group of units. The methods are not suitable for infinite groups. In 1972 Passi, Passman and Sehgal [1] obtained the following deep result:

Theorem. *Let $\mathbb{F}G$ be a noncommutative group algebra over the field $\mathbb{F}$. Then $\mathbb{F}G$ is Lie solvable if and only if one of the following conditions is satisfied:*

1. *$\mathbb{F}$ is of characteristic p, the commutator subgroup of G is a finite p-group;*

2. $\mathbb{F}$ *is of characteristic* 2, *the group* G *has a subgroup* H *of index* 2 *such that its commutator subgroup* H' *is a finite 2-group.*

Using the Lie solvability property of group algebras, Bovdi and Khripta [3] found a new method to investigate group algebras of torsion groups.

Let $t = 2^n$, $r = 2^{n-1}$ and $y_1, y_2, \dots, y_t \in U(\mathbb{F}G)$. Define by induction the following group commutators: $(y_1, y_2)^o = y_1^{-1}y_2^{-1}y_1y_2$, and $(y_1, y_2, \dots, y_t)^o = ((y_1, y_2, \dots, y_r)^o, (y_{r+1}, y_{r+2}, \dots, y_t)^o)$. Similarly we define for Lie commutators: $[y_1, y_2, \dots, y_t]^o$.

Let $z_1, z_2, \dots, z_t$ be arbitrary elements of $\mathbb{F}G$. Suppose that $v_1, v_2, \dots, v_t$ are nilpotent elements, which satisfy the following conditions: $v_i^{s+1} = 0$ for all i; $v_iv_j = v_jv_i$ for all i, j and $v_iz_j = z_jv_i$ for all i, j.

Then z_iv_i is a nilpotent element and $y_i = 1 - z_iv_i$ is a unit for all i. It is easy to see that the group commutator

$$(y_1, y_2) = 1 + \sum_{m=0}^{s}\sum_{k=0}^{s}(v_1z_1)^m(v_2z_2)^k[z_1, z_2]v_1v_2.$$

Using this formula we can conclude the following statement:

There exists a polynomial $f(x_1, x_2, \dots, x_t)$ over the group algebra $\mathbb{F}G$ of the form

$$c_1x_1x_2\dots x_t + \sum \gamma_{\alpha_1,\alpha_2,\dots,\alpha_t}x_1^{\alpha_1}x_2^{\alpha_2}\dots x_t^{\alpha_t}$$

with the properties

1. $c_1 = [z_1, z_2, \dots, z_t]^o$ is the Lie commutator defined before;
2. all coefficients $\gamma_{\alpha_1,\alpha_2,\dots,\alpha_t}$ commute with every v_i, and $\gamma_{\alpha_1,\alpha_2,\dots,\alpha_t}$ does not depend on $v_1, v_2, \dots, v_t$;
3. $1 \le \alpha_i$ for every i and $t + 1 \le \alpha_1 + \alpha_2 + \dots + \alpha_t \le ts$;
4. the group commutator $(y_1, y_2, \dots, y_t)^o = 1 + f(v_1, v_2, \dots, v_t)$.

Suppose that $U(\mathbb{F}G)$ is a solvable group of derived length n. If $t = 2^n$, then $(y_1, y_2, \dots, y_t)^o = 1 + f(v_1, v_2, \dots, v_t) = 1$. Therefore, we conclude that $f(v_1, v_2, \dots, v_t) = 0$ for arbitary elements $z_1, z_2, \dots, z_t$ of $\mathbb{F}G$ and $v_1, v_2, \dots, v_t$ satisfies the conditions, which we have defined before.

Suppose that $\mathbb{F}G$ is not Lie solvable. Then there exists elements $g_1, g_2, \dots, g_t$ such that $c_1 = [g_1, g_2, \dots, g_t]^o \neq 0$. Put $z_1 = g_1$, $z_2 = g_2, \dots,$ $z_t = g_t$. Define the subgroup $H = \bigcap_{i=1}^{t} C_G(g_i)$ and suppose that H

has an infinite abelian p-subgroup A. Using properties of the polynomial $f(x_1, x_2, \dots, x_t)$, we obtain that $c_1 = [g_1, g_2, \dots, g_t]^o \neq 0$ has the property $c_1(a-1) = 0$ for infinite number of elements a from A. It is easy too see that this is impossible. Bovdi and Khripta showed that if $p > 3$ then the group $U(\mathbb{F}G)$ is solvable if and only if $\mathbb{F}G$ is Lie solvable. The result for torsion groups is the following:

Theorem. *Let $\mathbb{F}$ be a field of characteristic $p > 2$ and let P be a nontrivial p-Sylow subgroup of the torsion group G. The nonabelian group $U(\mathbb{F}G)$ is solvable if and only if the commutator subgroup of G is a finite p-group or $\mathbb{F}$ is the field of three elements, the 3-Sylow subgroup P is a finite normal subgroup and the factor group $\bar{G} = G/P$ satisfies one of the following conditions:*

