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Orthogonality equation almost everywhere

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Dedicated to Professors Zoltán Daróczy
and Imre Kátai on their 60th birthdays

Abstract. For inner product spaces X and Y, we consider the *orthogonality equation*

$$\langle f(x) \mid f(z)\rangle = \langle x \mid z\rangle \quad \text{for } x, z \in X$$

as well as its restricted versions

$$\langle f(x) \mid f(z)\rangle = \langle x \mid z\rangle \quad \text{for } (x, z) \in X^2 \setminus M$$

and

$$\langle f(x) \mid f(z)\rangle = \langle x \mid z\rangle \quad \text{for } x, z \in X \setminus U$$

where $f : X \to Y$ and the sets M and U are, in some sense, *small* in X^2 and X, respectively. Under some additional assumptions we prove that for a solution f of the orthogonality equation postulated almost everywhere (in one of the two senses mentioned above) there exists a unique solution f_* of the (unrestricted) orthogonality equation such that f and f_* are equal almost everywhere on X.

1. Introduction

Let X and Y be (real or complex) inner product spaces. When dealing with *inner product preserving mappings*, i.e., with solutions $f : X \to Y$ of the *orthogonality equation*

$$\text{(OE)} \qquad \langle f(x) \mid f(z)\rangle_Y = \langle x \mid z\rangle_X \quad \text{for } x, z \in X$$

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one can consider a class of functions satisfying the above equation *almost everywhere*, i.e.,

$$\text{(A)} \qquad \langle f(x) \mid f(z)\rangle_Y = \langle x \mid z\rangle_X \quad \text{for } (x,z) \in X^2 \setminus M$$

where $M \subset X^2$ is supposed to be a "small" set in X^2.

The analogous problem for the Cauchy functional equation was raised by P. Erdös [3] and resulted in numerous papers some of which will be quoted in a sequel.

One can separate the variables x and z and consider the following variation of (A)

$$\text{(A')} \qquad \langle f(x) \mid f(z)\rangle_Y = \langle x \mid z\rangle_X \quad \text{for } x, z \in X \setminus U$$

where $U \subset X$ is "small" in X. Again, there is an analogy with similar problems for the Cauchy equation dealt with by S. Hartman [9] and others.

2. Linearly invariant ideals

Definition 1. For an arbitrary set X a family $\mathcal{J} \subset 2^X$ is called an *ideal* in X iff (cf. [12], p. 437)

(i) if $I \in \mathcal{J}$ and $J \subset I$, then $J \in \mathcal{J}$;

(ii) if $I, J \in \mathcal{J}$, then $I \cup J \in \mathcal{J}$.

If, in addition, $\mathcal{J}$ satisfies

(iii) $X \notin \mathcal{J}$,

then it is called *proper* ideal.

Now, suppose that $(X, \mathbb{K}, +, \cdot)$ is a vector space over $\mathbb{K}$.

Definition 2. We say that an ideal $\mathcal{J} \subset 2^X$ is *linearly invariant* iff – besides of (i) and (ii) – it satisfies

(iv) for every $x \in X$, $k \in \mathbb{K}$ and $I \in \mathcal{J}$ we have $x - kI \in \mathcal{J}$.

Here $x - kI = \{x - kz : \ z \in I\}$. Condition (iv) is similar but stronger than the one in [12] on p. 438. However, a group $(X, +)$ was considered there, and the condition was

(v) for every $x \in X$ and $I \in \mathcal{J}$ we have $x - I \in \mathcal{J}$.

In the case of vector space, (v) does not imply (iv). For let consider the vector space $(\mathbb{C}, \mathbb{C}, +, \cdot\,)$ and $\mathcal{J} \subset \mathcal{P}(\mathbb{C})$ where

$$I \in \mathcal{J} :\Longleftrightarrow \operatorname{card}\{\Im z : z \in I\} < \infty.$$

One can check that $\mathcal{J}$ is a proper ideal satisfying (v) but not (iv) as for $I = [0,1] \in \mathcal{J}$, $iI \notin \mathcal{J}$. Cf. also the final section in this paper.

For a subset $A \subset X^2$ of the product vector space X^2 and for $x \in X$ and $k \in \mathbb{K}$ define

$$A[x,k] := \{z \in X : (x + kz, z) \in A\}.$$

This generalizes the notion $A[x] = \{z \in X : (x,z) \in A\}$ used in [12]; for $k = 0$ we have $A[x,0] = A[x]$. For $X = \mathbb{R}$ we have a simple geometrical interpretation of $A[x,k]$

$$\alpha = \frac{1}{k}, \quad (k \neq 0);$$

$$\alpha = \frac{\pi}{2}, \quad (k = 0).$$

For $k \in \mathbb{K}$ we define a bijective mapping $\varphi_k : X^2 \to X^2$ by

$$\varphi_k(x,z) := (x + kz, z) \quad \text{for } (x,z) \in X^2.$$

Then we have

$$\begin{aligned} A[x,k] &= \{z \in X : \varphi_k(x,z) \in A\} \\ &= \{z \in X : (x,z) \in \varphi_k^{-1}(A)\} = \varphi_k^{-1}(A)[x]. \end{aligned} \tag{1}$$

According to [12], p. 439, two proper linearly invariant (*p.l.i.* for short) ideals $\mathcal{J}_1$ and $\mathcal{J}_2$ in X and X^2, respectively, are called *conjugate* iff

$$\forall\, I \in \mathcal{J}_2 \, \exists U \in \mathcal{J}_1 \, \forall\, x \in X \setminus U \quad I[x] \in \mathcal{J}_1,$$

i.e., if $I \in \mathcal{J}_2$, then $I[x] \in \mathcal{J}_1$ almost everywhere with respect to $\mathcal{J}_1$ (briefly: $\mathcal{J}_1$-a.e.) in X.

We propose a possibly stronger condition.

