Publ. Math. Debrecen
52 / 3-4 (1998), 309–316

Regular functions that preserve digital representation

By ZOLTÁN BOROS (Debrecen)

Dedicated to Professors Zoltán Daróczy and Imre Kátai
on the occasion of their 60th birthday

Abstract. A functional equation related to generalized number systems in Euclidean spaces is investigated. Reasonable sufficient conditions, under which every solution of the equation is continuous or every smooth solution of the equation is linear, are established.

Definitions. Let us consider a finite set P with $\{0\} \subsetneqq P \subset \mathbb{R}$, a positive integer N, and a sequence $(q_n)\colon \mathbb{Z} \to \mathbb{R}^N$ such that $\sum_{n=1}^{\infty} |q_n| < \infty$ (where $|u|$ denotes the Euclidean norm of $u \in \mathbb{R}^N$) and for every vector $x \in \mathbb{R}^N$ there exist $m \in \mathbb{Z}$ and $\varepsilon_n \in P$ $(n = m, m+1, \dots)$ satisfying $x = \sum_{n=m}^{\infty} \varepsilon_n q_n$. Such a pair $(P, (q_n))$ will be called a digital representation system in $\mathbb{R}^N$. We say that a function $f\colon \mathbb{R}^N \to \mathbb{R}$ preserves the digital representation with respect to $(P, (q_n))$ if

$$f\left(\sum_{n=m}^{\infty} \varepsilon_n q_n\right) = \sum_{n=m}^{\infty} \varepsilon_n f(q_n) \tag{1}$$

holds for every $m \in \mathbb{Z}$ and $\varepsilon_n \in P$ $(n = m, m+1, \dots)$.

Mathematics Subject Classification: Primary 39B22, Secondary 11K55.
Key words and phrases: number system, functional equation on restricted domain, representation preserver function.
The author's research was supported by the Grants OTKA T-016846 and FKFP 0310/1997.

Remark 1. Obviously every linear functional $f:\mathbb{R}^N\to\mathbb{R}$ satisfies (1). It is, however, an open problem whether there exist non-linear solutions f of (1). For the particular case $N=1$ analogous problems are investigated in [9], [5] and [1]; the main theorems in [5] and [1] suggest the conjecture that every solution of (1) has to be linear. We neither prove nor disprove this conjecture in this generality here. Our aim is to take the first step towards the characterization of representation preserver functions involving dimensions greater than (or equal to) one.

Proposition 1. *If f preserves the digital representation with respect to the digital representation system $(P,(q_n))$, then $\sum_{n=1}^{\infty}|f(q_n)|<\infty$.*

Proof. Let $p\in P\setminus\{0\}$, $\varepsilon_n=p$ if $f(q_n)\geq 0$, while $\varepsilon_n=0$ if $f(q_n)<0$, and $\delta_n=p-\varepsilon_n$ $(n\in\mathbb{N})$. It follows from equation (1) that the following series are convergent and

$$\begin{aligned}&f\left(\sum_{n=1}^{\infty}\varepsilon_n q_n\right)-f\left(\sum_{n=1}^{\infty}\delta_n q_n\right)\\ &=\sum_{n=1}^{\infty}\varepsilon_n f(q_n)-\sum_{n=1}^{\infty}\delta_n f(q_n)=p\sum_{n=1}^{\infty}|f(q_n)|.\end{aligned}$$

Notation. In the sequel we consider a fixed digital representation system $(P,(q_n))$ in $\mathbb{R}^N$ and the corresponding sets defined by

$$S_m=\left\{\sum_{n=m+1}^{\infty}\varepsilon_n q_n \;\middle|\; \varepsilon_n\in P \quad (n=m+1,m+2,\dots)\right\}\quad (m\in\mathbb{Z}),$$

$$R_k=\left\{\sum_{l=1}^{k}\varepsilon_l q_l \;\middle|\; \varepsilon_l\in P \quad (l=1,2,\dots,k)\right\}\quad (k\in\mathbb{N})\quad\text{and}$$

$$T_m=\left\{\sum_{l=m-k}^{m}\varepsilon_l q_l \;\middle|\; \varepsilon_l\in P \quad (l=m-k,\dots,m),k\in\mathbb{N}\right\}\quad (m\in\mathbb{Z}).$$

