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# On a generalized Lagrange's identity characterizing inner product spaces

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*Dedicated to Professor Zoltán Daróczy*

**Abstract.** In a real normed space $(E, \|\cdot\|)$ of dimension 3 we show that the existence of a bi-additive function $F$ from $E \times E$ into $E$, satisfying the generalized Lagrange's identity

$$\rho'_+ \left(F(x,y), F(z,v)\right) = \rho'_+(x,z)\rho'_+(y,v) - \rho'_+(x,v)\rho'_+(y,z),$$

where $\rho'_+(a,b)$ is $\|a\|$ multiplied by the right derivative of the norm, implies that the norm must be induced by an inner product.

In three dimensional inner product space $(\mathbb{R}^3, \langle\cdot\,,\cdot\rangle)$ one has the cross product $\times$ satisfying, among others, the bi-additive conditions

$$x \times (y+z) = x \times y + x \times z \quad \text{and} \quad (x+y) \times z = x \times z + y \times z$$

and the well-known Lagrange's identity

$$\langle x \times y, z \times v\rangle = \langle x, z\rangle \cdot \langle y, v\rangle - \langle x, v\rangle \langle y, z\rangle.$$

Let us assume that we have a real normed linear space $(E, \|\cdot\|)$ and the right derivative of the norm $\rho'_+(x,y) = \lim_{t\to 0^+}(\|x+ty\|^2 - \|x\|^2)/2t$ (functional that coincides with the inner product when the norm is induced by it). The norm derivatives play a crucial role in characterizations of inner

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product spaces (see, for example [1], [2], [3], [4]). Our main concern in this paper is to study when it is possible to have in a real normed space $(E, \|\cdot\|)$ an operation $F$ satisfying the bi-additivity conditions and the generalized Lagrange's identity

$$\rho'_+(F(x,y), F(z,v)) = \rho'_+(x,z)\rho'_+(y,v) - \rho'_+(x,v)\rho'_+(y,z).$$

The surprising result is that for dimension 3 the existence of such operations $F$ forces the space to be an inner product space.

Precisely, let $(E, \|\cdot\|)$ be a real normed linear space and consider the norm derivatives defined by

$$\rho'_\pm(x,y) = \lim_{t\to 0^\pm} \frac{\|x+ty\|^2 - \|x\|^2}{2t},$$

for every pair $x, y \in E$. The following properties of $\rho'_\pm$ are well-known (see [1]) and will be used in this paper:

(i) $\rho'_\pm(x,x) = \|x\|^2$ for all $x \in E$;

(ii) $\rho'_\pm(ax, by) = ab\rho'_\pm(x,y)$ if $a \cdot b \geq 0$ and $x, y \in E$;

(iii) $|\rho'_\pm(x,y)| \leq \|x\| \cdot \|y\|$ for all $x, y \in E$;

(iv) $\rho'_\pm(x, ax+y) = a\|x\|^2 + \rho'_\pm(x,y)$ if $a$ is any real and $x, y \in E$;

(v) $\rho'_+(\cdot\,,\cdot)$ is continuous and subadditive in the second variable and $\rho'_-(\cdot\,,\cdot)$ is continuous and superadditive in the second variable and, moreover $\rho'_-(x,y) \leq \rho'_+(x,y)$, for all $x, y$ in $E$;

(vi) If the norm $\|\cdot\|$ is induced by an inner product $\langle\cdot\,,\cdot\rangle$, then $\rho'_+(x,y) = \rho'_-(x,y) = \langle x, y\rangle$, for all $x, y$ in $E$.

Let us mention that $\rho'_+(x,y) = \rho'_+(y,x)$ for all $x, y$ in normed space $(E, \|\cdot\|)$ if and only if the norm derives from an inner product, i.e., very weak conditions on $\rho'_\pm$ may characterize inner products.

Our aim in this paper is to determine in a real normed linear space $(E, \|\cdot\|)$ functions $F$ from $E \times E$ into $E$ satisfying the following conditions for all $x$, $y$, $z$, $v$ in $E$:

$$F(x, y+z) = F(x,y) + F(x,z), \tag{1}$$

$$F(x+y, z) = F(x,z) + F(y,z), \tag{2}$$

and

$$\rho'_+(F(x,y),F(z,v)) = \rho'_+(x,z)\rho'_+(y,v) - \rho'_+(x,v)\rho'_+(y,z). \tag{3}$$