Definition 3. Let $\mathcal{S}$ be a subset of $\mathbb{K}$; we say that ideals $\mathcal{J}_1$ and $\mathcal{J}_2$ (in X and X^2, resp.) are $\mathcal{S}$*-conjugate* iff

$$\forall I \in \mathcal{J}_2 \, \exists U \in \mathcal{J}_1 \, \forall x \in X \setminus U \quad \forall k \in \mathcal{S} \quad I[x,k] \in \mathcal{J}_1. \tag{2}$$

Therefore, "$\{0\}$-conjugate" means "conjugate" in the terminology of [12]. Because of (1), one can give an equivalent definition: $\mathcal{J}_1$ and $\mathcal{J}_2$ are $\mathcal{S}$-conjugate iff

$$\forall I \in \mathcal{J}_2 \, \exists U \in \mathcal{J}_1 \, \forall x \in X \setminus U \quad \forall k \in \mathcal{S} \quad \varphi_k^{-1}(I)[x] \in \mathcal{J}_1. \tag{3}$$

Remark 1. If $\mathcal{J}_1$ and $\mathcal{J}_2$ are conjugate (i.e., $\{0\}$-conjugate) and

$$\forall I \in \mathcal{J}_2 \, \forall k \in \mathcal{S} \quad \varphi_k^{-1}(I) \in \mathcal{J}_2, \tag{4}$$

then $\mathcal{J}_1$ and $\mathcal{J}_2$ are $\mathcal{S}$-conjugate. However, (4) is not necessary for $\mathcal{S}$-conjugacy of $\mathcal{J}_1$ and $\mathcal{J}_2$. Namely, one can consider the ideal $\mathcal{J}_1$ of all Lebesgue nullsets in $\mathbb{R}$ and the ideal $\mathcal{J}_2$ consisting of all Lebesgue nullsets A in $\mathbb{R}^2$ with the additional property: $A[x]$ is bounded for all $x \in \mathbb{R}$. Ideals $\mathcal{J}_1$ and $\mathcal{J}_2$ are p.l.i. $\mathbb{R}$-conjugate ideals (compare Example 3 below) but for $A := \{(x,x) : x \in \mathbb{R}\}$ we have $\varphi_1^{-1}(A)[0] = \mathbb{R}$ – unbounded, whence $\varphi_1^{-1}(A) \notin \mathcal{J}_2$.

Following the definition of Ger (cf. [4] or [12] p. 440) for a given p.l.i. ideal $\mathcal{J}$ in X we define

$$\Omega(\mathcal{J}) := \{A \subset X^2 : A[x] \in \mathcal{J} \quad \mathcal{J} - \text{a.e. in } X\}$$

and

$$\Pi(\mathcal{J}) := \{A \subset X^2 : A \subset (U \times X) \cup (X \times U) \quad \text{for some } U \in \mathcal{J}\}.$$

Lemma 17.5.3 in [12] states that both $\Omega(\mathcal{J})$ and $\Pi(\mathcal{J})$ are p.l.i. ideals in X^2 conjugate to $\mathcal{J}$. This remains true under our definition of linear invariance (condition (iv)). Namely, we have to ensure that $kA \in \Omega(\mathcal{J})$ and $kA \in \Pi(\mathcal{J})$ whenever $A \in \Omega(\mathcal{J})$ or $A \in \Pi(\mathcal{J})$, respectively. If $A \in \Omega(\mathcal{J})$, then there exists $U \in \mathcal{J}$ such that for all $x \in X \setminus U$ we have $A[x] \in \mathcal{J}$. For $k \neq 0$, $kU \in \mathcal{J}$ and for $x \in X \setminus kU$ we have $\frac{1}{k}x \in X \setminus U$ whence $A\left[\frac{1}{k}x\right] \in \mathcal{J}$ and consequently $k\left(A\left[\frac{1}{k}x\right]\right) \in \mathcal{J}$. However, $k\left(A\left[\frac{1}{k}x\right]\right) = (kA)[x]$ whence we have: for all $x \in X \setminus kU$, $(kA)[x] \in \mathcal{J}$, i.e., $kA \in \Omega(\mathcal{J})$. For $k = 0$ the proof is trivial.

Now, if $A \in \Pi(\mathcal{J})$, i.e., for some $U \in \mathcal{J}$, $A \subset (U \times X) \cup (X \times U)$, then $kA \subset (kU \times X) \cup (X \times kU)$ and $kU \in \mathcal{J}$.

Moreover, one can prove that $\mathcal{J}$ and $\Pi(\mathcal{J})$ are $\mathbb{K}$-conjugate. Indeed, we already know that they are $\{0\}$-conjugate so let $k \neq 0$ be fixed. For $A \in \Pi(\mathcal{J})$ there exists $U \in \mathcal{J}$ such that $A \subset (U \times X) \cup (X \times U)$. Let $x \in X$; then we have

$$\begin{aligned} A[x,k] = \{z \in X : (x+kz, z) \in A\} &\subset \{z \in X : \ x + kz \in U \quad \text{or } z \in U\} \\ &= \left(-\frac{1}{k}x + \frac{1}{k}U\right) \cup U \in \mathcal{J} \end{aligned}$$

whence $\{k\}$-conjugacy holds true. Thus $\Pi(\mathcal{J})$ and $\mathcal{J}$ are $\mathbb{K}$-conjugate whereas it is not true for $\Omega(\mathcal{J})$ and $\mathcal{J}$ as can be seen in the following example.