We will also write $\|P\|=\max\{|p| \mid p\in P\}$,

$$\sigma_m=\|P\|\sum_{n=m+1}^{\infty}|q_n|,\quad\text{and}\quad \varrho_m=\|P\|\sum_{n=m+1}^{\infty}|f(q_n)|\quad (m\in\mathbb{Z}).$$

We will denote the open ball with radius r and centered at x by $B_r(x)$, the interior of the set H by H°, and the (metric) closure of the set H by $\overline{H}$.

Remark 2. It follows from $0 \in P$ that $\mathbf{0} \in S_m \cap R_k \cap T_m$ for every $m \in \mathbb{Z}$ and $k \in \mathbb{N}$, where $\mathbf{0}$ denotes the zero vector in $\mathbb{R}^N$. Moreover, if $f : \mathbb{R}^N \to \mathbb{R}$ preserves the digital representation with respect to $(P, (q_n))$, then

$$f(t+s) = f(t) + f(s) \qquad (t \in T_m\,,\ s \in S_m)\ (m \in \mathbb{Z})$$

and

$$f(z+u+w) = f(z) + f(u) + f(w) \quad (z \in T_0,\ u \in R_k,\ w \in S_k)\ (k \in \mathbb{N}).$$

In particular, $f(\mathbf{0}) = 0$. Let us also observe that

$$S_m \subset \overline{B_{\sigma_m}(\mathbf{0})} \ \text{ and } \ f(S_m) \subset \overline{B_{\varrho_m}(0)} \ \ (m \in \mathbb{Z}).$$

We may (and will) assume that $q_n \neq \mathbf{0}$ $(n \in \mathbb{Z})$.

Definition. We call the digital representation system $(P, (q_n))$ *non-accumulative* if ${T_0}' = \emptyset$.

Proposition 2. *The digital representation system $(P, (q_n))$ is non-accumulative if and only if the set $T_m \cap B$ contains finitely many elements for every $m \in \mathbb{Z}$ and for every bounded set $B \subset \mathbb{R}^N$.*

Proof. The finite intersection property is clearly sufficient: one only has to apply it for $m = 0$ and $B = B_r(x)$ with arbitrary $r > 0$ and $x \in \mathbb{R}^N$. Conversely, if $(P, (q_n))$ is non-accumulative and $B \subset \mathbb{R}^N$ is bounded, then $T_0 \cap B$ is finite (cf. the Bolzano-Weierstrass theorem). If $m < 0$, then we have $T_m \subset T_0$, hence $T_m \cap B \subset T_0 \cap B$, thus $T_m \cap B$ is also finite. If $m > 0$, let $r_m = \|P\| \sum_{l=1}^{m} |q_l|$ and $B_1 = B + B_{2r_m}(\mathbf{0})$. If $u \in T_m \cap B$, then there exist $t \in T_0$ and $w \in R_m$ such that $u = t + w$. In this case $t \in B_1$, since $|t - u| = |w| \leq r_m < 2r_m$. Obviously B_1 is bounded, thus $T_0 \cap B_1$ is finite. The set R_m is also finite and $T_m \cap B \subset (T_0 \cap B_1) + R_m$, hence $T_m \cap B$ is also finite.

The first part of the following result is proved for geometric sequences in [4] and in full generality in [2]. It is, however, reasonable to involve the short proof, which is due to Maksa (cf. [10]), into this presentation as well.

Lemma 1. *The set S_m , corresponding to a digital representation system $(P,(q_n))$, is compact and satisfies $S_m = \overline{S_m^\circ}$ for every $m \in \mathbb{Z}$.*

Proof. The set of sequences with values in the finite set P can be considered as a topological product of infinitely many copies of the compact set P, hence it is also compact. The mappings $\phi_n(\varepsilon_{m+1}, \varepsilon_{m+2}, \dots) = \varepsilon_n q_n$ are continuous and the sum $\phi = \sum_{n=m+1}^{\infty} \phi_n$ is uniformly convergent, hence ϕ is also continuous and the codomain S_m of ϕ is compact.

