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Some remarks on a lemma of A.E. Ingham

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To the 60th birthday of Professor Imre Kátai

Abstract. A mean-value lower bound over "short intervals" is given for the absolute value of the logarithmic derivative of the Riemann zeta-function and more general Dirichlet series.

1. Introduction

First we wish to give a rough idea of how we prove our Theorems 1, 2 and 4 using a certain important lemma of A.E. Ingham. A well-known principle (called the argument principle) is the result

$$\frac{1}{2\pi i}\int_C F_0'(z)(F_0(z))^{-1}dz = N_0$$

which is stated precisely in the remark immediately following Theorem 2. We will be interested in the case where C is the anticlockwise boundary of the rectangle (we call this "The Ingham rectangle") obtained by joining the points

$$\sigma_0 + iT_1,\ \beta + i\,T_1,\ \beta + i\,T_2,\ \sigma_0 + i\,T_2,\ \sigma_0 + i\,T_1$$

in this order. Here $\sigma_0 < \beta$ (where β is a large positive constant), $T_1 = T + \Delta_1$, $T_2 = T + H - \Delta_2$, $(100 \le H \le T)$. The numbers Δ_1 and Δ_2 are

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certain positive real numbers which are bounded. It is of importance to have H as small a function of T as possible in Theorems 1, 2 and 4 (since their truth for any H implies their truth for a bigger value of H). The fact that (on the horizontal sides of the Ingham rectangle with suitable Δ_1, Δ_2) we have $F_0'(z)(F_0(z))^{-1} = O((\log T)^2)$, for a wide class of functions $F_0(z)$ is essentially due to Ingham. (See Section 2. He considered only the case $F_0(z) = \zeta(z)$). Hence for such functions the difference between "The arguments" in $\log F_0(z)$ (got by continuous variation) at the points $z_1 = \sigma_0 + iT_1$ and $z_2 = \sigma_0 + iT_2$ is equal to $2\pi N_0 + O((\log T)^2)$, provided that on the right vertical boundary the contribution is $O((\log T)^2)$. This is easily the case if $F_0(z)$ is a non-trivial generalised Dirichlet series with constant term 1 (in fact it is $O(1)$). However it is a different story to obtain a non-trivial lower bound for N_0, the number of zeros inside the Ingham rectangle (such results are fairly involved and treated (for the first time) for some class of generalised Dirichlet series in a series of papers by R. Balasubramanian and K. Ramachandra. One may refer to the first five references listed in the end). This leads to Theorems 1 and 2 (As is clear from our proof it actually shows that the real part of

$$\frac{1}{2\pi i(T_2 - T_1)} \int_{\sigma_0 + iT_1}^{\sigma_0 + iT_2} F_0'(z)(F_0(z))^{-1} dz$$

is bounded below if $1 - \delta \geq \sigma_0 \geq \frac{1}{2} - \delta$ (with $0 < \delta < \frac{1}{2}$) and greater than a positive constant multiple of $\log T$ if $-\delta^{-1} \leq \sigma \leq \frac{1}{2} - \delta$ provided H exceeds a certain positive *constant* multiple of T^{ϵ}, the *constant* depending on ϵ). The next kind of result is a simple one but tricky. It consists in integrating thigs like

$$\frac{\zeta'(z)}{\zeta(z) - 2} \quad \text{and} \quad \frac{2^z \zeta'(z)}{\zeta(z) - 2}$$

on the Ingham rectangle and comparing the results. This leads to Theorem 4 which is more general and covers things like zeta and L-functions associated with number fields and also things like the Hurwitz zeta-functions. Theorem 4 is proved in Section 3 in some generality. It is somewhat surprising that in Theorm 4 we can take $f_0(z) = \zeta(z)$. (The letter C in Theorem 4 denotes a certain positive constant and should not be confused with the boundary C which appears in the begining of the introduction). It should be stated that we are unable to prove things like

$$\frac{1}{H}\int_{T}^{T+H}\left|\frac{\zeta'(\frac{3}{4}+it)}{\zeta(\frac{3}{4}+it)-1}\right|dt > D_1,$$

with $H = C_1(\log T)^2$ for suitable constants $C_1 > 0$, $D_1 > 0$. See Remark 2 and the conjecture in the end.