Example 1. Let $X = \mathbb{R}$ and let $\mathcal{J}_1$ be the family of all bounded subsets of $\mathbb{R}$. Obviously, $\mathcal{J}_1$ is a p.l.i. ideal in $\mathbb{R}$. Let $\mathcal{J}_2 = \Omega(\mathcal{J}_1)$; thus $\mathcal{J}_2$ is a p.l.i. ideal in $\mathbb{R}^2$, and $\mathcal{J}_1$ and $\mathcal{J}_2$ are conjugate. Let $k \neq 0$ be arbitrary and fixed and define

$$M_k := \left\{(u,v) \in \mathbb{R}^2 : 0 \leq v \leq \frac{1}{k}u\right\}.$$

We have

$$M_k[x] = \begin{cases} \emptyset & \text{for } x < 0, \\ \left[0, \frac{1}{k}x\right] & \text{for } x \geq 0 \end{cases}$$

whence $M_k[x] \in \mathcal{J}_1$ for all $x \in \mathbb{R}$ and consequently $M_k \in \mathcal{J}_2$. On the other hand

$$M_k[x,k] = \begin{cases} \emptyset & \text{for } x < 0, \\ [0, \infty) & \text{for } x \geq 0 \end{cases}$$

whence $M_k[x,k] \notin \mathcal{J}_1$ on $[0,\infty) \notin \mathcal{J}_1$. Thus $\mathcal{J}_1$ and $\mathcal{J}_2$ are p.l.i. ideals in $\mathbb{R}$ and $\mathbb{R}^2$, respectively, conjugate but not $\{k\}$-conjugate for any $k \neq 0$.

3. Additivity

Theorem 1. *Let X and Y be inner product spaces over $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$. Let $\mathcal{S} := \{0,1,i\}\cap\mathbb{K}$ and let $\mathcal{J}_1$ and $\mathcal{J}_2$ be $\mathcal{S}$-conjugate p.l.i. ideals in X and X^2, respectively. Finally, let $M\in\mathcal{J}_2$ and satisfy*

$$M[\mathbf{0},1]\in\mathcal{J}_1 \quad \text{and (for } \mathbb{K}=\mathbb{C}) \quad M[\mathbf{0},i]\in\mathcal{J}_1. \tag{5}$$

If $f : X\to Y$ satisfies (A) *with M as above, then there exists a unique additive function $f_* : X\to Y$ such that there exists a set $U\in\mathcal{J}_1$ such that for $x\in X\setminus U$:*

$$f_*(x)=f(x) \quad (\text{i.e., } f_*=f\ \mathcal{J}_1\text{-a.e. in } X); \tag{6}$$

$$\|f_*(x)\|=\|x\|; \tag{7}$$

$$f_*(ix)=if(x) \quad (\text{in the complex case}). \tag{8}$$

Proof. $\mathcal{J}_1$ and $\mathcal{J}_2$ are $\{0,1\}$-conjugate whence there exist $U_0,U_1\in\mathcal{J}_1$ such that

$$M[x]\in\mathcal{J}_1 \quad \text{for } x\in X\setminus U_0,$$

$$M[x,1]\in\mathcal{J}_1 \quad \text{for } x\in X\setminus U_1.$$

Define

$$U := U_0\cup U_1\cup M[\mathbf{0},1]\cup M[\mathbf{0},i]\cup(-iM[\mathbf{0},1]) \tag{9}$$

(in the real case the last two summands have to be omitted) and

$$V_x := M[x]\cup M[\mathbf{0},1]\cup(M[x]\cup M[\mathbf{0},1]-x)\cup M[x,1]. \tag{10}$$

We have $U\in\mathcal{J}_1$ and, for $x\in X\setminus U$, also $V_x\in\mathcal{J}_1$. Let $x\in X\setminus U$ and $z\in X\setminus V_x$ be arbitrary and fixed. We have:

$$x\notin U \Longrightarrow x\notin M[\mathbf{0},1]\Longrightarrow (x,x)\notin M;$$

$$z\notin V_x \Longrightarrow z\notin M[\mathbf{0},1]\Longrightarrow (z,z)\notin M;$$

$$z\notin V_x \Longrightarrow z\notin M[x]\Longrightarrow (x,z)\notin M;$$

$$z \notin V_x \Longrightarrow z \notin (M[x] \cup M[\mathbf{0},1] - x) \Longrightarrow x+z \notin M[\mathbf{0},1] \\ \Longrightarrow (x+z, x+z) \notin M;$$

$$z \notin V_x \Longrightarrow z \notin (M[x] \cup M[\mathbf{0},1] - x) \Longrightarrow x+z \notin M[x] \\ \Longrightarrow (x, x+z) \notin M;$$

$$z \notin V_x \Longrightarrow z \notin M[x,1] \Longrightarrow (x+z, z) \notin M.$$

Using the above properties and (A) we have for $x \in X \setminus U$, $y \in X \setminus V_x$:

$$\begin{aligned} \|f(x+z) - f(x) - f(z)\|^2 &= \langle f(x+z) \mid f(x+z)\rangle + \langle f(x) \mid f(x)\rangle \\ &\quad + \langle f(z) \mid f(z)\rangle + 2\Re\langle f(x) \mid f(z)\rangle \\ &\quad - 2\Re\langle f(x) \mid f(x+z)\rangle - 2\Re\langle f(x+z) \mid f(z)\rangle \\ &= \langle x+z \mid x+z\rangle + \langle x \mid x\rangle + \langle z \mid z\rangle \\ &\quad + 2\Re\langle x \mid z\rangle - 2\Re\langle x \mid x+z\rangle - 2\Re\langle x+z \mid z\rangle = 0 \end{aligned}$$

whence

$$f(x+z) = f(x) + f(z) \quad \text{for } x \in X \setminus U,\ z \in X \setminus V_x.$$

Let

$$M_1 := \left\{(x,z) \in X^2 : x \in U \text{ or } (x \in X \setminus U \text{ and } z \in V_x)\right\}.$$

We have $M_1 \in \Omega(\mathcal{J}_1)$, $\Omega(\mathcal{J}_1)$ being the largest ideal in X^2 conjugate (cf. [12] p. 441) with $\mathcal{J}_1$. Thus we get

$$f(x+z) = f(x) + f(z) \quad \text{for } (x,z) \in X^2 \setminus M_1 \tag{11}$$

where $M_1 \in \Omega(\mathcal{J}_1)$. Moreover, we have

$$M_1[x] \in \mathcal{J}_1 \quad \text{for } x \in X \setminus U. \tag{12}$$

By virtue of the theorem of de Bruijn–Ger ([12] p. 444) there exists a unique additive function $f_* : X \to Y$ such that

$$f_*(x) = f(x) \quad \text{for } x \in X \setminus U. \tag{13}$$

(From the statement of the theorem we know that $f_* = f$ $\mathcal{J}_1$-a.e. in X; one has to look into Ger's proof to see that the equality $f_*(x) = f(x)$ holds on the set $X \setminus U$ – because of (12).)