1. *$\bar{G}$ is an extension of an elementary abelian 2-group A by a group $\langle b\rangle$ of order 2;*
2. *$\bar{G}$ is an extension of an abelian group A of exponent 4 by a group $\langle bA\rangle$ of order 2 and $bab^{-1} = a^{-1}$ for all $a \in A$;*
3. *$\bar{G}$ is an extension of an abelian group A of exponent 8 by a group $\langle b\rangle$ of order 2 and $bab = a^3$ for all $a \in A$;*
4. *$\bar{G}$ is a direct product of the group*

$$\langle a, b \mid a^4 = b^4 = 1,\ (a, b)^2 = 1,\ (a, b, a) = (a, b, b) = 1\rangle$$

of order 32 and the elementary abelian 2-group.

Theorem. *Let $\mathbb{F}$ be a field of characteristic 2 and assume that 2-Sylow subgroup of the torsion group G is nontrivial. The nonabelian $U(\mathbb{F}G)$ is solvable if and only if there exists a finite normal 2-subgroup N such that the factorgroup $\bar{G} = G/N$ satisfies one of the following conditions:*

1. *$\bar{G}$ is abelian;*
2. *$\bar{G}$ is a direct product of an abelian group W having no element of order 2, and a 2-group B with following property:*
   a) *B has an abelian normal subgroup of index 2;*
   b) *$B/\zeta(B)$ is a group of finite exponent, where $\zeta(B)$ is the center of B;*
3. *$|\mathbb{F}| = 2$ and $\bar{G}$ is an extension of an abelian group A by a group $\langle b\rangle$ of order 2, where A is a direct product of a bounded abelian 2-group of finite exponent and an elementary 3-group W and $bab = a^3$ for all $a \in W$.*

The problem of the solvability of $U(\mathbb{F}G)$ has been settled when G is any nilpotent group by A. Bovdi [8], but for locally nilpotent G the problem is still open.

Theorem. *Let $\mathbb{F}$ be a field of characteristic p and let P be a nontrivial p-Sylow subgroup of the non-torsion nilpotent group G. The nonabelian group $U(\mathbb{F}G)$ is solvable if and only if G satisfies one of the following conditions:*

1. *the commutator subgroup of G is a finite p-group;*
2. *$p = 2$ and there exists a subgroup H in G of index 2 and a finite normal 2-subgroup N such that the group H/N is abelian, and $G/\zeta(G)$ is a 2-group of finite exponent;*
3. *the torsion subgroup of G is the direct product of the finite p-Sylow subgroup P and an abelian subgroup A. If A is a not central in G then the torsion subgroup of $U(\mathbb{F})$ has order $p^t - 1$ and for any $g \in G \setminus C_G(A)$ and $a \in A$ there exists r such $t \mid r$ and $gag^{-1} = a^{p^r}$.*

Since we do not know the answer for the Mal'cev–Kaplansky problem on the triviality of the group of units for torsion-free groups G for solvable groups this problem is extremely difficult.

Modular group algebras with polycyclic group of units were described by Bovdi and Khripta [6].

22. The derived length of the group of units

By Smirnov and Zalesskii's result [1], a Lie solvable ring R has a nilpotent ideal I such that R/I is a Lie centrally metabelian ring (i.e. the identity $[x_1, x_2, [x_3, x_4], x_5] = 0$ holds).

We know only a few facts concerning the Lie derived length of a Lie solvable group ring and the derived length of a solvable group of units. The first result in this direction has been obtained by Levin and Rosenberger [1]:

Theorem. *Let $\mathbb{F}$ be a field and let G be a noncommutative group such that $\mathbb{F}G$ is strong Lie metabelian. Then G is nilpotent of class 2, and either $|G'| = 3$ if $\mathbb{F}$ has characteristic 3, or G' is elementary abelian of order 2 or 4 if $\mathbb{F}$ has characteristic 2.*

Denote by $\mathrm{dl}(R)$ the derived length of the associated Lie algebra of the ring R.

Let $\mathbb{F}$ be a field of characteristic $p > 2$. Then $\mathbb{F}G$ is Lie solvable if and only if G' is a finite p-group. Clearly the augmentation ideal of $\mathbb{F}G'$ is a nilpotent and denote by $t(G')$ its nilpotency index. If $\{\delta_n(\mathbb{F}G)\}$ is the derived series of $\mathbb{F}G$ then $\delta_n(\mathbb{F}G) \subseteq \mathfrak{I}_{\mathbb{F}}(G')^{2^{n-1}}$, so that the inequality $\mathrm{dl}(\mathbb{F}G) \le [\log_2(2t(G'))]$ hold, where $[c]$ the upper integral part of real number c.

Shalev [8] described the derived length of $\mathbb{F}G$ for some metabelian groups G and proved, if $\mathrm{dl}(\mathbb{F}G)$ is at most n and $p > 2^n$, then G is an abelian group.