Now it follows easily that the interior of S_m is non-void for every $m \in \mathbb{Z}$. Indeed, if we assume that $S_m^\circ = \emptyset$ for some $m \in \mathbb{Z}$, then $\overline{S_m} = S_m$ yields that S_m is a nowhere dense set. Since T_m is a countable set and $\mathbb{R}^N = T_m + S_m$, we obtain that $\mathbb{R}^N$ is a set of first category, which contradicts the completeness of $\mathbb{R}^N$ (cf. Baire's theorem).

Let us begin the proof of $S_m = \overline{S_m^\circ}$ (for arbitrary $m \in \mathbb{Z}$) with the the trivial inclusion $\overline{S_m^\circ} \subset \overline{S_m} = S_m$. In order to prove the reversed inclusion we consider arbitrary $x \in S_m$ and $r > 0$. Then there exist ε_n $(n = m+1, m+2, \dots)$ and $k \in \mathbb{N}$ such that $x = \sum_{n=m+1}^{\infty} \varepsilon_n q_n$, $k > m$, and $\sigma_k < r/2$. Let $x_k = \sum_{n=m+1}^{k} \varepsilon_n q_n$ and $s = \sum_{n=k+1}^{\infty} \varepsilon_n q_n = x - x_k$. Since $S_k^\circ \neq \emptyset$, there exists $u \in S_k^\circ$. Let $y = x_k + u$. Then $y \in x_k + S_k^\circ \subset S_m^\circ$ and $|y - x| = |u - s| \leq 2\sigma_k < r$, which completes the proof.

Having enumerated some interesting properties of digital representation systems we begin the investigation of representation preserver functions.

Theorem 1. *If the digital representation system $(P,(q_n))$ is non-accumulative and $f : \mathbb{R}^N \to \mathbb{R}$ preserves the digital representation with respect to $(P,(q_n))$, then f is continuous.*

Proof. Let us fix $x_0 \in \mathbb{R}^N$ and $\sigma > 0$ arbitrarily. Due to Proposition 1 there exists $m \in \mathbb{N}$ such that $\varrho_m < \sigma/2$. Let $x_1 \in B_{\sigma_m}(x_0)$. If $u \in T_m$ with $|u - x_0| \geq 2\sigma_m$, then $|x_1 - u| > \sigma_m$, hence $x_1 - u \notin S_m$. The intersection $W = T_m \cap B_{2\sigma_m}(x_0)$ is finite, therefore $W_1 + S_m$ is compact for every $W_1 \subset W$. Thus there exists $r \in]0, \sigma_m[$ such that for every $t \in W$ we have either $x_0 \in t + S_m$ or $(t + S_m) \cap B_r(x_0) = \emptyset$. Let us now consider an arbitrary $x \in B_r(x_0)$. Then there exists $t \in W$ such that $x - t \in S_m$ and $x_0 - t \in S_m$, hence (cf. Remark 2)

$$\begin{aligned} |f(x) - f(x_0)| &= |\,(f(t) + f(x-t)) - (f(t) + f(x_0 - t))\,| \\ &= |f(x-t) - f(x_0 - t)| \leq 2\varrho_m < \sigma. \end{aligned}$$

Example 1. If (θ, P) is a canonical number system in the ring of Gaussian integers and $q_n = \theta^{-n}$ $(n \in \mathbb{Z})$, then $(P, (q_n))$ is a digital representation system in $\mathbb{C}$ (now regarded as $\mathbb{R}^2$) by [7] and obviously $T_0' = \emptyset$ (since $T_0 = \mathbb{Z} + i\mathbb{Z}$). Further examples are provided in [6] and [8].

The following generalizations of the notion of directional derivatives, which will serve as powerful devices in our investigations, are closely related to Clarke's generalized directional derivatives (cf. [3]). Let us note that our generalized directional derivatives are not necessarily finite.