Thus (6) holds.

To prove (7) it is enough to notice that if $x \in X \setminus U$, then $x \notin M[\mathbf{0}, 1]$ whence $(x, x) \notin M$ and

$$\|f(x)\| = \sqrt{\langle f(x) \mid f(x)\rangle} = \sqrt{\langle x \mid x\rangle} = \|x\|$$

which together with (13) gives

$$\|f_*(x)\| = \|x\| \quad \text{for } x \in X \setminus U.$$

Observe that so far we have not needed $\{i\}$-conjugacy of $\mathcal{J}_1$ and $\mathcal{J}_2$ (even in the complex case). This will be needed to prove (8). For $x \in X \setminus U$ we have, in particular,

$$\begin{aligned} x \notin -iM[\mathbf{0}, 1] &\Longrightarrow ix \notin M[\mathbf{0}, 1] \Longrightarrow (ix, ix) \notin M; \\ x \notin M[\mathbf{0}, i] &\Longrightarrow (ix, x) \notin M. \end{aligned}$$

Thus for $x \in X \setminus U$, because of (A), we obtain:

$$\begin{aligned} \|f(ix) - if(x)\|^2 &= \langle f(ix) \mid f(ix)\rangle + \langle f(x) \mid if(x)\rangle - 2\Re\langle f(ix) \mid if(x)\rangle \\ &= \langle ix \mid ix\rangle + \langle x \mid x\rangle - 2\Re\left(-i\langle ix \mid x\rangle\right) = 0 \end{aligned}$$

whence

$$f(ix) = if(x) \quad \text{for } x \in X \setminus U.$$

As we have (13), f_* inherits the above property and (8) follows. □

4. Boundedness and continuity

Let $\mathcal{B}(X, Y)$ denote the family of all subsets $B \subset X$ with the property that each additive function $f : X \to Y$ bounded on B is continuous, i.e.,

$$\begin{aligned} \mathcal{B}(X, Y) := \{B \subset X : (f : X \to Y &\text{ additive}, \\ f \text{ bounded on } B) &\Rightarrow f \text{ continuous}\}. \end{aligned} \tag{14}$$

One can put the above definition into a more general setting (e.g., for topological groups); however, with respect to our main result, we restrict ourselves to considering this class for X and Y being inner product spaces.

Such a class $\mathcal{B}$ (for particular X and Y) was considered by Kuczma [11], [12] p. 206 ff, Ger–Kuczma [8], Ger–Kominek [7]. A characterization of $\mathcal{B}(\mathbb{R}^n, \mathbb{R})$ was given by Smital [16].

Let us note a simple

Lemma 1. *If X and Y are arbitrary i.p.s. over $\mathbb{K}$, $Y \neq \{0\}$, then*

$$\mathcal{B}(X,Y) \subset \mathcal{B}(X,\mathbb{K}).$$

PROOF. Let $B \in \mathcal{B}(X,Y)$ and let $f_1 : X \to \mathbb{K}$ be an additive function bounded on B. Define $f : X \to Y$ by $f(x) := f_1(x) \cdot y_0$ (for some arbitrary but fixed $y_0 \in Y \setminus \{0\}$). The function f is additive on X and $\|f(x)\| = |f_1(x)| \cdot \|y_0\|$ whence it is bounded on B. As $B \in \mathcal{B}(X,Y)$, f is continuous and consequently f_1 is continuous whence $B \in \mathcal{B}(X,\mathbb{K})$. □

Proposition 1. *For an inner product space X and a Hilbert space $Y \neq \{0\}$ over $\mathbb{K}$ we have*

$$\mathcal{B}(X,Y) = \mathcal{B}(X,\mathbb{K}).$$

PROOF. We need only prove the reverse inclusion to that in Lemma 1, i.e.,

$$\mathcal{B}(X,\mathbb{K}) \subset \mathcal{B}(X,Y).$$

We will use the following result

Let X be an inner product space and Y a Hilbert space over $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$. Let $f : X \to Y$ be an additive mapping. Then the following two conditions are equivalent:

(I) *f is $\mathbb{R}$-linear, continuous on X;*

(II) *for every $y \in Y$, the functional $f_y : X \to \mathbb{K}$ defined by $f_y(x) := \langle f(x) \mid y\rangle$ for $x \in X$, is continuous.*

This theorem was proved in [15] in the case $X = Y$, but the proof runs also in the general case, and the completeness of the target space only is essential.

Now, let $B \in \mathcal{B}(X,\mathbb{K})$ and let $f : X \to Y$ be an additive function bounded on B. For an arbitrary $y \in Y$ the mapping f_y is additive and for all $x \in X$ we have

$$\|f_y(x)\| = |\langle f(x) \mid y\rangle| \leq \|y\| \cdot \|f(x)\|.$$

As f is bounded on B so is f_y whence f_y is continuous. Thus we proved (II) in the theorem quoted above whence (I) holds, i.e., f is continuous. That means $B \in \mathcal{B}(X,Y)$, and the proof is completed. □

According to [7], each nonvoid open subset of a real linear topological Baire space belongs to $\mathcal{B}(X,\mathbb{R})$ as well as each second category subset with the Baire property (i.e., of the form $A=(G\cup P)\setminus R$ where G is open and P,R are of the first category) of X does (cf. also [14]). There are generalizations of classical theorems of Bernstein-Doetsch and Mehdi, respectively (cf. [12]).

