Publ. Math. Debrecen
52 / 3-4 (1998), 299–308

Translation equation and some new geometries

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Dedicated to his friend Zoltán Daróczy
on the occasion of his 60th birthday

Abstract. All translation groups T with an axis which satisfy the translation equation over a (left) vector space X are determined. The group $\{T \cup L\}$ generated by T and a group L of bijective linear transformations of X leads to a geometry $(X, \{T \cup L\})$ which in most cases turns out to be new. However, euclidean and hyperbolic geometries may be constructed this way.

1. It is well-known that the group $I(n, \mathbb{R})$ of euclidean isometries of $\mathbb{R}^n$ is the product of the orthogonal group $O(n, \mathbb{R})$ and of $(\mathbb{R}^n, +)$. If $e \neq 0$ is a fixed element of $\mathbb{R}^n$ and if T^n is the group of translations of the form $x \to x + te$ with $t \in \mathbb{R}$, then already

$$O(n, \mathbb{R}) \cup T^n \tag{1}$$

generates $I(n, \mathbb{R})$.

The idea now, we would like to propose in this note, is to replace T^n in (1) by a suitable group T of bijections

$$x \to T(x, t), \quad t \in \mathbb{R},$$

Mathematics Subject Classification: 39B40, 39B50, 39B70, 51B99, 51M10, 51M99.
Key words and phrases: translation equation, translation group, parameter group, geometry of a group acting on a set, hyperbolic geometry, Lorentz boosts, entire Cremona transformations.

of $\mathbb{R}^n$ satisfying the translation equation (see J. Aczél [1], pp. 245–253)

$$T(x, t+s) = T\big(T(x,t), s\big)$$

with $x \in \mathbb{R}^n$ and real t and s. The geometry

$$\Big(\mathbb{R}^n, \{O(n,\mathbb{R}) \cup T\}\Big) \tag{2}$$

(and also generalizations, see Section 2) then will be of interest, where $\{K\}$ denotes (see [3], p. 10) the group generated by K. For many details of a theory of geometries (S,G), where G is a group of permutations of the set $S \neq \emptyset$, and for many connections of this theory with functional equations, see W. Benz [2]. In Section 2 we will prove a theorem which characterizes a reasonable large class of groups T useful for the definition of new geometries, and, on the other hand, not too far away from the classical case. In Section 3 we will show that a certain group T even leads to n-dimensional hyperbolic geometry. By means of some non-trivial entire Cremona transformations we define in Section 4 a concrete example of a new geometry (2). This geometry is a Cremona geometry in the sense of [2], p. 271. Some general statements about geometries (2) in the case of our translation groups T are also included in Section 4. An open problem will finally be posed.

2. Suppose that X is a (left) vector space over a (commutative or non-commutative) field F of (finite or infinite) dimension $\dim_F X$ at least 2. Let

$$T : F \to \operatorname{Perm} X$$

be a mapping of F into the group of all permutations of X, and let $e \neq 0$ be a fixed element of X. We will call T a *translation group* of X with *axis* (or *direction*) e if, and only if, the following properties hold true.

(i) $T_{t+s} = T_t \cdot T_s$ *for all* $t, s \in F$.

(ii) *For all* $x, y \in X$ *satisfying* $y - x \in Fe$ *there exists exactly one* $t \in F$ *with* $T_t(x) = y$.

(iii) $T_t(x) - x$ *is in* Fe *for all* $x \in X$ *and all* $t \in F$.

Here T_t designates the image of $t \in F$ under T, and $T_t(x)$ denotes the image of $x \in X$ under the permutation T_t of X. Property (i) is the translation equation.

If we put

$$T_t(x) := x + te \tag{3}$$

we obviously get the classical example of a translation group with axis $e \neq 0$.
Other examples in $\mathbb{R}^2$ with axis $(1, 0)$ are

$$T_t(x_1, x_2) = \begin{cases} (x_1 + t, x_2) & \text{for } x_2 \neq 0 \\ (x_1 - t, x_2) & \text{for } x_2 = 0 \end{cases}$$

or

$$T_t(x_1, x_2) = \Big((\sqrt[3]{x_1} + t)^3, x_2 \Big).$$

Theorem 1. *Let H be a maximal subspace of X with*

$$H \oplus Fe = X$$

and let $\varrho : H \times F \to F$ satisfy

(∗) *For all $h \in H$ and $\xi \in F$ there exists exactly one $t = t(h, \xi)$ in F with $\varrho(h, t) = \xi$.*

Then for all $h \in H$ and all $t, \tau \in F$

$$T_t\big(h + \varrho(h, \tau)e\big) := h + \varrho(h, \tau + t)e \tag{4}$$

defines a translation group of X with axis e. There are no other such groups.