Shalev determined the minimal derived length for a non-commutative $\mathbb{F}G$ group algebra of characteristic $p > 2$: $\mathrm{dl}(\mathbb{F}G) \ge [\log_2(p+1)]$. This bound is actually the correct one when the commutator subgroup of G is central of order p.

Shalev [11] obtain for a finite nilpotent group G of class two that

1. *if the commutator subgroup is cyclic of order p^k then*

$$\mathrm{dl}(\mathbb{F}G) = [\log_2(p^k + 1)];$$

2. *if G is p-group which is abelian-by-cyclic then*

$$\mathrm{dl}(\mathbb{F}G) = [\log_2(t(G') + 1)].$$

Question (Shalev). Whether the Lie derived length of $\mathbb{F}G$ is approximately $\log_2 t(G')$ for odd characteristic.

Most recent related results concerning the Lie derived length of group algebras are the following: in [1] Sharma and Srivastava, and in [2] Külshammer and Sharma described Lie centrally metabelian group rings of characteristic $p > 3$ and $p = 3$, respectively; in [1] Sahai determined group algebras of derived length 3 for $p > 2$ (to our best knowledge, the previous two problems for $p = 2$ are still open).

Let R be a nil-ring over the field $\mathbb{F}$ of characteristic 2 and $|\mathbb{F}| > 2$. Smirnov [2] has obtained that if R is a Lie centrally metabelian ring and R satisfies the identity $x^4 = 0$ then the adjoint group $(R, \circ)$ is a centrally metabelian group. Riley and Tasic [1] constructed an example of a finite dimensional nilpotent algebra R which is Lie centrally metabelian such that R satisfies the indentities $x^{2p} = 0$ when $p > 2$ and $x^8 = 0$ when $p = 2$ and the adjoint group of R is not centrally metabelian.

As we have seen before it is natural to ask:

Question. Describe Lie centrally metabelian group rings of characteristic 2. Is the group of units of such a group ring a centrally metabelian group?

The following result of Smirnov [1] can be used to obtain some bound for the derived length of the group of units of a group ring.

Theorem. *Let R be a Lie solvable ring such that if $2r=0$ then $r=0$. Then the group of units $U(R)$ is solvable and*

$$\mathrm{dl}(U(R) \leq 4 \cdot \mathrm{dl}(R) + 3,$$

if $\mathrm{dl}(R) > 2$ *and* $\mathrm{dl}(U(R) \leq 3$ *for* $\mathrm{dl}(R) = 2$.

Shalev [6] first characterized group algebras of finite groups with metabelian groups of units for $p > 2$. For characteristic $p > 2$ the unit group $U(\mathbb{F}G)$ is metabelian if and only if one of the following conditions holds:

1. G is abelian;
2. $p = 3$ and G is nilpotent with $|G'| = 3$.

For $p = 2$ this question was solved by Kurdics [1] and also Coleman and Sandling independently in [1]:

Theorem. *Let G be a finite group and let $\mathbb{F}$ be a field of characteristic 2. The group of units $U(\mathbb{F}G)$ is metabelian if and only if one of the following conditions holds:*

1. *G is abelian;*
2. *G is nilpotent of class 2 and has an elementary abelian commutator subgroup of order 2 or 4;*
3. *$\mathbb{F} = \mathbb{F}_2$, the field of two elements, and G is an extension of an elementary abelian 3-group H by the group $\langle b \rangle$ of order 2 with $b^{-1}ab = a^{-1}$ for every $a \in H$.*

23. Engel properties and group of units

If for any $x, y \in R$ there exists a positive integer $n = n(x, y)$ such that $[x, y, n] = 0$ then we say that R is Engel. If $[x, y, n] = 0$ is an identity in R then we call R n-Engel, in which case we also say that R is of Engel length n. R is called bounded Engel if it is n-Engel for some positive integer n.

The determination of the structure of Engel group algebras is very difficult. Only bounded Engel group algebras were described by Sehgal [4, Theorem V.6.1]

Theorem. *Let $\mathbb{F}$ be a field of characteristic $p \geq 0$ and let G be a group. Then $\mathbb{F}G$ is bounded Engel if and only if one of the following conditions is satisfied:*

1. *G is abelian;*
2. *$p > 0$, G is nilpotent and there exists a normal subgroup A of G such that A' and G/A are of p-power orders.*

The first information on the Engel length of a bounded Engel group algebra was obtained by Rips and Shalev [1].

Theorem. *Let $\mathbb{F}$ be a field of characteristic p and G a group such that $\mathbb{F}G$ is n-Engel. Then*

1. *if $n < p$ then G is abelian;*
2. *if $n = p$ then G' is of order 1 or p;*
3. *if $n \leq 2p - 2$ then G' is finite.*

Kurdics [2] extends this result and describes group algebras of Engel length 3.