For $X=\mathbb{R}^n$ we have the theorem of Ostrowski (cf. [12] p. 210) stating that each subset of $\mathbb{R}^n$ of positive inner Lebesgue measure belongs to $\mathcal{B}(\mathbb{R}^n,\mathbb{R})$.
Notice, that the quoted results are far more general; we quoted what we actually will need in the sequel.

One can quote also Smital's result. He proved in [16] (Th. 4) that $B\subset\mathbb{R}^n$ belongs to $\mathcal{B}(\mathbb{R}^n,\mathbb{R})$ if and only if $\mathbb{Q}$-convex hull of the set $(B-B)$ contains a ball.

Bearing in mind our Proposition 1 and the results mentioned above we can state:

Proposition 2. *Let X be a real Baire inner product space and let $Y\neq\{0\}$ be a real Hilbert space. If $B\subset X$ satisfies one of the following conditions:*

(i) *$B\neq\emptyset$ and B is open,*

(ii) *B is of the second category with the Baire property,*

(iii) *$X=\mathbb{R}^n$, and $B\subset\mathbb{R}^n$ is of positive inner Lebesgue measure,*

then $B\in\mathcal{B}(X,Y)$. Moreover, if $X=\mathbb{R}^n$, we have for $B\subset\mathbb{R}^n$

(iv) *$B\in\mathcal{B}(\mathbb{R}^n,Y)\iff\mathbb{Q}$-convex hull of $(B-B)$ contains a ball.*

An inner product space need not be a Baire space; the space $\mathbb{K}^{(\mathbb{N})}$ (of all sequences $x=(\xi_1,\xi_2,\dots)$ such that $\xi_i=0$ for almost every $i\in\mathbb{N}$) is a suitably example. A Hilbert space is a Baire space; however, completeness is not a necessary condition for a Baire inner product space as was shown by Hausdorff [10].

5. Main result

One can pose a question: *does there exist a function $g:X\to Y$ satisfying the orthogonality equation* (OE)*, equal $\mathcal{J}_1$-almost everywhere on X to a given function $f:X\to Y$ satisfying* (A)*?* As each function satisfying

(OE) is additive and because of the uniqueness in the assertion of Theorem 1 the question, actually, is: *does the function* f_* *(from the assertion of Theorem 1 satisfy the orthogonality equation?* This – as it will turn out – can be reduced to a question as to whether the condition $\|f_*(x)\| = \|x\|$ holds for all $x \in X$. A positive answer, under some additional assumptions, will be given.

Theorem 2. *Let* X *and* Y *be inner product spaces over* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. *Let* $\mathcal{S} = \{0, 1, i\} \cap \mathbb{K}$ *and let* $\mathcal{J}_1$ *and* $\mathcal{J}_2$ *be p.l.i.* $\mathcal{S}$*-conjugate ideals in* X *and* X^2, *respectively. Moreover, assume that* $\mathcal{J}_1$ *satisfies two additional conditions:*

$$\forall V \in \mathcal{J}_1 \ \exists B \subset X \setminus V \quad \text{such that } B \text{ is bounded and } B \in \mathcal{B}(X, Y); \tag{15}$$

$$\forall V \in \mathcal{J}_1 \quad \forall 0 \neq x \in V \quad \exists \{x_n\}_{n=1}^{\infty} \subset X \quad \exists \{k_n\}_{n=1}^{\infty} \subset \mathbb{R} \setminus \{0\}$$
$$\text{such that } x_n \to x \ (n \to \infty) \text{ and } k_n \cdot x_n \notin V \ (n = 1, 2, \dots). \tag{16}$$

Let $M \in \mathcal{J}_2$ *and satisfy* $M[\mathbf{0}, 1] \in \mathcal{J}_1$ *and (for* $\mathbb{K} = \mathbb{C}$*)* $M[\mathbf{0}, i] \in \mathcal{J}_1$.
Then, if a function $f : X \to Y$ *satisfies*

$$\langle f(x) \mid f(z) \rangle = \langle x \mid z \rangle \quad \text{for } (x, z) \in X^2 \setminus M, \tag{A}$$

then there exists a unique function $f_* : X \to Y$ *satisfying the orthogonality equation* (OE) *and such that*

$$f_* = f \quad \mathcal{J}_1\text{-a.e. on } X.$$

Moreover, $\dim X \leq \dim Y$.

Proof. All the assumptions of Theorem 1 are satisfied, so we can consider the unique additive function f_* from the assertion of this theorem – equal to f on the set $X \setminus U$ ($U \in \mathcal{J}_1$ and was defined by (9)). According to (15), there exists a set $B \subset X \setminus U$, bounded and belonging to $\mathcal{B}(X, Y)$. Because of the condition (7) that holds on $X \setminus U$, function f_* is bounded on B whence (definition (14)) f_* is continuous on X and, consequently, $\mathbb{R}$-linear. For $x \in X \setminus U$ we have $\|f_*(x)\| = \|x\|$. This is also true for $x = 0$ as f_* is additive. Let us consider an $x \in U \setminus \{0\}$. From (16) there exists $\{x_n\}$ a sequence in X convergent to x and $\{k_n\}$ a sequence of reals (different from zero) such that $k_n x_n \notin U$ ($n \in \mathbb{N}$) which implies $\|f_*(k_n x_n)\| = \|k_n x_n\|$ for all $n \in \mathbb{N}$. From the $\mathbb{R}$-homogeneity of f_* we have $\|f_*(x_n)\| = \|x_n\|$ for all $n \in \mathbb{N}$, and continuity of f_* implies $\|f_*(x)\| = \|x\|$. Thus we have