Definition. If $D \subset \mathbb{R}^N$, $f : D \to \mathbb{R}$, $x_0 \in D^\circ$, $v \in \mathbb{R}^N$, and $\delta > 0$ such that $B_\delta(x_0) \subset D$, let

$$\partial_0^\delta f(x_0, v) = \left\{ \frac{1}{t}(f(x+tv) - f(x)) \;\middle|\; t \in \mathbb{R} \text{ with } x, x+tv \in B_\delta(x_0) \right\},$$

$$\partial_L^\delta f(x_0, v) = \inf \partial_0^\delta f(x_0, v), \quad \partial_U^\delta f(x_0, v) = \sup \partial_0^\delta f(x_0, v),$$

$$\partial_L f(x_0, v) = \lim_{\delta \to 0} \partial_L^\delta f(x_0, v), \quad \text{and} \quad \partial_U f(x_0, v) = \lim_{\delta \to 0} \partial_U^\delta f(x_0, v).$$

Lemma 2. *If* $\mathbf{0} \in S_m^\circ$ *for every* $m \in \mathbb{Z}$ *and* $f : \mathbb{R}^N \to \mathbb{R}$ *preserves the digital representation with respect to the digital representation system* $(P, (q_n))$, *then* $\partial_L f(x, v) \le \partial_U f(y, v)$ *for every* $x, y, v \in \mathbb{R}^N$.

Proof. Let $x, y, v \in \mathbb{R}^N$ and $\varepsilon > 0$. We may assume that $v \neq \mathbf{0}$ (the $v = \mathbf{0}$ case is trivial). Our definitions yield $\lim_{m\to\infty} \sigma_m = 0$, hence there exists $m \in \mathbb{Z}$ such that $\sigma_m < \varepsilon/2$. Since $\mathbb{R}^N = T_m + S_m$, we can choose $x', y' \in T_m$ such that $x \in x' + S_m$ and $y \in y' + S_m$, which implies $\max\{|x' - x|, |y' - y|\} \le \sigma_m < \varepsilon/2$. By our assumption there exists $r > 0$ with $B_r(\mathbf{0}) \subset S_m$. Let $\delta \in \,]0, \frac{1}{|v|} \min\{r, \varepsilon/2\}[$ and $t \in \,]-\delta, \delta[$. This choice yields $tv \in S_m$, hence

$$|(x' + tv) - x| \le |tv| + |x' - x| = |t|\,|v| + |x' - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

i.e. $x' + tv \in B_\varepsilon(x)$ and, similarly, $y' + tv \in B_\varepsilon(y)$. Due to Remark 2 we have

$$\frac{f(tv)}{t} = \frac{f(x') + f(tv) - f(x')}{t} = \frac{f(x' + tv) - f(x')}{t} \in \partial_0^\varepsilon f(x, v)$$

and, analogously, $\frac{f(tv)}{t} \in \partial_0^\varepsilon f(y,v)$, thus

$$\partial_L^\varepsilon f(x,v) \le \frac{f(tv)}{t} \le \partial_U^\varepsilon f(y,v).$$

Letting ε tend to 0 we obtain the statement.

Proposition 3. *If $D \subset \mathbb{R}^N$ and $f : D \to \mathbb{R}$ is continuously differentiable at $x_0 \in D^\circ$, then $\partial_L f(x_0,v) = \partial_U f(x_0\,,v) = f'(x_0)v$ for every $v \in \mathbb{R}^N$.*

Proof. Under our assumption there exists $r > 0$ such that f is differentiable on $B_r(x_0)$. If $x \in B_r(x_0)$, $v \in \mathbb{R}^N$, and $t \in \mathbb{R} \setminus \{0\}$ such that $x + tv \in B_r(x_0)$, then we can apply Lagrange's mean value theorem, which yields $(f(x+tv) - f(x))/t = f'(x+t_1v)v$ with some t_1 between 0 and t. Since f' is continuous at x_0, this implies the statement.