Remark. If we define $\varrho(h, t) := t$ for $h \in H$ and $t \in F$, then, obviously, we get the classical case.

Theorem 1 consists of two statements. Concerning the first part we have to show that (4) leads to a translation group with axis e. We observe that for $x \in X$

$$x = h + te \quad \text{with } h \in H \text{ and } t \in F$$

implies that $h \in H$ and $t \in F$ are uniquely determined since $H \oplus Fe$ is a direct sum. First of all we have to show that T_t must be a bijection of X for every given $t \in F$. Suppose that

$$x = h_1 + \varrho(h_1, \tau)e, \quad y = h_2 + \varrho(h_2, \sigma)e$$

hold true together with $T_t(x) = T_t(y)$ for $h_1, h_2 \in H$ and $\tau, \sigma \in F$. Then

$$h_1 + \varrho(h_1, \tau + t)e = h_2 + \varrho(h_2, \sigma + t)e$$

yields $h_1 = h_2$ and hence $\tau + t = \sigma + t$ with $(*)$, i.e. $x = y$. The mapping T_t must thus be injective. It is also surjective: let

$$y = h_2 + \varrho(h_2, \sigma)e$$

be given. We have to solve $T_t(x) = y$ with respect to x. If there exists such an

$$x = h_1 + \varrho(h_1, \tau)e,$$

then (4) leads to

$$h_1 + \varrho(h_1, \tau + t)\, e = h_2 + \varrho(h_2, \sigma)e,$$

i.e. to $h_1 = h_2$ and hence to $\tau = \sigma - t$. On the other hand we have

$$T_t\big(h_2 + \varrho(h_2, \sigma - t)e\big) = h_2 + \varrho(h_2, \sigma)e,$$

in view of (4).

In order now to prove

$$T_{t+s}(x) = T_t\big(T_s(x)\big)$$

for all $x \in X$ and all $t, s \in F$, we put

$$x =: h + \xi e \quad \text{with } h \in H \text{ and } \xi \in F.$$

Then

$$T_s(x) = h + \varrho(h, \tau + s)e$$

if $\tau \in F$ satisfies $\varrho\,(h, \tau) = \xi$. Hence

$$T_t\big(T_s(x)\big) = h + \varrho(h, \tau + s + t)e,$$

on account of (4). But

$$T_{t+s}\big(x + \varrho(h, \tau)e\big)$$

is also equal to $h + \varrho(h, \tau + t + s)e$, in view of (4). The translation equation is thus satisfied.

In order to prove (ii) we put

$$x = h_1 + \xi e \quad \text{and} \quad y = h_2 + \eta e$$

with $h_1, h_2 \in H$ and $\xi, \eta \in F$. Now

$$y - x \in Fe$$

implies $h_1 = h_2 =: h$. With

$$\xi = \varrho(h, \tau) \quad \text{and} \quad \eta = \varrho(h, \sigma)$$

for uniquely determined $\tau, \sigma \in F$, then

$$h + \varrho(h, \sigma)e = y = T_t(x) = h + \varrho(h, \tau + t)e$$

leads to $\varrho(h, \sigma) = \varrho(h, \tau + t)$, i.e. to the unique solution $t = \sigma - \tau$, because of $(*)$. Property (iii) follows immediately from (4).

The other part of the proof of Theorem 1 is to show that a translation group of X with axis $e \neq 0$ can be represented by equation (4) with a function ϱ satisfying $(*)$. So suppose that H is a direct summand of Fe. If $h \in H$ and $t \in F$, we define

$$\varrho(h, t)e := T_t(h) - h \tag{5}$$

by observing (iii). In order to prove $(*)$, assume that $h \in H$ and $\xi \in F$ are given. Put $y := h + \xi e$. Since $y - h \in Fe$, (ii) yields that there is exactly one $t \in F$ with $T_t(h) = y$. Hence

$$\varrho(h, t)e = T_t(h) - h = y - h = \xi e.$$

Assume, furthermore, $\varrho(h, t') = \xi$ with $t' \in F$. Then

$$T_{t'}(h) - h = \varrho(h, t')e = \xi e,$$

i.e. $T_t(h) = h + \xi e = T_{t'}(h)$, i.e. $t' = t$ in view of (ii).

Finally we must prove equation (4). In order to do this we will apply definition (5) two times:

$$T_t\big(h + \varrho(h, \tau)e\big) = T_t T_\tau(h) = T_{t+\tau}(h) = h + \varrho(h, t + \tau)e.$$

This finishes the proof of Theorem 1.