Theorem. *Let $\mathbb{F}$ be a field of characteristic p, G an arbitrary group. Then the group algebra $\mathbb{F}G$ is 3-Engel if and only if one of the following conditions holds:*

1. *G is abelian;*
2. *$p = 2$ and G is nilpotent of class 2 with an elementary abelian commutator subgroup of order 2 or 4;*
3. *$p = 2$ and G is nilpotent of class 2 such that its commutator subgroup is an elementary abelian 2-group of either finite order greater than 4 or of infinite order, and there exists an abelian subgroup of index 2 in G;*
4. *$p = 3$ and G is nilpotent with a commutator subgroup of order 3.*

It is well-known that 3-Engel Lie algebras are nilpotent except for the characteristic 2 and 5 cases. If G is of type 3. in the previous theorem with infinite commutator subgroup and $\mathbb{F}$ is a field of characteristic 2 then $\mathbb{F}G$ is 3-Engel but not Lie nilpotent.

The problem of characterization of group algebras with Engel or bounded Engel groups of units, raised by Sehgal [4], was in particular solved by A. Bovdi and Khripta [8, 9].

Theorem. *Let $T(G)$ be the set of all elements of finite order of the group G and let the characteristic of the field $\mathbb{F}$ not divide the orders of the elements of $T(G)$. If the group $U(\mathbb{F}G)$ is Engel then $T(G)$ is a subgroup of the Engel group G and one of the following conditions holds:*

1. *$T(G)$ is subgroup of the center of G, or*
2. *$\mathbb{F}$ is a prime field of characteristic $p = 2^t - 1$, $T(G)$ is an abelian group of exponent that divides $p^2 - 1$ and $g^{-1}ag = a^p$ for all $a \in t(G)$ and for every $g \in G \setminus C_G(T(G))$.*

Theorem. *Let $\mathbb{F}$ be a field of characteristic p and let G be a solvable group with a nontrivial p-Sylow subgroup P. Then $U(\mathbb{F}G)$ is Engel if and only if G is locally nilpotent and G' is a p-group.*

Theorem. *Let $\mathbb{F}$ be a field of characteristic p and let G be a group with a nontrivial p-Sylow subgroup P. If $U(\mathbb{F}G)$ is n-Engel and P is finite then $T(G)$ is a subgroup, the commutator subgroup G' is a p-group and $G/\zeta(G)$ is a p-group of finite exponent, where $\zeta(G)$ is the center of G. Moreover:*

1. *if G is a solvable group with a nontrivial finite p-Sylow subgroup P then $U(\mathbb{F}G)$ is n-Engel if and only if G is nilpotent and the commutator subgroup G' is a finite p-group;*
2. *if G is a solvable group with an infinite p-Sylow subgroup P then $U(\mathbb{F}G)$ is n-Engel if and only if the following conditions hold:*
   a) *G is nilpotent and it has a normal subgroup H such that $|G : H| = p^m$;*
   b) *the commutator subgroup of H is a finite p-group and $G/\zeta(G)$ is a p-group of finite exponent.*

Shalev [10] obtains the following interesting result: Let A be an n-Engel associative algebra over a field of prime characteristic. Then $U(A)$ is m-Engel for some m depending on n.

Let $f(n)$ be the smallest possible such m in the last statment. For group algebras it is easy to see $f(2) = 2$ and Kurdics [2] proved that if $\mathbb{F}G$ is a 3-Engel group algebra then the group of units $U(\mathbb{F}G)$ is also 3-Engel.

It would be interesting to assess the values $f(n)$ for greater n. Kurdics obtains [3] the following lower bounds on the Engel length of the group of units:

Theorem. *Let $\mathbb{F}$ be a field of characteristic $p > 2$ and let G be a locally nilpotent nonabelian group with a nontrivial p-Sylow subgroup.*

1. *The group of units $U(\mathbb{F}G)$ is p-Engel if and only if G is nilpotent with the commutator subgroup of order p;*
2. *$U(\mathbb{F}G)$ is not $(p-2)$-Engel, and $U(\mathbb{F}G)$ is $(p-1)$-Engel if and only if G is nilpotent with $G' = T_p(G)$ of order p, where $T_p(G)$ is the set of all p-elements of G.*

Corollary. *Let $\mathbb{F}$ be a field of characteristic $p > 2$ and let G be a nonabelian nilpotent group with G' a finite p-group. Then $U(\mathbb{F}G)$ is nilpotent of class greater than $p-2$ and*

1. *$U(\mathbb{F}G)$ is nilpotent of class $p-1$ if and only if $G' = T_p(G)$ is of order p;*
2. *$U(\mathbb{F}G)$ is nilpotent of class p if and only if G' is of order p and $G' \neq T_p(G)$.*

24. Groups of units of Engel length 2 and 3

Theorem (Kurdics [3]). *Let $\mathbb{F}$ be a field of characteristic p and let G be a group with a nontrivial p-Sylow subgroup. Then the group of units $U(\mathbb{F}G)$ is 2-Engel if and only if one of the following conditions holds:*