Here we record a few results on the mean-value of functions like $|\zeta'(s)(\zeta(s)-a)^{-1}|$ ($s = \sigma + it$, $a =$ any non-zero complex constant) and $|\zeta''(s)(\zeta'(s)-a)^{-1}|$ ($a =$ any complex constant). Typical theorems are

Theorem 1. *Let a be any non-zero complex constant, $G(s) = \zeta'(s)(\zeta(s)-a)^{-1}$ and $F(s) = G(s)$ or $G(s) + \log 2$ according as $a \neq 1$ or $a = 1$. Then for all σ such that $\frac{1}{2} \leq \sigma \leq 1-\delta$ we have uniformly (for $T \geq T_0$ (δ, a))*

$$\frac{1}{H}\int_{T}^{T+H}|F(\sigma+it)|dt \gg_{\delta,\epsilon,a} 1$$

where ϵ $(0 < \epsilon < 1)$ and δ $(0 < \delta \leq \frac{1}{2})$ are arbitrary constants and $H = T^{\epsilon}$. Next let $0 < \delta \leq \frac{1}{2}$ and $-\frac{1}{\delta} \leq \sigma \leq \frac{1}{2} - \delta$. Then there holds

$$\frac{1}{H}\int_{T}^{T+H}|F(\sigma+it)|dt \geq C(\delta)\log T + O(H^{-1}(\log T)^2)$$

where H exceeds a positive constant depending on a and δ and $C(\delta)$ $(= C(\delta, a) > 0)$, depends only on δ and a. (Here a may be zero and in this case $F(s) = \zeta'(s)(\zeta(s))^{-1}$).

Theorem 2. *Let $j (\geq 1)$ be any integer constant, and a any complex constant. Let $G(s) = \zeta^{(j+1)}(s)(\zeta^{(j)}(s)-a)^{-1}$, and $F(s) = G(s)$ or $G(s) + \log 2$ according as $a \neq 0$ or $a = 0$. Then there exists a constant η $(= \eta(j, a) > 0)$ such that for all σ satisfying $\frac{1}{2} \leq \sigma \leq \frac{1}{2} + \eta$, we have*

$$\frac{1}{H}\int_{T}^{T+H}|F(\sigma+it)|dt \gg_{\epsilon,j,a} 1.$$

Here ϵ $(0 < \epsilon < 1)$ is any arbitrary constant and $H = T^{\epsilon}$.

Next let $0 < \delta \leq \frac{1}{2}$ and $-\frac{1}{\delta} \leq \sigma \leq \frac{1}{2} - \delta$. Then we have

$$\frac{1}{H} \int_{T}^{T+H} |F(\sigma + it)| dt \geq C(\delta) \log T + O(H^{-1} (\log T)^2)$$

where $C(\delta)$ $(= C(\delta, a, j) > 0)$ and H exceeds a positive constant depending only on δ, a and j.

Remark. We can state many more general theorems, but these are typical of a large class. All these are corollaries to the main results of [RB,KR,2], [RB,KR,3] and [KR,1] (also [RB,KR,4] and [RB,KR,1]) which follow on applying the following result. Let C be the smooth boundary (oriented in the anti-clockwise direction) of a simply connected region in the complex plane. Let $z = x + iy$, and $F_0(z)$ be analytic inside and on C and $\neq 0$ on C. Then

$$\frac{1}{2\pi i} \int_{C} F_0'(z) (F_0(z))^{-1} \, dz = N_0$$

where N_0 is the number of zeros of $F_0(z)$ inside C. This follows from Cauchy's Theorem of residues. However a certain procedure is necessary to deduce theorems like 1 and 2. The procedure will be explained in Section 2. It is due to (late) Professor A.E. Ingham, who explained it to the second author in connection with explicit formula for the summatory function of the coefficients of $-\zeta'(s)(\zeta(s))^{-1}$. His procedure is published here for the first time.

Notation. The notation employed in this paper is all standard. $O(\dots)$ means less than a constant times $\dots \gg \dots$ means greater than a constant times (where the constant depends on $\dots$). Some times when we do not specify the constants we write $\gg$.

2. Procedure

We begin with

Theorem 3 (Bórel–Caratheódory). *Suppose $f(z) (z = x + iy)$ is analytic in $|z - z_0| \leq R$ and on the circle $z = z_0 + Re^{i\theta}$ $(0 \leq \theta \leq 2\pi)$ we have $Ref(z) \leq U$. Then in $|z - z_0| \leq r < R$ we have*

$$|f(z) - f(z_0)| \leq \frac{2r(U - Ref(z_0))}{R - r}$$

and for integers $j \geq 1$ we have also

$$\left|\frac{f^{(j)}(z)}{j!}\right| \leq \frac{2R(U - Re f(z_0))}{(R-r)^{(j+1)}}.$$

Remark. For the proof of this theorem see page 26 of [KR,2].

