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On orthogonally additive functions

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Abstract. Under appropriate conditions we decompose orthogonally additive functions into an additive and a quadratic part.

In this paper, G means an abelian group, and V is a vector space over a field Λ, where char $\Lambda \neq 2$. We also consider some binary relation $\perp$ in V, which we call orthogonality. In fact, the literature offers a lot of possibilities for $\perp$, all of them reflecting some properties of the ordinary orthogonality, which stems from an inner product. For our purposes we only need two properties of $\perp$, which are valid in many of the existing orthogonality spaces $(V, \perp)$; we quote them below as (O), (P).

A function $f : V \to G$ is called *orthogonally additive* (cf. the survey article by Paganoni and Rätz [6]), if

$$f(x+y) = f(x) + f(y) \qquad (x, y \in V;\ x \perp y), \tag{1}$$

it is called *additive*, if

$$f(x+y) = f(x) + f(y) \qquad (x, y \in V),$$

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and it is called *quadratic*, if

$$f(x+y)+f(x-y)=2f(x)+2f(y) \qquad (x,y\in V). \tag{2}$$

Now we are able to formulate (O) and (P):

(O) $0\perp 0$, and from $x\perp y$ the relations $-x\perp -y$, $\frac{x}{2}\perp\frac{y}{2}$ follow.

(P) If an orthogonally additive function from V to G is odd, then it is additive; if it is even, then it is quadratic.

Orthogonality spaces fulfilling (O), (P) (the abelian group G in (P) being arbitrary) can be found in the papers by Rätz [7]–[9], Rätz and Szabó [10], Szabó [12]–[19]. Very often (P) is true because the space V under consideration has the following property (which trivially implies (P)):

(Q) Orthogonally additive functions from V to G are additive.

For example, if V is a real normed space which is not an inner product space, and if $\perp$ is the orthogonality in the sense of Birkhoff and James, then (Q) holds. The proof of this fact has been a longer story, as can be realized from the series of papers by Sundaresan [11], Gudder and Strawther [4], Lawrence [5], Rätz [7], and Szabó [12]; cf. Szabó [16]. It will be clear in a moment that in the present paper (Q) is not of interest. So we are rather concerned with orthogonality spaces V satisfying (P) but not (Q); the papers of Rätz and Szabó [10] and of Szabó [14] are good references for this.

Under the assumptions (O), (P) we show in Theorem 1 that every orthogonally additive function $f:V\to G$ has the form

$$f(x)=a(x)+q(x) \qquad (x\in V), \tag{3}$$

a being additive and q being quadratic. This theorem holds without further assumptions on the abelian group G: The case of a 2-torsion-free group has already been treated by Rätz and Szabó [10], whereas the case of an inner product space V (again no restriction upon G) can be found in [2]. Let us also mention that, with exception of Remark 2, the vector space V always can be replaced by an abelian group V, which is uniquely divisible by two; concerning the orthogonality $\perp$ in V, nothing has to be changed.

Let us start with two lemmas where the relation $\perp$ in V is not needed.

Lemma 1. *Let $f : V \to G$ satisfy $f(0) = 0$, and suppose* (3) *to hold, a being an additive function and q a quadratic one. Then a, q are uniquely determined, viz.*

$$a(x) = f\left(\frac{x}{2}\right) - f\left(-\frac{x}{2}\right) \qquad (x \in V), \tag{4}$$

$$q(x) = 2\left[f\left(\frac{x}{2}\right) + f\left(-\frac{x}{2}\right)\right] \qquad (x \in V). \tag{5}$$

Proof. Since $f(0) = a(0) = 0$, we get from (3) that $q(0) = 0$. So $x = 0$ in

$$q(x + y) + q(x - y) = 2q(x) + 2q(y) \qquad (x, y \in V) \tag{6}$$

implies that q is an even function. Hence we get from (3) the relation

$$f(-x) = -a(x) + q(x) \qquad (x \in V). \tag{7}$$

Subtracting this from (3), replacing x by $\frac{x}{2}$, and using the additivity of a gives (4). Now, addition of (3), (7) and multiplication by 2 yields

$$4q(x) = 2(f(x) + f(-x)) \qquad (x \in V).$$

When using (6) with $y = x$, we can replace $4q(x)$ by $q(2x)$, and finally we replace x by $\frac{x}{2}$ to get (5).

Remark 1. From (3), (4), (5) it follows (after replacing x by $2x$) that

$$f(2x) = 3f(x) + f(-x) \qquad (x \in V).$$

Lemma 2. *Let $f : V \to G$ be a function such that $f(0) = 0$. Then f is a solution of* (2) *if and only if*

$$f(x) = b(x, x) \qquad (x \in V) \tag{8}$$

for some biadditive, symmetric $b : V \times V \to G$. The function b is uniquely determined by f, viz.

$$b(x, y) = f\left(\frac{x + y}{2}\right) - f\left(\frac{x - y}{2}\right) \qquad (x, y \in V). \tag{9}$$

Proof. Suppose first that $f : V \to G$ satisfies (8) with some biadditive, symmetric $b : V \times V \to G$. Then (9) is easily established:

$$f\left(\frac{x+y}{2}\right) - f\left(\frac{x-y}{2}\right) = b\left(\frac{x}{2}+\frac{y}{2}, \frac{x}{2}+\frac{y}{2}\right) - b\left(\frac{x}{2}-\frac{y}{2}, \frac{x}{2}-\frac{y}{2}\right)$$
$$= 4b\left(\frac{x}{2}, \frac{y}{2}\right) = b(x,y).$$

Now let $f : V \to G$ be a solution of (2), such that $f(0) = 0$. Then f is even (put $x = 0$ in (2)). In this step of the proof we define the function $b : V \times V \to G$ by (9), and we show its biadditivity (by a routine argument; cf. [1], pp. 419, 420): (2) implies

$$b(z_1 + z_2, y) + b(z_1 - z_2, y) = 2b(z_1, y), \tag{10}$$

whence $b(2z_1, y) = 2b(z_1, y)$. Replacing the right hand side of (10) by this and then setting $z_1 = \frac{x_1+x_2}{2}$, $z_2 = \frac{x_1-x_2}{2}$ gives

$$b(x_1, y) + b(x_2, y) = b(x_1 + x_2, y).$$

The rest of the proof (of this lemma) is easy.