Theorem 2. *If $\mathbf{0} \in S_m^\circ$ for every $m \in \mathbb{Z}$, $f : \mathbb{R}^N \to \mathbb{R}$ is continuously differentiable, and f preserves the digital representation with respect to the digital representation system $(P,(q_n))$, then f is linear.*

Proof. Combining Lemma 2 with Proposition 3 we obtain that $f'(x)v \le f'(y)v$ for every $x,y,v \in \mathbb{R}^N$. The reversed inequality follows by interchanging x and y, therefore f' is constant, hence f is an affine function with $f(\mathbf{0}) = 0$ (cf. Remark 2), i.e. f is linear.

Definition. The digital representation system $(P,(q_n))$ will be called *uniform* if there exist $K > 0$, $R > 0$, and a mapping $d\colon B_R(\mathbf{0}) \setminus \{\mathbf{0}\} \to \mathbb{Z}$ such that for every $x \in B_R(\mathbf{0}) \setminus \{\mathbf{0}\}$ we have $x \in S_{d(x)}$ and $K|x| \ge \sigma_{d(x)}$.

Lemma 3. *If the digital representation system $(P,(q_n))$ is uniform, then there exists $M > 0$ such that for every $v \in \mathbb{R}^N$ and $k \in \mathbb{Z}$ we have $|v^\top q_n| \ge M|v|\,|q_n|$ for some $n > k$.*

Proof. We shall use the notation introduced in the above definition. Let $M = \frac{1}{2K}$ and assume that, on the contrary, there exist $v \in \mathbb{R}^N$ and $k \in \mathbb{Z}$ such that $|v^\top q_n| < M|v|\,|q_n|$ for every $n > k$ (whence $v \ne \mathbf{0}$). Obviously one can replace v with $u = \lambda v$ in the above inequality if $\lambda \in \mathbb{R}\setminus\{0\}$. We can choose $m_0 \in \mathbb{Z}$ such that $m_0 \ge k$ and $\sigma_{m_0} < R$. Let $\delta = \sigma_{m_0}/K$. For any $x \in B_R(\mathbf{0}) \setminus \{\mathbf{0}\}$ obviously $|x| \le \sigma_{d(x)}$, hence $K \ge 1$. Then $\delta < R$ and for every $x \in B_\delta(\mathbf{0}) \setminus \{\mathbf{0}\}$ we have $\sigma_{m_0} = K\delta > K|x| \ge \sigma_{d(x)}$, thus $d(x) > m_0$.

Let $u = \frac{\delta}{2|v|}v$. Then $u \in B_\delta(\mathbf{0}) \setminus \{\mathbf{0}\}$, hence there exist $m = d(u) \in \mathbb{Z}$ and $\varepsilon_n \in P$ $(n = m+1, m+2, \dots)$ such that $u = \sum_{n=m+1}^{\infty} \varepsilon_n q_n$ and

$$|u|^2 = |u^\top u| = \big|u^\top \sum_{n=m+1}^{\infty} \varepsilon_n q_n\big| = \big|\sum_{n=m+1}^{\infty} \varepsilon_n u^\top q_n\big|$$

$$\leq \sum_{n=m+1}^{\infty} |\varepsilon_n|\, |u^\top q_n| < \sum_{n=m+1}^{\infty} \|P\|\, M\, |u|\, |q_n| = M|u|\sigma_m \leq MK|u|^2\,,$$

i.e. $MK > 1$, which contradicts the above given formula for M.

Theorem 3. *If $(P, (q_n))$ is a uniform digital representation system in $\mathbb{R}^N$, $f : \mathbb{R}^N \to \mathbb{R}$ preserves the digital representation with respect to $(P, (q_n))$, and $x_0, y_0 \in \mathbb{R}^N$ such that f is differentiable at x_0 and y_0, then $f'(x_0) = f'(y_0)$.*

In particular, if f is a differentiable representation preserver function with respect to a uniform digital representation system, then f is linear.