$$\|f_*(x)\| = \|x\| \quad \text{for } x \in X. \tag{17}$$

In the case $\mathbb{K} = \mathbb{C}$ we have also – according to (8) – $f_*(ik_n x_n) = if_*(k_n x_n)$ for all $n \in \mathbb{N}$ whence (by $\mathbb{R}$-homogeneity of f_*) $f_*(ix_n) = if_*(x_n)$ for all $n \in \mathbb{N}$ and finally (by continuity of f_*) $f_*(ix) = if_*(x)$. The last equality holds also for $x \in X \setminus U$ and for $x = 0$ whence we have

$$f_*(ix) = if_*(x) \quad \text{for } x \in X. \tag{18}$$

Now, for arbitrary $x, z \in X$ we have, because of additivity of f_* and (17),

$$\begin{aligned}
\Re\langle f_*(x) \mid f_*(z)\rangle &= \frac{1}{4}\left(\|f_*(x) + f_*(z)\|^2 - \|f_*(x) - f_*(z)\|^2\right) \\
&= \frac{1}{4}\left(\|f_*(x+z)\|^2 - \|f_*(x-z)\|^2\right) \\
&= \frac{1}{4}\left(\|x+z\|^2 - \|x-z\|^2\right) = \Re\langle x \mid z\rangle.
\end{aligned}$$

Similarly, making use of (18),

$$\begin{aligned}
\Im\langle f_*(x) \mid f_*(z)\rangle &= \frac{1}{4}\left(\|f_*(x) + if_*(z)\|^2 - \|f_*(x) - if_*(z)\|^2\right) \\
&= \frac{1}{4}\left(\|f_*(x+iz)\|^2 - \|f_*(x-iz)\|^2\right) \\
&= \frac{1}{4}\left(\|x+iz\|^2 - \|x-iz\|^2\right) = \Im\langle x \mid z\rangle.
\end{aligned}$$

Therefore we have

$$\langle f_*(x) \mid f_*(z)\rangle = \langle x \mid z\rangle \quad \text{for } x, z \in X,$$

i.e., f_* is an inner product preserving mapping.

As f_* is injective and linear, $\dim X \leq \dim Y$. □

Remark 2. If we replaced the condition $M[\mathbf{0}, i] \in \mathcal{J}_1$ by $M[\mathbf{0}, \gamma] \in \mathcal{J}_1$ for some arbitrary $\gamma \in \mathbb{C} \setminus \mathbb{R}$ and if we then replaced in (9) the summand $-iM[\mathbf{0}, 1]$ by $\gamma^{-1}M[\mathbf{0}, 1]$ we would obtain instead of (8): $f_*(\gamma x) = \gamma f_*(x)$ for $x \in X \setminus U$. From the last equality, and from the continuity of f_*, one can easily derive $f_*(ix) = if_*(x)$ on $X \setminus U$. That means, our assumptions in Theorem 2 can be weakened a little bit.

Notice that if $f = f_*$ $\mathcal{J}_1$-a.e. on X for f_* satisfying (OE) and

$$M := \{(x,z) : \langle x \mid z\rangle \neq \langle f(x) \mid f(z)\rangle\}$$

then $M[x,k] \in \mathcal{J}_1$ for any $x \in X$, $k \in \mathbb{K} \setminus \{0\}$.

As a corollary from Theorem 1 and Theorem 2 we are able to state

Theorem 3. *Let X and Y be inner product spaces over $\mathbb{K}$, let $\mathcal{J}$ be a p.l.i. ideal in X and let $f : X \to Y$ satisfy*

$$\text{(A')} \qquad \langle f(x) \mid f(z)\rangle = \langle x \mid z\rangle \quad \text{for } x,z \in X \setminus U$$

where $U \in \mathcal{J}$. Then there exists a unique additive mapping $f_* : X \to Y$ such that $f_* = f$ $\mathcal{J}$-a.e. in X. Moreover, under the assumption that $\mathcal{J}$ satisfies (15) *and* (16) *function f_* is an inner product preserving mapping, i.e., it satisfies* (OE).

Proof. Define $M := (U \times X) \cup (X \times U)$; then $M \in \Pi(\mathcal{J})$. As we know $\Pi(\mathcal{J})$ and $\mathcal{J}$ are $\mathbb{K}$-conjugate. Moreover, we have

$$M[\mathbf{0},1] = \{x : (x,x) \in M\} = U \in \mathcal{J};$$
$$M[\mathbf{0},i] = \{x : (ix,x) \in M\} = (-iU) \cup U \in \mathcal{J}$$

whence all the assumptions of Theorem 1 are satisfied and therefore a unique additive function f_*, equal almost everywhere to f, exists. If we assume $\mathcal{J}$ satisfies (15), (16), then we can apply Theorem 2 with $\mathcal{J}_1 = \mathcal{J}$, $\mathcal{J}_2 = \Pi(\mathcal{J})$ and M defined above to obtain the assertion. $\square$

Remark 3. In the case of the Cauchy equation, if $f(x+y) = f(x)+f(y)$ for $x,y \in X \setminus U$ where U belongs to an ideal, then ([12] p. 446) the function f itself is additive. This is not the case for the orthogonality equation as one can consider an arbitrary inner product space X with a p.l.i. ideal $\mathcal{J}$, an arbitrary set $\{0\} \neq U \in \mathcal{J}$ and a function $f : X \to X$ given by

$$f(x) = x \quad \text{for } x \notin U \quad \text{and } f = 0 \quad \text{on } U.$$

Obviously, $\langle f(x) \mid f(z)\rangle = \langle x \mid z\rangle$ for $x,z \in X \setminus U$ whereas f is not a solution of (OE) and need not be additive.

6. Applications

In this section we provide some examples of applications of Theorems 2 and 3. We restrict ourselves to a real Baire inner product space X and to Y being a real Hilbert space. In particular, we consider the case $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$ with – *a posteriori* – $m \geq n$.