Lemma 3. *Suppose* (O), (P) *to hold, and let* $f : V \to G$ *be an orthogonally additive function satisfying* $2f = 0$. *Then* $f = 0$.

Proof. From (1) and (O) we get $f(0) = 0$. Now define $g : V \to G$ by

$$g(x) = f(x) - f(-x) \qquad (x \in V).$$

This function is odd, and using (O), we obtain that it is orthogonally additive. Then by (P) it is additive, in particular

$$g(2x) = 2g(x) = 2f(x) - 2f(-x) = 0 \qquad (x \in V),$$

whence $g = 0$. So f is an even function, and applying (P) once more gives (2). Now $y = x$ in this formula yields $f(2x) = 4f(x) = 0$ $(x \in V)$, whence $f = 0$.

Theorem 1. *Suppose* (O), (P) *to hold, and let* $f : V \to G$ *be given. Then* f *is orthogonally additive, if and only if the following condition is fulfilled:*

$$f(x) = a(x) + b(x,x) \qquad (x \in V), \tag{11}$$

where $a : V \to G$ *is additive,* $b : V \times V \to G$ *biadditive and symmetric, and*

$$b(x,y) = 0 \qquad (x,y \in V;\ x \perp y). \tag{12}$$

Moreover, in this case the functions a*,* b*, and* $q(x) = b(x,x)$ $(x \in V)$ *are uniquely determined; they are given by* (4),

$$b(x,y) = 2\left[f\left(\frac{x+y}{4}\right) + f\left(\frac{-x-y}{4}\right) - f\left(\frac{x-y}{4}\right) - f\left(\frac{-x+y}{4}\right)\right] \qquad (x,y \in V),$$

and (5), *respectively.*

Proof. It is easy to see that (11) defines an orthogonally additive function f, provided a, b are as in the above condition. So, let now f be an orthogonally additive function, i.e., a solution of (1). Define $a, q_0 : V \to G$ by

$$a(x) = f\left(\frac{x}{2}\right) - f\left(-\frac{x}{2}\right), \quad q_0(x) = f\left(\frac{x}{2}\right) + f\left(-\frac{x}{2}\right) \qquad (x \in V).$$

Obviously a is odd, q_0 is even and, due to (O), both functions are orthogonally additive and $q_0(0) = 0$. Applying (P) we obtain additivity of a and quadraticity of q_0. Lemma 2 shows the existence of a biadditive, symmetric function $b_0 : V \times V \to G$ such that

$$q_0(x) = b_0(x,x) \qquad (x \in V).$$

Since q_0 is orthogonally additive, we get

$$2b_0(x,y) = 0 \qquad (x,y \in V;\ x \perp y).$$

Consequently the function $b : V \times V \to G$ defined by

$$b(x,y) = 2b_0(x,y) \qquad (x,y \in V)$$

is biadditive, symmetric and satisfies (12). On the other hand,

$$2f(x) = a(2x) + q_0(2x) = 2(a(x) + 2q_0(x)) \qquad (x \in V),$$

i.e., the orthogonally additive function $f_0 = f - a - 2q_0$ satisfies $2f_0 = 0$. By Lemma 3 we get $f_0 = 0$, hence

$$f(x) = a(x) + 2q_0(x) = a(x) + b(x, x) \qquad (x \in V).$$

This shows that f really fulfils the condition of the theorem. From Lemmas 1, 2 we get the uniqueness of a, b, q, as well as the formulas for their representation.

Remark 2. Assume Λ is a euclidean ordered field (cf. [3]), $\dim V \geq 2$, and $\varphi : V \times V \to \Lambda$ is bilinear, symmetric, positive definite. According to Rätz [8] the orthogonality $\perp$ defined by

$$x \perp y \Leftrightarrow \varphi(x, y) = 0$$

satisfies (O), (P), and every even orthogonally additive function $q : V \to G$ has the form $q(x) = g(\varphi(x, x))$ $(x \in V)$ with an additive $g : \Lambda \to G$. Together with Theorem 1 it follows that every orthogonally additive function $f : V \to G$ has the form

$$f(x) = a(x) + g(\varphi(x, x)) \qquad (x \in V) \tag{13}$$

with additive functions $a : V \to G$, $g : \Lambda \to G$. Moreover,

$$a(x) = f\left(\frac{x}{2}\right) - f\left(-\frac{x}{2}\right) \qquad (x \in V),$$

$$g(\lambda) = f\left(\frac{u\sqrt{\lambda}}{\sqrt{2\varphi(u, u)}}\right) + f\left(-\frac{u\sqrt{\lambda}}{\sqrt{2\varphi(u, u)}}\right) \qquad (\lambda \in \Lambda,\ \lambda \geq 0)$$

with an arbitrary (but fixed) $u \in V \setminus \{0\}$. When choosing $\Lambda = \mathbb{R}$, we get a real inner product space V, and (13) can be read as $f(x) = a(x) + g(\|x\|^2)$; this was the result obtained in [2].

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