Suppose $f(z)$ is analytic in $Re z \geq \beta$ (≥ 1) where β is a constant and there $|f(z) - 1| \leq \frac{1}{10}$. Suppose $f(z)$ can be continued analytically in $|z - z_0| \leq 10R$ where $R(\geq 2)$ is a constant and there $|f(z)| \leq T^A$ (where $T \geq 3$) and $A(\geq 1)$ is a constant. Suppose that ρ runs over all the zeros of $f(z)$ in $|z - z_0| \leq R$. Put

$$F(z) = f(z)\pi_\rho \left(1 - \frac{z - z_0}{\rho - z_0}\right)^{-1}.$$

By maximum modulus principle we have

$$\max_{|z-z_0|\leq R} |F(z)| \leq \max_{|z-z_0|\leq 10R} |f(z)| \leq T^A. \tag{1}$$

Also

$$\frac{9}{10} \leq |F(z_0)| \leq \max_{|z-z_0|\leq 10R} |F(z)| \leq T^A 3^{-N_1} \tag{2}$$

where N_1 is the number of zeros ρ. From (2) we obtain

$$N_1 \leq \log(2T^A) \leq (A+1)\log T. \tag{3}$$

Now putting $g(z) = \log F(z_0)$ in $|z - z_0| \leq R$ we find that $g(z)$ is analytic in $|z - z_0| \leq R$ and there we have

$$Re\, g(z) \leq A \log T. \tag{4}$$

Now applying Theorem 3 (with $j = 1$, $r = \frac{R}{2}$) we obtain

$$\left|\frac{F'(z)}{F(z)}\right| \leq \frac{4}{R}(A\log T + \log 2)$$

i.e.

$$\left|\frac{f'(z)}{f(z)} - \sum \frac{1}{z-\rho}\right| \leq \frac{4}{R}(A\log T + 1) \tag{5}$$
$$\leq \frac{8A}{R}\log T$$

in $|z-z_0| \leq \frac{1}{2}R$. Total number of zeros $\rho(\text{in}|z-z_0| \leq R)$ is $\leq 2A\log T$. Hence if we divide the $y-$ range $|y-y_0| \leq \frac{1}{4}R$ into $[20A\log T]$ equal intervals, (by Dirichlets box principle) at least one of these intervals is free from zeros ρ. We fix one such $y-$ interval. Let $y = y_1$ be the middle (horizontal) line in this interval. Then we have (for $z = x+iy_1$, $|z-z_0| \leq \frac{1}{2}R$, $|y_1-y_0| \leq \frac{1}{4}R$, i.e. certainly for $z = x+iy_1$, $|y_1-y_0| \leq \frac{1}{4}R$, $|x-\beta| \leq \frac{1}{4}R$)

$$\left|\frac{1}{z-\rho}\right| \leq 1000A\log T \tag{6}$$

and so

$$\sum \frac{1}{|z-\rho|} \leq 2000A^2(\log T)^2. \tag{7}$$

Hence (for $z = x+iy, |x-\beta| \leq \frac{1}{4}R, |y_1-y_0| \leq \frac{1}{4}R$) we have

$$\left|\frac{f'(z)}{f(z)}\right| \leq 4000A^2(\log\ T)^2. \tag{8}$$

We now come to a more specific application of (8). We revert back to the notation $s = \sigma + it$. Let $1 = \lambda_1 < \lambda_2 < \lambda_3 < \ldots$ with $\lambda_{n+1} - \lambda_n \gg 1$. Suppose that

$$f(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s} \quad (a_n \text{ being complex numbers with } a_1 = 1)$$

converges absolutely in $\sigma \geq \beta$ (≥ 2) and there $|f(s)-1| \leq \frac{1}{10}$. Also suppose that $f(s)$ can be continued analytically in $\sigma \geq -10\beta$ and there $|f(s)| \leq |t|^A$ for all t such that $|t| \geq t_0$ (≥ 100), and for a suitable constant $A(\geq 1)$. (We assume β to be a constant). Let J be the boundary of the rectangle with the corners $\beta+iT$, $\beta+i(T+H)$, $-2\beta+i(T+H)$, $-2\beta+iT$, $\beta+iT$ oriented in this order where $T \geq T_0$ and $T \geq H \geq H_0$. We can increase T and decrease H both with suitable bounded quantitites such that on the horizontal portions of J we have $|f'(s)(f(s))^{-1}| = O((\log T)^2)$ uniformly. Also for the integral on the vertical line $\sigma = \beta$ we have

$$\int f'(s)(f(s))^{-1} ds = O(1).$$

Let K be the boundary of the rectangle with the corners $\beta + iT$, $\beta + i(T + H)$, $\sigma + i(T + H)$, $\sigma + iT$, $\beta + iT$ oriented in this order. As mentioned already

$$\frac{1}{2\pi i} \int_K f'(s)(f(s))^{-1} ds$$

is the number N of zeros of $f(s)$ inside K, provided that the left hand vertical portion of K is free from zeros of $f(s)$. Thus putting $F(s) = f'(s)(f(s))^{-1}$, we have

$$\frac{1}{H} \int_T^{T+H} |F(\sigma + it)| dt \gg NH^{-1} + O(H^{-1}(\log T)^2).$$

If however the left hand vertical portion of K contains a zero of $f(s)$ the mean-value is clearly infinite. This proves Theorems 1 and 2 on using lower bounds for N given by the papers [RB,KR,2], [RB,KR,3], [KR,1], [RB,KR,4], and [RB,KR,1].