Remarks. a) Instead of (4) we also may write

$$T_t(x) = x + \big[\varrho(h, \tau + t) - \varrho(h, \tau)\big]e \tag{6}$$

in the case that $x = h + \varrho(h, \tau)e$ with $h \in H$ and $\tau \in F$.

b) Let $\tau : H \to F$ be an arbitrary function and let $\varrho : H \times H \to F$ satisfy $(*)$. Then also

$$\varrho'(h, t) := \varrho\big(h, \tau(h) + t\big)$$

satisfies $(*)$, and ϱ and ϱ' define, according to (4), the same mapping $T : F \to \operatorname{Perm} X$.

If we define $\tau(h)$ by $\varrho\big(h, \tau(h)\big) = 0$, then for all $h \in H$

$$\varrho'(h, 0) = \varrho\big(h, \tau(h) + 0\big) = 0.$$

In addition to $(*)$ we hence may assume, without loss of generality, that also $\varrho(h, 0) = 0$ holds true for all $h \in H$.

c) We already were speaking of a group in connection with the mapping $T : F \to \operatorname{Perm} X$. But, obviously, T is an abelian group, isomorphic to the additive group of the field F: since $T_0(x) = x$ for all $x \in X$ in view of (6), T_0 is the identity element of T. If T_t were equal to $T_{t'}$ for $t \neq t'$ we would get a contradiction to $(*)$, namely

$$\varrho(h, \tau + t) = \varrho(h, \tau + t'),$$

and this from (6) for all $h \in H$ and $\tau \in F$.

The general definition of a new geometry in our context is now as follows. We still would like to work with our vector space X as already introduced. Let L be a group of bijective and linear mappings of X and let T be a group of translations of X with a given axis $e \neq 0$. Then the geometry

$$\big(X, \{L \cup T\}\big) \tag{7}$$

in the sense of [2], chapter 1, is of interest. If we assume that L acts transitively on the set of lines through the origin $0 \in X$, then we get translation groups of the form

$$\lambda T \lambda^{-1}, \quad \lambda \in L,$$

for every line Fa with $0 \neq a \in X$. Given a geometry (7) the problem is to find *invariants, invariant notions, defining invariants*, and so on, of this geometry (see [2], chapter 1).

3. Suppose that our vector space X of Section 2 is now $\mathbb{R}^n$, n a positive integer. We put

$$e := (1, 0, \dots, 0) \tag{8}$$

and

$$H := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 = 0\}. \tag{9}$$

We define

$$\varrho\big((0, x_2, \dots, x_n), t\big) := \sinh t \cdot \sqrt{1 + x_2^2 + \ldots + x_n^2}.$$

Obviously, property $(*)$ is satisfied. We now would like to designate the corresponding translation group with axis e by Δ.

Theorem 2. *The geometry*

$$\Big(\mathbb{R}^n, \{O(n, \mathbb{R}) \cup \Delta\}\Big) \tag{10}$$

is isomorphic to the n-dimensional hyperbolic geometry.

Proof. In view of (4) we get

$$T_t(x) = \Big(x_1 \cosh t + \sqrt{1 + x^2} \sinh t, x_2, x_3, \dots, x_n\Big)$$

written by means of the euclidean scalar product, i.e. with

$$x^2 = x_1^2 + \ldots + x_n^2.$$

Replacing

$$x = (x_1, \dots, x_n) \in \mathbb{R}^n$$

by its Weierstraa coordinates

$$\omega(x) = \Big(x_1, \dots, x_n, \sqrt{1 + x^2}\,\Big)$$

we may write for $x \to T_t(x)$

$$\omega(x) \to \omega(x) \cdot H(t)$$

with the Lorentz boost

$$H(t) = \begin{pmatrix} \cosh t & & & & \sinh t \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ \sinh t & & & & \cosh t \end{pmatrix}$$

which is represented as a $(n+1) \times (n+1)$-matrix. To an element of $O(n, \mathbb{R})$ we assign its induced Lorentz matrix. The group

$$\{O(n, \mathbb{R}) \cup \Delta\}$$

is then isomorphic to the group Π of all linear and orthochronous Lorentz transformations of $\mathbb{R}^{n+1}$. We hence get with this group isomorphism and with the mapping ω that (10) is isomorphic (see [2], p. 32) to the hyperbolic geometry (S, Π) with

$$S = \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; x_{n+1} = \sqrt{1 + x_1^2 + \ldots + x_n^2} \right\}$$

and with Π as defined above (see [2], pp. 60–63).