Note that, in particular, (3) implies taking $z=x$ and $v=y$ that

$$\|F(x,y)\|^2 = \|x\|^2\|y\|^2 - \rho'_+(x,y)\rho'_+(y,x) \tag{4}$$

**Lemma 1.** *If $F$ satisfies* (1), (2) *and* (4) *then*

(6) (i) $F(x,x)=0$, *for all $x$ in $E$;*

(7) (ii) $F(y,x)=-F(x,y)$, *for all $x,y$ in $E$;*

(8) (iii) $F(x,ay+bz)=aF(x,y)+bF(x,z)$, *for all real $a,b$ and for all $x,y,z$ in $E$.*

Proof. The substitution $y=x$ into (4) yields (i). Next by (i) $F(x+y, x+y)=0$ and by (1) and (2) one gets (ii). Finally by (4) and the properties of $\rho'_+$, $F(x,\cdot)$ is continuous at $y=0$ and by (1) condition (iii) follows.

**Lemma 2.** *If $F$ satisfies* (1), (2) *and* (3) *then:*

$$\rho'_+(x,y) = \rho'_-(x,y) \quad \text{for all } x,y \text{ in } E. \tag{9}$$

Proof. By (3) and Lemma 1 we have

$$\begin{aligned} 0 &= \rho'_+(F(x,-y),F(y,-y)) = \rho'_+(x,y)\rho'_+(-y,-y) - \rho'_+(x,-y)\rho'_+(-y,y) \\ &= \rho'_+(x,y)\|y\|^2 - (-\rho'_-(x,y))(-\rho'_-(y,y)) = (\rho'_+(x,y)-\rho'_-(x,y))\|y\|^2, \end{aligned}$$

whence for $y\neq 0$, $\rho'_+(x,y)=\rho'_-(x,y)$ and since this last equality is obvious for $y=0$ we can conclude (9).

Now we prove our main result

**Theorem 1.** *If $(E,\|\cdot\|)$ is a real normed linear space of dimension 3 and there exists a function $F$ from $E\times E$ into $E$ satisfying* (1), (2) *and* (3) *then necessarily the norm $\|\cdot\|$ is induced by an inner product.*

Proof. Assume that $F$ from $E\times E$ into $E$ satisfies (1), (2) and (3). By the previous lemmas for all $x,z$ in $E$ and $a,b$ in $\mathbb{R}$ we have

$$\|F(z,x+bz)\|^2 = \|F(z,x+az+bz)\|^2,$$

i.e.,

$$
\begin{aligned}
&\|z\|^2\|x+bz\|^2-\rho'_+(z,x+bz)\rho'_+(x+bz,z)\\
&=\|z\|^2\|x+(a+b)z\|^2-\rho'_+(z,x+(a+b)z)\rho'_+(x+(a+b)z,z)
\end{aligned}
\tag{10}
$$

We can rewrite (10) in the form

$$
\begin{aligned}
&\|z\|^2\left[\|x+(a+b)z\|^2-\|x+bz\|^2\right]=\rho'_+(z,x+(a+b)z)\\
&\times\rho'_+(x+(a+b)z,z)-\rho'_+(z,x+bz)\rho'_+(x+bz,z).
\end{aligned}
\tag{11}
$$

Let us fix $x$ and $z$ two independent vectors in $E$ and let us introduce the function $f$ from $\mathbb{R}$ into $\mathbb{R}$ defined by

$$
f(t)=\rho'_+(x+tz,z)=\rho'_-(x+tz,z).
\tag{12}
$$

Thus by means of (11) and (12) we can write

$$
\begin{aligned}
&\|z\|^2\left[\|x+(a+b)z\|^2-\|x+bz\|^2\right]\\
&=\left[(a+b)\|z\|^2+\rho'_+(z,x)\right]f(a+b)-\left[b\|z\|^2+\rho'_+(z,x)\right]f(b)\\
&=\left[b\|z\|^2+\rho'_+(z,x)\right]\left[f(a+b)-f(b)\right]+a\|z\|^2f(a+b).
\end{aligned}
\tag{13}
$$

Since the norm is continuous and $|f(a+b)|\le\|x+(a+b)z\|\,\|z\|$, taking limits in (13) when $a\to 0\pm$ we obtain

$$
\left[b\|z\|^2+\rho'_+(z,x)\right]\lim_{a\to 0\pm}(f(a+b)-f(b))=0,
$$

i.e., for any real $b$, $b\neq b_0:=-\rho'_+(z,x)/\|z\|^2$, we obtain

$$
\lim_{a\to 0\pm}f(b+a)=f(b).
\tag{14}
$$

Note that by (13) at point $b_0$ we have

$$
\begin{aligned}
\lim_{a\to 0\pm}f(b_0+a)&=\lim_{a\to 0^\pm}\frac{\|x+b_0z+az\|^2-\|x+b_0z\|^2}{a}\\
&=2\rho'_\pm(x+b_0z,z)=2f(b_0).
\end{aligned}
\tag{15}
$$