1. *G is abelian;*
2. *$p = 2$ and G is nilpotent with a commutator subgroup of order 2;*
3. *$p = 2$ and G is nilpotent of class 2 with an elementary abelian 2-Sylow subgroup $T_2(G) = G'$ of order 4;*
4. *$p = 3$ and G is nilpotent with $G' = T_3(G)$ of order 3.*

Theorem (Kurdics [3]). *Let $\mathbb{F}$ be a field of characteristic p and G a group with a nontrivial p-Sylow subgroup. Then the group of units $U(\mathbb{F}G)$ is 3-Engel if and only if one of the following conditions holds:*

1. *$\mathbb{F}G$ is Lie 3-Engel;*
2. *$p = 2$ and G is nilpotent of class 2 such that G' is elementary abelian of order 8, and $T_2(G)$ is of order 8 or 16 and central in G;*
3. *$p = 2$ and G is nilpotent of class 2 with G' elementary abelian of order 8, and $T_2(G) = \langle G', a\rangle$ is of order 16 such that $|G : C_G(a)| = 2$ and $C'_G(a) = (a, G)$;*
4. *$p = 2$ and G is nilpotent of class 2 with $G' = T_2(G)$ elementary abelian of order 16;*
5. *$p = 2$ and G is nilpotent with $G' = T_2(G)$ cyclic of order 4;*
6. *$p = 2$ and G is nilpotent of class 3 with $G' = T_2(G)$ elementary abelian of order 4.*

25. Lie nilpotency and the group of units

The lower Lie central series of R is defined by $\gamma_1(R) = R$ and $\gamma_k(R) = [\gamma_{k-1}(R), R]$ for $k \geq 2$.

Similarly define the upper Lie central series of R: let $\zeta_1(R) = \zeta(R)$ and $\zeta_k(R) = \{r \in R \mid [r, R] \subseteq \zeta_{k-1}(R)\}$ for $k \geq 2$.

It is well-known that $[\gamma_i(R), \gamma_j(R)] \subseteq \gamma_{i+j}(R)$.

We shall denote by $R^{[k]}$ the associative ideal generated by $\gamma_k(R)$, and by $R^{(k)}$ the associative ideal generated by all Lie products $[x, r]$ with $x \in R^{(k-1)}$ and $r \in R$, where $R^{(1)} = R$. Clearly, $R^{[k]} = \gamma_k(R)R$, $R^{(k)} = [R^{(k-1)}, R]R$ and $R^{[k]} \subseteq R^{(k)}$ for all k. The values $t_L(R) = \min\{i \mid R^{[i]} = \{0\}\}$ and $t^L(R) = \min\{i \mid R^{(i)} = \{0\}\}$ (which may be infinity) are called the lower and upper Lie nilpotency indices, respectively.

If $t_L(R)$ or $t^L(R)$ is finite then R is called Lie nilpotent and strongly Lie nilpotent, respectively, and in the former case we also say that R is Lie nilpotent of class $t_L(R) - 1$.

Passi and Sehgal [2] proved that

$$[R^{(k)}, R^{(l)}] \subseteq R^{(k+l)}; \qquad R^{(k)}R^{(l)} \subseteq R^{(k+l-1)},$$

and by Gupta and Levin's [1]

$$R^{[k]}R^{[l]} \subseteq R^{[k+l-2]}.$$

Let $\{x_i\}$ be a sequence of elements of R. By induction we define the following Lie commutator: $[x_1, x_2, \ldots, x_n] = [[x_1, x_2, \ldots, x_{n-1}], x_n]$.

Definition. If for any sequence $\{x_i\}$ of elements of R there exists some n such that $[x_1, x_2, \ldots, x_n] = 0$ then R will be called *Lie hypercentral*.

Clearly, if n exists such that it is independent of the sequence then R is Lie nilpotent.

Theorem (A. Bovdi, Khripta [5], Passi, Passman, Sehgal [1]). *Let RG be a noncommutative group ring over the commutative ring R. Then the following statements are equivalent:*

1. *RG is Lie hypercentral;*
2. *RG is Lie nilpotent;*

3. *the characteristic of R is p^t, G is a nilpotent group such that its commutator subgroup is a finite p-group. Furthermore, in this case RG is also strongly Lie nilpotent.*

A group G is said to be *hypercentral* if the ascending central series reaches G after some, possibly infinite, ordinal. Chernikov showed this to be equivalent to the property that for every sequence $a_1, a_2, \dots, a_m, \dots$ of elements in G, there exists an index m such that the group commutator $(a_1, a_2, \dots, a_m) = 1$. So in particular, if m can be chosen independently of the sequence then G is nilpotent.

The problem of classifying group algebras with unit groups satisfying some generalized nilpotency property has not been studied with the exception of group algebras whose unit groups are hypercentral which was inspired by the previous theorem.