4. We again would like to work with $X := \mathbb{R}^n$ and with (8) and (9). Put

$$\varrho\big((0, x_2, \ldots, x_n), t\big) := t \cdot \left(1 + x_2^2 + \ldots + x_n^2\right).$$

Property $(*)$ is then satisfied. In view of (4) we hence get

$$T_t(x_1, \ldots, x_n) = \Big(x_1 + t \cdot \left[1 + x_2^2 + \ldots + x_n^2\right], x_2, \ldots, x_n\Big).$$

These are entire Cremona transformations of $\mathbb{R}^n$ for every $t \in \mathbb{R}$ and they are, obviously, in all cases $t \neq 0$ non-trivial examples of those transformations. It would be nice to draw up more intensively the corresponding geometry

$$\Big(\mathbb{R}^n, \{O\,(n, \mathbb{R}) \cup T\}\Big) \tag{11}$$

and to determine some of its defining invariants and invariant notions.

Suppose now that an arbitrary geometry (2) is given such that T is a translation group with axis (8) and with maximal subspace (9) according to Theorem 1. Denote by G^n the group

$$\{O(n,\mathbb{R}) \cup T\}.$$

Proposition 1. *G^n acts transitively on $\mathbb{R}^n$.*

Proof. Let $x \neq 0$ be an element of $\mathbb{R}^n$. We have to show that there exists γ in G^n with $\gamma(0) = x$. Since T acts transitively on the line $\mathbb{R} \cdot e$, in view of (ii), there exists

$$\tau \in T < G^n$$

with $\tau(0) = ||x||e$. Because of $||\tau\,(0)|| = ||x||$ there furthermore exists $\delta \in O\,(n,\mathbb{R})$ with

$$\delta\big(\tau\,(0)\big) = x.$$

Now put $\gamma := \delta\tau$. $\square$

The stabilizer of G^n in 0 needs not to be the group $O(n,\mathbb{R})$ as in the classical case or in the hyperbolic case of Section 3. If we take for instance that example of the beginning of this section in the case $n = 2$, then for

$$\pi := T_{-\frac{1}{\sqrt{2}}} \cdot R\left(-\frac{\pi}{2}\right) \cdot T_{-\frac{\sqrt{2}}{3}} \cdot R\left(\frac{\pi}{4}\right) \cdot T_1$$

we have $\pi(0) = 0$ and $\pi(1,0) = \left(\frac{1}{\sqrt{2}}, 0\right)$, and hence $\pi \notin O(2,\mathbb{R})$, where $R\,(\alpha)$ designates the rotation (in the positive sense) about 0 with angle α.

Proposition 2. *Assume that the stabilizer of G^n in 0 is $O(n,\mathbb{R})$. Then to every $\gamma \in G^n$ there exist $\alpha, \beta \in O(n,\mathbb{R})$ and $\tau \in T$ with $\gamma = \alpha\tau\beta$.*

Proof. Suppose that $\gamma(0) = a \neq 0$. Take α in $O(n,\mathbb{R})$ and $\tau \in T$ with $\alpha(||a||e) = a$ and $\tau(0) = ||a||e$. Hence

$$\tau^{-1}\alpha^{-1}\gamma(0) = 0$$

and thus $\beta := \tau^{-1}\alpha^{-1}\gamma \in O(n,\mathbb{R})$. $\square$

Proposition 3. *Suppose that*

$$G^n = O(n, \mathbb{R}) \cdot T \cdot O(n, \mathbb{R}). \tag{12}$$

The stabilizer of G^n in 0 is then $O(n, \mathbb{R})$.

Proof. Assume that $\gamma(0) = 0$ for $\gamma \in G^n$. Since γ is of the form $\alpha\tau\beta$ with $\tau \in T$ and α, β in $O(n, \mathbb{R})$, we get

$$\tau\beta(0) = \alpha^{-1}(0),$$

i.e. $\tau(0) = 0$, i.e. $\tau = T_0$, in view of (ii). Hence $\gamma = \alpha\beta$ must be in $O(n, \mathbb{R})$. □

Remark. In the case (12), the stabilizer of G^n in every $x \in \mathbb{R}^n$ must be, of course, isomorphic to $O(n, \mathbb{R})$, in view of Proposition 1.

The problem we finally would like to pose is the following. Given two functions ϱ_1 and ϱ_2 satisfying $(*)$, and given their corresponding translation groups T_1 and T_2. Find necessary and sufficient conditions in ϱ_1 and ϱ_2 such that the geometries

$$\Big(\mathbb{R}^n, \{O(n, \mathbb{R}) \cup T_i\}\Big), \quad i = 1, 2,$$

are isomorphic.

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(Received October 28, 1997)