3. Proof of Theorem 4 (to be stated)

Let as usual $s = \sigma + it$, $100 \leq H \leq T$ and let $f(s)$ be as in Section 2 (after equation (8)). The only conditions on $f(s)$ are

$$f(s) = 1 + \sum_{n=2}^{\infty} a_n \lambda_n^{-s}, \quad (a_2 \neq 0)$$

where $1 < \lambda_2 < \lambda_3 < \ldots$, $\lambda_{n+1} - \lambda_n \gg 1$ and a_n are complex numbers with absolute value bounded above by a positive constant power of λ_n. Further in $\sigma \geq \beta (\geq 2)$, $|f(s) - 1| \leq \frac{1}{10}$ and $f(s)$ can be continued analytically in $\sigma \geq -10\beta$, $t \geq 100$ (100 is unimportant. We can replace it by any positive constant) and there $|f(s)| \leq t^A$. Here β and $A > 0$ are constants. Then we prove this following theorem.

Theorem 4. *For all σ_0, T, H with $\frac{1}{2}\beta \geq \sigma_0 \geq -\beta$, $100 \leq H \leq T$, we have*

$$\frac{1}{H}\int_T^{T+H} |f'(\sigma_0+it)(f(\sigma_0+it))^{-1}|dt \geq C + O\left(\frac{(\log T)^2}{H}\right) \tag{9}$$

where $C > 0$ is a constant independent of σ_0.

Proof. Let K be the boundary of the rectangle with corners $\beta+iT$, $\beta+i(T+H)$, $\sigma_0+i(T+H)$, σ_0+iT, $\beta+iT$ oriented in this order. By Cauchy's Theorem we have

$$\frac{1}{2\pi i}\int_K f'(s)(f(s))^{-1}ds = N$$

where N is the number of zeros of $f(s)$ in K (we may assume without loss of generality that there are no zeros on the boundary of K. Also we can change T and H by bounded quantities say to T, and H_1, such that on the horizontal sides the integral has absolute value $O(\log T)^2))$. Clearly the right vertical side contributes $O(1)$. Denoting the left vertical side by L we have

$$-\frac{1}{2\pi i}\int_L f'(s)(f(s))^{-1}ds = N + O((\log T)^2) \tag{10}$$

also by a similar procedure we have

$$\begin{aligned} &\frac{1}{2\pi i}\int_L \lambda_2^s f'(s)(f(s))^{-1}ds \\ &= -\sum_\rho \lambda_2^\rho + \frac{H_2}{2\pi} + O((\log T)^2) = \frac{H_2}{2\pi} + \theta\lambda_2^\beta N + O((\log T)^2) \end{aligned} \tag{11}$$

where $H_2 = H_1 a_2 \log\lambda_2$, ρ runs over all the zeros of $f(s)$ in K, and θ is real with absolute value ≤ 1. If $N \leq \frac{1}{4\pi}\lambda_2^{-\beta}|H_2|$, then we are through by (11). If on the other hand $N \geq \frac{1}{4\pi}\lambda_2^{-\beta}|H_2|$ then (10) gives the theorem. Hence the theorem is completely proved.

Remark 1. Examples of functions like $f(s)$ are to be found in papers by us at several places (see for example [RB,KR,1]). Plainly we can take $f(s) = \zeta(s) - a$ or $f(s) = \zeta^{(j)}(s) - a$ where a is any complex number including zero. But we need the condition $a \neq 1$ for $\zeta(s) - a$. We have only the result given by Theorem 4 for $f(s) = 2^s(\zeta(s) - 1)$, if $a = 1$.

Remark 2. Using mean square of $|\zeta(\sigma+it)|$ and $|\zeta'(\sigma+it)|$ over T, $T+T^{\frac{1}{3}}$ it is not hard to prove in a simple way things like

$$T^{-\frac{1}{3}}\int_T^{T+T^{\frac{1}{3}}}\left|\zeta'\left(\tfrac{3}{4}+it\right)\left(\zeta\left(\tfrac{3}{4}+it\right)-1\right)^{-1}\right|dt$$

are bounded below by a positive constant. But is is hard to extend this proof to cover zeta and L-functions of algebraic number fields.

Conjecture. *Prove (or disprove) that for all fixed $\sigma>\frac{1}{2}$, and for $H=C_2(\log T)^2$,*

$$\frac{1}{H}\int_T^{T+H}\left|\zeta'(\sigma+it)(\zeta(\sigma+it)-1)^{-1}\right|dt>D_2$$

where C_2 and D_2 are certain positive constants independent of T.

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