Theorem (A. Bovdi and Khripta [6], Khripta [1], Riley [1]). *Let $\mathbb{F}G$ be a noncommutative modular group algebra over the field $\mathbb{F}$ of characteristic p. Then the following statements are equivalent:*

1. *$U(\mathbb{F}G)$ is a hypercentral;*
2. *$U(\mathbb{F}G)$ is a nilpotent;*
3. *G is a nilpotent group such that its commutator subgroup is a finite p-group.*

Bhandari and Passi [2] obtained that if $p > 3$ then $t_L(\mathbb{F}G) = t^L(\mathbb{F}G)$. It is very likely that this also holds for the characteristic $p = 2, 3$ cases.

Let R be a radical ring. Then $(R, \circ)$ is a group called the adjoint (or circle) group of R where $x \circ y = x + y + xy$. The following result of Du [1] verifies a conjecture of Jennings on radical rings:

Theorem. $\zeta_i(R, \circ) = \zeta_i(R)$ $(i \geq 0)$.

This immediately implies $\mathrm{cl}(R, \circ) + 1 = t_L(R)$, which has been reproved recently by Krasilnikov [1], independently.

Recall that if G is a nilpotent p-group with G' finite then the group of units $V(\mathbb{F}G)$ with augmentation 1 is isomorphic to the adjoint group of a radical ring, namely the augmentation ideal $\mathfrak{I}_{\mathbb{F}}(G)$. Hence,

$$\mathrm{cl}(U(\mathbb{F}G)) + 1 = t_L(\mathbb{F}G),$$

the value of which is determined by Bhandari and Passi's theorem in case $p > 3$. Moreover, by Gupta and Levin's [1] $\gamma_k(U(R)) \subseteq 1 + R^{[k]}$.

These results enable us to reduce the computation of the class of $U(\mathbb{F}G)$ to computation of the upper and lower Lie nilpotency indices of group algebras.

The following Lie properties are very useful to study the group of units:

1. (Levin, Sehgal [1]). For any $x, y, z, u, v \in R$ and $m \geq 1$

$$[x, y, y]R^{[m]} \subseteq R^{[m+2]}; \quad [x, y][y, z]R^{[m]} \subseteq R^{[m+2]};$$
$$[x, y][y, u, v] \in R^{[4]}.$$

2. (Levin, Sehgal [1]). Let $x, y, x_i, y_i, z_i \in U(R), m \geq 1, k \geq 1$. Then

$$\prod_{i=1}^{k} \big((x_i, y_i) - 1\big)\big((y_i, z_i) - 1\big)R^{[m]} \subseteq R^{[m+2k]}; \quad ((x, y) - 1)^k \in R^{[k+1]}.$$

3. (A. Bovdi, Kurdics [1]). Let $m \geq 1$. Then the following statements hold:
   (i) $[x, \gamma_m(R)][x, y] \subseteq R^{[m+2]}$ for any $x, y \in R$;
   (ii) $[x, y]^k R^{[m]} \subseteq R^{[m+k]}$ for any $x, y \in R$ and $k > 1$;
   (iii) $((x, y) - 1)^k R^{[m]} \subseteq R^{[m+k]}$ for any $x, y \in U(R)$ and $k > 1$;
   (iv) $(a - 1)R^{[m]} \subseteq R^{[m+p^k]}$ for any $a \in U(R)'^{p^k}$, if R is of characteristic p and $k \geq 1$;
   (v) $(a - 1)R^{[m]} \subseteq R^{[m+2]}$ for any $a \in \gamma_3(U(R))$, if 3 is a unit in R.

Let $\mathbb{F}G$ be a modular group algebra and $V(\mathbb{F}G)$ a nilpotent group. Denote by $\mathrm{cl}(V), \mathrm{cl}(G)$ respectively the nilpotency class of $V(\mathbb{F}G)$ and G.

By Khripta's result $U(\mathbb{F}G)$ is nilpotent if and only if G is nilpotent and the derived subgroup G' is a finite group of order p^n. Then by Sharma and Vikast Bist's [1] $\mathrm{cl}(V) \leq p^n$. Shalev [9] shows that if G is a finite p-group and $p > 3$, then $\mathrm{cl}(V) = |G'| = p^n$ if and only if G' is cyclic.

Question. Which values are attained by the function $f(G) = \mathrm{cl}(V) - \mathrm{cl}(G)$? Determine all groups G for which $f(G) = n$.

The following result describes groups for which $f(G) = 0$.

Theorem (Khripta [1]). *Let $\mathbb{F}$ be a field of characteristic p and G a nonabelian nilpotent group with G' a finite p-group. Then the nilpotency classes of G and $U(\mathbb{F}G)$ coincide if and only if*

1. $p = 3$, $G' = \mathrm{Syl}_p(G)$ *is of order* 3;
2. $p = 2$, $\mathrm{cl}(G) = 2$, $G' = \mathrm{Syl}_p(G)$ *is elementary abelian of order* 4
3. $p = 2$, $\mathrm{cl}(G) = 3$, $G' = \mathrm{Syl}_p(G)$ *is of order* 4;
4. $p = 2$, G' *is of order* 2.