We claim that $f(b_0) = 0$. To see this consider the following chain of equalities for any real $\lambda$ and for our fixed $x, z$:

$$\begin{aligned}
&\lambda^2\|F(x,z)\|^2 = \|F(x,\lambda z)\|^2 = \|F(x+b_0z,\lambda z)\|^2 \\
&\quad = \|F(x+b_0z, x+b_0z+\lambda z)\|^2 = \|x+b_0z\|^2 \cdot \|x+b_0z+\lambda z\|^2 \\
&\qquad - \rho'_+(x+b_0z, x+b_0z+\lambda z)\rho'_+(x+b_0z+\lambda z, x+b_0z+\lambda z-\lambda z) \\
&\quad = \|x+b_0z\|^2\|x+b_0z+\lambda z\|^2 \\
&\qquad - \left[\|x+b_0z\|^2 + \rho'_+(x+b_0z,\lambda z)\right]\left[\|x+b_0z+\lambda z\|^2\right. \\
&\qquad \left. + \rho'_+(x+b_0z+\lambda z, -\lambda z)\right] \\
&\quad = -\|x+b_0z\|^2\rho'_+(x+(b_0+\lambda)z, -\lambda z) \\
&\qquad - \|x+b_0z+\lambda z\|^2\rho'_+(x+b_0z,\lambda z) \\
&\qquad - \rho'_+(x+b_0z,\lambda z)\rho'_+(x+(b_0+\lambda)z,-\lambda z)).
\end{aligned} \tag{16}$$

Then taking into account that we have already proved that in our case $\rho'_+ = \rho'_-$, division by $\lambda < 0$ in (16) yields

$$\begin{aligned}
&\lambda\|F(x,z)\|^2 = +\|x+b_0z\|^2\rho'_+(x+(b_0+\lambda)z,z) \\
&+\|x+b_0z+\lambda z\|^2\rho'_+(x+b_0z,z) - \lambda\rho'_+(x+b_0z,z)\rho'_+(x+(b_0+\lambda)z,z)
\end{aligned} \tag{17}$$

and taking limits when $\lambda \to 0-$ we obtain using (15)

$$0 = \|x+b_0z\|^2 2f(b_0) + \|x+b_0z\|^2 f(b_0),$$

and since $x$ and $z$ are independent, $f(b_0) = 0$. Therefore by (17), for any $\lambda < 0$ it is

$$\lambda\|F(x,z)\|^2 = \|x+b_0z\|^2 f(b_0+\lambda)$$

i.e., for any $t < b_0$:

$$f(t) = f(b_0+(t-b_0)) = \frac{\|F(x,z)\|^2}{\|x+b_0z\|^2}(t-b_0),$$

so $f$ is an affine function on $(-\infty, b_0]$. Since $f(b_0) = 0$ by (16) we also have for $\lambda > 0$

$$\lambda^2\|F(x,z)\|^2 = \lambda\|x+b_0z\|^2 f(b_0+\lambda),$$

i.e., for any $t > b_0$:

$$f(t) = f(b_0 + (t - b_0)) = \frac{\|F(x,z)\|^2}{\|x + b_0 z\|^2}(t - b_0),$$

so $f$ is an affine function on $\mathbb{R}$ vanishing at $b_0$. Thus for all real $t$

$$\rho'_+(x + tz, z) = \frac{\|F(x,z)\|^2}{\|x - \frac{\rho'_+(z,x)}{\|z\|^2} z\|^2}\left(t + \frac{\rho'_+(z,x)}{\|z\|^2}\right),$$

and for $t = 0$ we obtain:

$$\rho'_+(x,z) = \frac{\|x\|^2\|z\|^2 - \rho'_+(x,z)\rho'_+(z,x)}{\|x - \frac{\rho'_+(z,x)}{\|z\|^2} z\|^2} \cdot \frac{\rho'_+(z,x)}{\|z\|^2},$$

i.e., $\rho'_+(x,z) = 0$ if and only if $\rho'_+(z,x) = 0$ and the symmetry of the orthogonality relation $\rho'_+(x,z) = 0$ yields that necessarily (in dimension 3) the norm derives from an inner product (see [1]). This completes the proof.

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