Remark 2. Let $\mathcal{J}_1^{\aleph_0}$ and $\mathcal{J}_2^{\aleph_0}$ consist of all at most countable subsets of X and X^2, respectively. Of course, $\mathcal{J}_1^{\aleph_0}$ and $\mathcal{J}_2^{\aleph_0}$ are $\mathbb{R}$-conjugate p.l.i. ideals. The ideal $\mathcal{J}_1^{\aleph_0}$ satisfies (15); indeed, if $V \in \mathcal{J}_1^{\aleph_0}$ and B_1 denotes the open unit ball centered at the origin, then $B := B_1 \setminus V$ is a bounded set of the second category with the Baire property (here we use the fact that X is a Baire space whence B_1 is of the second category). Hence, according to Proposition 2 (ii), $B \in \mathcal{B}(X, Y)$. The condition (16) holds trivially (each ball contains uncountably many elements). Of course if $M \in \mathcal{J}_2^{\aleph_0}$, then $M[\mathbf{0}, 1] \in \mathcal{J}_1^{\aleph_0}$.

From Theorem 2 we derive

Corollary 1. *Let X be a real Baire inner product space and let Y be a real Hilbert space. If $f : X \to Y$ satisfies the condition*

$$\langle f(x) \mid f(z) \rangle = \langle x \mid z \rangle \quad \textit{for } (x, z) \in X^2 \setminus M$$

where M is an at most countable set, then there exists exactly one function $f_* : X \to Y$ satisfying

$$\langle f_*(x) \mid f_*(z) \rangle = \langle x \mid z \rangle \quad \textit{for } x, z \in X$$

and such that $f_* = f$ on X except for an at most countable subset. Moreover, $\dim X \leq \dim Y$.

Example 3. Let $\mathcal{J}_1^0$ and $\mathcal{J}_2^0$ consist of all Lebesgue nullsets in $\mathbb{R}^n$ and $\mathbb{R}^{2n}$, respectively. Then $\mathcal{J}_1^0$ and $\mathcal{J}_2^0$ are p.l.i. ideals and they are conjugate (Fubini's theorem). Moreover, for an arbitrary $k \in \mathbb{R}$ a mapping $\varphi_k^{-1}(x, z) = (x - kz, z)$ is linear and for its matrix $\Phi_k^{-1} = \begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$ we have $\left|\det \Phi_k^{-1}\right| = 1$. Thus if $A \in \mathcal{J}_2^0$, then $\varphi_k^{-1}(A) \in \mathcal{J}_2^0$ (mapping φ_k^{-1} preserves the Lebesgue measure, cf. [13] Th. 5.4.8. p. 105) and (4) holds. According to Remark 1, $\mathcal{J}_1^0$ and $\mathcal{J}_2^0$ are $\mathbb{R}$-conjugate. If $V \in \mathcal{J}_1^0$, then $B = B_1 \setminus V$ is of positive Lebesgue measure whence (Proposition 2 (iii))

$B \in \mathcal{B}(\mathbb{R}^n, Y)$ and (15) holds. A Lebesgue nullset cannot contain an open ball, thus for $V \in \mathcal{J}_1^0$ the set $\mathbb{R}^n \setminus V$ is dense in $\mathbb{R}^n$ and (16) is satisfied.

Notice that for an arbitrary $M \in \mathcal{J}_2^0$ the condition $M[\mathbf{0}, 1] \in \mathcal{J}_1^0$ need not be satisfied. Indeed, one can consider $M_0 := \{(x, x) \in \mathbb{R}^{2n} : \ x \in \mathbb{R}^n\}$; then $M_0 \in \mathcal{J}_2^0$ whereas $M_0[\mathbf{0}, 1] = \mathbb{R}^n \notin \mathcal{J}_1^0$. Thus if we want to apply Theorem 2, we have to make an additional assumption on the set M.

Corollary 2. *Let Y be a real Hilbert space. If $f : \mathbb{R}^n \to Y$ satisfies the condition*

$$\langle f(x) \mid f(z)\rangle = \langle x \mid z\rangle \quad \textit{for } (x, z) \in \mathbb{R}^{2n} \setminus M$$

where M is a Lebesgue nullset in $\mathbb{R}^{2n}$ such that the set $\{x \in \mathbb{R}^n : (x, x) \in M\}$ is a Lebesgue nullset in $\mathbb{R}^n$, then there exists a unique function $f_* : \mathbb{R}^n \to Y$ satisfying the orthogonality equation and such that $f_* = f$ Lebesgue-almost everywhere in $\mathbb{R}^n$. Moreover, $\dim Y \geq n$.

Example 4. Let $\mathcal{J}_1^f$ and $\mathcal{J}_2^f$ consist of all first category subsets of X and X^2, respectively. Then $\mathcal{J}_1^f$ and $\mathcal{J}_2^f$ are p.l.i. ideals (this was stated in [12] and remains true under our terminology of linear invariance; indeed, if A is of the first category, so is kA for $k \in \mathbb{R}$) and they are conjugate ([12] Th. 2.1.7 p. 30). For $k \in \mathbb{R}$ the mapping φ_k is a homeomorphism whence $\varphi_k^{-1}(A)$ is of the first category whenever A is. Hence (4) holds and consequently (Remark 1) $\mathcal{J}_1^f$ and $\mathcal{J}_2^f$ are $\mathbb{R}$-conjugate.

If V is of the first category in X (X is a Baire space), then the set $B := B_1 \setminus V$ is of the second category with the Baire property whence $B \in \mathcal{B}(X, Y)$ according to Proposition 2 (ii), and (15) holds.

To prove (16) notice that if V is of the first category in X, then $X \setminus V$ has to be dense – otherwise V would contain an open ball – a second category set in X. (If we assumed X being a compact space we could just quote the Baire theorem (cf. [17] Baire's Theorem 1, p. 12): if M is of the first category in a compact topological space X, then $X \setminus M$ is dense in X.)

Again, if one wants to apply Theorem 2 one has to assume that $M[\mathbf{0}, 1] \in \mathcal{J}_1^f$ as it does not follow from $M \in \mathcal{J}_2^f$ and the other assumptions (M_0 defined in the previous example can be considered here as well).