Let G be a finite noncommutive p-group. The values of the function are determined in the following cases:

1. (Baginski [1], Mann and Shalev [1]) $\mathrm{cl}(V) = p$ if and only if $|G'| = p$.
2. (Shalev [9]) If $\mathrm{cl}(V) > p$ and $p > 3$, then $\mathrm{cl}(V) \geq 2p-1$. The equality holds if G' is elementary abelian of order p^3 and $\mathrm{cl}(G) = 2$.
3. (Shalev [9]) If $\mathrm{cl}(V) > 2p-1$ and $p > 3$, then $\mathrm{cl}(V) \geq 3p-2$. The equality holds if G' is elementary abelian and one of the following conditions holds:
   a) $|G'| = p^3$ and $\mathrm{cl}(G) = 2$;
   b) $|G'| = p^2$ and $\mathrm{cl}(G) = 3$.
4. (Shalev [9]) If G' is a central elementary subgroup of order p^n, then $\mathrm{cl}(V) = n(p-1)+1$.
5. (Konovalov [1]) If G is 2-group of maximal class then $\mathrm{cl}(V) = |G'|$.
6. (A. Bovdi, Kurdics [1]) If $p > 2$ and G is an arbitrary nilpotent group of class greater than 2 with G' of p-power order then $f(G) \geq p$.

Parts of the next theorem were proved by Shalev in [9] for $p \geq 5$.

Theorem (A. Bovdi, Kurdics [1]). *Let $\mathbb{F}$ be a field of characteristic p and let G be a nilpotent group such that the commutator subgroup G' is a finite abelian p-group with invariants $(p^{m_1}, p^{m_2}, \dots, p^{m_s})$. Then the following statements hold:*

1. $t_L(G) \geq t(G') + 1 = 2 + \sum_{i=1}^{s}(p^{m_i} - 1)$;
2. $t_L(G) = t^L(G) = t(G') + 1$ *if* $\gamma_3(G) \subseteq G'^p$;
3. $\mathrm{cl}(U(\mathbb{F}G)) = t(G')$ *if G is a p-group and* $\gamma_3(G) \subseteq G'^p$.

Let P be a finite abelian p-group,

$$P = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_s \rangle, \quad |a_i| = p^{m_i}, \quad m_1 \geq m_2 \geq \cdots \geq m_s.$$

We call $\{a_i\}$ a basis of P. Any $g \in P$ can be written uniquely as

$$g = a_1^{k_1} a_2^{k_2} \cdots a_s^{k_s} \quad \text{and} \quad 0 \leq k_i < p^{m_i}.$$

Let H be a proper subgroup of P with $|HP^p/P^p| = p^r$, $r > 0$. Then the function

$$\nu : H \setminus P^p \to \{1, 2, \dots, s\},\ h \mapsto \nu(h) = \min\{j \mid \gcd(k_j, p) = 1\}$$

takes r distinct values $v_1 < v_2 < \cdots < v_r$. Let

$$\{1, 2, \dots, s\} = \{u_1, u_2, \dots, u_{s-r}, v_1, v_2, \dots, v_r\}.$$

Then the subgroup $A = \langle a_{u_1} \rangle \times \langle a_{u_2} \rangle \times \cdots \times \langle a_{u_{s-r}} \rangle$ will be called the weak complement of H in P relative to the basis $\{a_i\}$.

It can be proved that the weak complements of H in P, relative to any basis, are all isomorphic to each other.

Theorem (A. Bovdi, Kurdics [1]). *Let $\mathbb{F}$ be a field of characteristic $p \neq 3$ and let G be a nilpotent group of class greater than 2 such that the commutator subgroup G' is a finite abelian p-group*

$$G' = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_s \rangle, \quad |a_i| = p^{m_i}, \quad m_1 \geq m_2 \geq \cdots \geq m_s,$$

and $|\gamma_3(G)G'^p/G'^p| = p^r$, $0 < r < s$. Furthermore, let $A = \langle a_{u_1} \rangle \times \langle a_{u_2} \rangle \times \cdots \times \langle a_{u_{s-r}} \rangle$, $u_1 < u_2 < \cdots < u_{s-r}$ be the weak complement of $\gamma_3(G)$ in G' relative to the basis $\{a_i\}$. Let

$$\{1, 2, \dots, s\} = \{u_1, u_2, \dots, u_{s-r}, v_1, v_2, \dots, v_r\}.$$

Then

1. $t_L(G) \geq t(G') + t(G'/A) = 2 + \sum_{i=1}^{s}(p^{m_i} - 1) + \sum_{j=1}^{r}(p^{m_{v_j}} - 1)$;
2. $t_L(G) = t^L(G) = t(G') + t(G'/A)$ *if G is of class 3;*
3. $\mathrm{cl}(U(\mathbb{F}G)) = t(G') + t(G'/A) - 1$ *if G is a p-group of class 3.*

A few facts are known of the nilpotency class of the group of units when G is not a p-group.