Proof. Let α_n denote the coefficient of q_n in the digital representation of x_0 and choose $\alpha'_n \in P \setminus \{\alpha_n\}$ for every $n \in \mathbb{Z}$ (where, of course, $\alpha_n = 0$ for almost all, i.e. except finitely many, negative integers n). Since f is differentiable at x_0, for arbitrary $\varepsilon > 0$ there exists $m_1 \in \mathbb{Z}$ such that for $n > m_1$ and $x = \sum_{k \in \mathbb{Z}\setminus\{n\}} \alpha_k q_k + \alpha'_n q_n$ we have

$$\varepsilon > \frac{|f(x) - f(x_0) - f'(x_0)(x - x_0)|}{|x - x_0|}$$

$$= \frac{|(\alpha'_n - \alpha_n) f(q_n) - f'(x_0)(\alpha'_n - \alpha_n) q_n|}{|(\alpha'_n - \alpha_n) q_n|} = \frac{|f(q_n) - f'(x_0) q_n|}{|q_n|},$$

i.e., $|f(q_n) - f'(x_0)q_n| < \varepsilon |q_n|$. Analogously, there exists $m_2 \in \mathbb{Z}$ such that for every $n > m_2$ we have $|f(q_n) - f'(y_0)q_n| < \varepsilon|q_n|$. Applying these inequalities and choosing $n > \max\{m_1, m_2\}$ as in Lemma 3 with $v = (f'(x_0) - f'(y_0))^\top$, we obtain

$$2\varepsilon|q_n| > |(f'(x_0) - f'(y_0))q_n| \geq M|(f'(x_0) - f'(y_0))^\top|\,|q_n|.$$

Since $\varepsilon > 0$ was arbitrary, this yields $f'(x_0) = f'(y_0)$.

Example 2. It follows from the proof of Theorem 5 in [4] that for any non-real $q \in \mathbb{C}$ with $0 < |q| < 1$ there exist $N \in \mathbb{N}$ and $r_0 > 0$ such that with $P_N = \{0, 1, \dots, N\}$ the pair $(P_N, (q^n))$ is a digital representation system in $\mathbb{C}$ (as $\mathbb{R}^2$) and $B_{|q|^k r_0}(\mathbf{0}) \subset S_k = q^k S_0$ $(k \in \mathbb{Z})$, therefore $(P_N, (q^n))$ is uniform.

References

[1] Z. Boros, On completely P-additive functions with respect to interval-filling sequences of type P, *Acta Math. Hung.* **65** no. 1 (1994), 17–26.

[2] Z. Boros, Sequences of connected spectrum and the Vilenkin group, *Publ. Math. Debrecen* **47** no. 3-4 (1995), 403–410.

[3] F. H. Clarke, Generalized gradients and applications, *Trans. Am. Math. Soc.* **205** (1975), 247–262.

[4] Z. Daróczy and I. Kátai, Generalized number systems in the complex plane, *Acta Math. Hung.* **51** (1988), 409–416.

[5] Z. Daróczy, I. Kátai and T. Szabó, On completely additive functions related to interval-filling sequences, *Arch. Math. (Basel)* **54** no. 2 (1990), 173–179.

[6] I. Kátai and I. Környei, On number systems in algebraic number fields, *Publ. Math. Debrecen* **41** no. 3-4 (1992), 289–294.

[7] I. Kátai and J. Szabó, Canonical number systems for complex integers, *Acta Sci. Math. (Szeged)* **37** (1975), 255–260.

[8] B. Kovács, Representation of complex numbers in number systems, *Acta Math. Hung.* **58** no. 1-2 (1991), 113–120.

[9] Gy. Maksa, On completely additive functions, *Acta Math. Hung.* **48** no. 3-4 (1986), 353–355.

[10] Gy. Maksa, Interval-filling sequences and the dyadic group, *Grazer Math. Ber.* **315** (1991), 69–74.

ZOLTÁN BOROS
INSTITUTE OF MATHEMATICS AND INFORMATICS
LAJOS KOSSUTH UNIVERSITY
H–4010 DEBRECEN, P.O. BOX 12
HUNGARY

(Received December 1, 1997)