Theorem (A. Bovdi, Kurdics [1]). *Let G be a nilpotent group and $\mathbb{F}$ a field of prime characteristic p.*

1. *Let the commutator subgroup G' be cyclic of order $p^n > 2$. Then $U(\mathbb{F}G)$ is nilpotent of class $p^n - 1$ if $\mathrm{Syl}_p(G) = G'$, and of class p^n if $\mathrm{Syl}_p(G) \neq G'$.*
2. *Let the commutator subgroup G' be an elementary abelian subgroup of order p^2. Then the following statements hold:*

(i) *if* G *is of class* 2 *then* $\mathrm{cl}(U(\mathbb{F}G)) = 2p-1$ *provided* $\mathrm{Syl}_p(G) \neq G'$, *and* $\mathrm{cl}(U(\mathbb{F}G)) = 2p-2$ *provided* $\mathrm{Syl}_p(G) = G'$;

(ii) *if* G *is of class* 3 *then* $\mathrm{cl}(U(\mathbb{F}G)) = 3p-2$ *provided* $\mathrm{Syl}_p(G) \neq G'$, *and* $\mathrm{cl}(U(\mathbb{F}G)) = 3p-3$ *provided* $\mathrm{Syl}_p(G) = G'$.

In [1] Rao and Sandling have characterized modular group algebras of finite p-groups with unit groups nilpotent of class 3. We extend this result.

Theorem (A. Bovdi, Kurdics [1]). *Let* $\mathbb{F}$ *be a field of characteristic* p, G *a nilpotent group with* G' *of* p*-power order. Then* $U(\mathbb{F}G)$ *is nilpotent of class* 3 *if and only if one of the following conditions holds:*

1. $p=2$, $\mathrm{cl}(G)=2$, G' *is elementary abelian of order* 4, $G' \neq \mathrm{Syl}_2(G)$;
2. $p=2$, $\mathrm{cl}(G)=2$, $G' = \mathrm{Syl}_2(G)$ *is elementary abelian of order* 8;
3. $|G'| = 8$, $\mathrm{Syl}_2(G)$ *is elementary abelian of order* 16 *and central in* G, *and the orders of the conjugacy classes in* G *do not exceed* 4;
4. $p=2$, $G' = \mathrm{Syl}_2(G)$ *is cyclic of order* 4;
5. $p=2$, $\mathrm{cl}(G)=3$, $G' = \mathrm{Syl}_2(G)$ *is elementary abelian of order* 4;
6. $p=3$, G' *is of order* 3, $G' \neq \mathrm{Syl}_3(G)$.

26. The units of the small group algebra

Let G' be the commutator subgroup of a locally finite p-group G and $\mathfrak{I}(G')$ be the ideal in $\mathbb{F}G$ generated by elements of the form $g-1$ with $g \in G'$. If $\mathfrak{I}_{\mathbb{F}}(G)$ is the augmentation ideal of $\mathbb{F}G$, then $\mathfrak{I}_{\mathbb{F}}(G)$ is locally nilpotent and $G \cap (1+\mathfrak{I}_{\mathbb{F}}(G)\mathfrak{I}(G')) = (G')'(G')^p$ is the Frattini subgroup of G'.

Definition. The quotient $\mathbb{F}G/\mathfrak{I}_{\mathbb{F}}(G)\mathfrak{I}(G')$ is called *the small group algebra of* G *over* $\mathbb{F}$ and the p-Sylow subgroup

$$S(\mathbb{F}G) = V(\mathbb{F}G/\mathfrak{I}_{\mathbb{F}}(G)\mathfrak{I}(G') \cong V(\mathbb{F}G)/1+\mathfrak{I}_{\mathbb{F}}(G)\mathfrak{I}(G')$$

of the group of units of the small group algebra is called the group of normalised units of the small group algebra $\mathbb{F}G/\mathfrak{I}_{\mathbb{F}}(G)\mathfrak{I}(G')$.

Clearly, if G is a finite p-group and $\mathbb{F} = \mathbb{F}_p$ is the field of p elements, then $S(\mathbb{F}G)$ is also a finite p-group and has structure which mimics that of the original group G.

Denote by $S_n(\mathbb{F}_pG)$ the n-th term of the lower central series of $S(\mathbb{F}_pG)$ and by $\zeta_n(S(\mathbb{F}_pG))$ the n-th term of the upper central series of $S(\mathbb{F}_pG)$.

Let G be a finite p-group and let the Frattini subgroup $(G')'(G')^p$ of G' equal to $\{1\}$ (i.e. the commutator subgroup of G is elementary abelian). Baginski and Caranti [1] proved that G and $S(\mathbb{F}_pG)$ have the same nilpotency class. Moreover, Salim and Sandling [1] conclude that

1. $S_n(\mathbb{F}_pG) = G_n$ for $n \geq 2$;
2. $\zeta_n(S(\mathbb{F}_pG)) = \zeta_n(G)$ for all n.

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