Corollary 3. *Let X be a real Baire inner product space and let Y be a real Hilbert space. If $f : X \to Y$ satisfies*

$$\langle f(x) \mid f(z)\rangle = \langle x \mid z\rangle \quad \text{for } (x, z) \in X^2 \setminus M$$

where M is of the first category in X^2 and such that the set $\{x \in X : (x, x) \in M\}$ is of the first category in X, then there exists a unique function $f_* : X \to Y$ satisfying the orthogonality equation and such that $f_* = f$ on a residual subset of X. Moreover, $\dim Y \geq \dim X$.

Example 5. Let $\mathcal{J}_1^b$ and $\mathcal{J}_2^b$ consist of all bounded subsets of X and X^2, respectively. They are p.l.i. ideals and they are $\mathbb{R}$-conjugate.
If $V \subset X$ is bounded, then one can easily find an open ball B such that $B \subset X \setminus V$. Then $B \in \mathcal{B}(X, Y)$ (Proposition 2 (i)), and (15) is satisfied.
If $V \in \mathcal{J}_1^b$ and $0 \neq x \in V$, then there exists $k \in \mathbb{R} \setminus \{0\}$ such that $kx \notin V$, i.e., (16) holds.

Finally, for an $M \in \mathcal{J}_2^b$ we have $M[\mathbf{0}, 1] \in \mathcal{J}_1^b$.

Thus one can derive from Theorem 2

Corollary 4. *Let X be a real Baire inner product space and let Y be a real Hilbert space. If $f : X \to Y$ satisfies*

$$\langle f(x) \mid f(z)\rangle = \langle x \mid z\rangle \quad \text{for } (x, z) \in X^2 \setminus M$$

where M is bounded in X^2, then there exists a unique function $f_* : X \to Y$ satisfying the orthogonality equation on X and such that $f_* = f$ on X except for some bounded subset. Moreover, $\dim Y \geq \dim X$.

Example 6. Let $\mathcal{J}_1^m$ and $\mathcal{J}_2^m$ consist of all subsets of $\mathbb{R}^n$ and $\mathbb{R}^{2n}$, respectively, of *finite* n-dimensional ($2n$-dimensional, respectively) outer measure. $\mathcal{J}_1^m$ and $\mathcal{J}_2^m$ are p.l.i. ideals and they are conjugate ([12] p. 440). The mappings φ_k^{-1} preserve the outer measure as we have $\left|\det \Phi_k^{-1}\right| = 1$ (cf. [13] Corollary 5.4.1). Thus $\mathcal{J}_1^m$ and $\mathcal{J}_2^m$ are $\mathbb{R}$-conjugate.

If $V \in \mathcal{J}_1^m$, i.e., if V is of finite outer measure in $\mathbb{R}^n$, then there exists $n \in \mathbb{N}$ such that for B_n being the ball centered at the origin of radius n, the set $B_n \setminus V$ is of positive inner measure whence (Proposition 2 (iii)) belongs to $\mathcal{B}(\mathbb{R}^n, Y)$. Thus (15) is satisfied.

Suppose that (16) is violated. There would exists $V \in \mathcal{J}_1^m$, $x \in V \setminus \{0\}$ and $\varepsilon > 0$ such that

$$\forall\ z \in B(x, \varepsilon)\ \ \forall\ k \in \mathbb{R} \setminus \{0\}\ \ kz \in V.$$

But that would imply $n \cdot B(x, \varepsilon) \subset V$ for all $n \in \mathbb{N}$ which contradicts the fact that V is of finite outer measure. Therefore (16) holds.

From Theorem 2 we obtain

Corollary 5. *Let Y be a real Hilbert space. If $f : \mathbb{R}^n \to Y$ satisfies the condition*

$$\langle f(x) \mid f(z) \rangle = \langle x \mid z \rangle \quad \textit{for } (x, z) \in \mathbb{R}^{2n} \setminus M$$

where M is of finite outer measure in $\mathbb{R}^{2n}$ and such that the set $\{x \in \mathbb{R}^n : (x, x) \in M\}$ is of finite outer measure in $\mathbb{R}^n$, then there exists a unique function $f_* : \mathbb{R}^n \to Y$ satisfying the orthogonality equation and such that $f_* = f$ on $\mathbb{R}^n$ except for some set of finite outer measure. Moreover, $\dim Y \geq n$.

Remark 4. The above Corollaries do not cover – in particular – the following conditional orthogonality equation

$$\langle f(x) \mid f(z) \rangle = \langle x \mid z \rangle \quad \text{for } x \neq z.$$

In this case $M = \{(x, x) : x \in X\}$ and $M[\mathbf{0}, 1] = X$. It can be proved (cf. [2]) that, in the case $X = Y = \mathbb{R}^n$, each solution of the above conditional equation is, in fact, a solution of the orthogonality equation itself. This is no longer true if we consider an infinite-dimensional space X or if we take $X \neq Y$. In those cases, the open question is if there exists a function f_* satisfying the unconditional orthogonality equation and equal to f almost everywhere with respect to some ideal in X.

From Theorem 3 and from the fact that all the ideals considered in Examples 2–6 satisfy (15) and (16) we have

Corollary 6. *Let X be a real Baire inner product space and let Y be a real Hilbert space. If $f : X \to Y$ satisfies*

$$\langle f(x) \mid f(z) \rangle = \langle x \mid z \rangle \quad \textit{for } x, z \in X \setminus U$$

and U is an at most countable [first category / bounded / of Lebesgue measure zero ($X = \mathbb{R}^n$) / of finite outer measure ($X = \mathbb{R}^n$)] *set, then there exists a unique solution $f_* : X \to Y$ of* (OE) *such that $f_* = f$ on X except for an at most countable* [first category / bounded / of Lebesgue measure zero / of finite outer measure – respectively] *subset.*

7. Final remarks

At the beginning of the present paper we gave the definition of linear invariance of an ideal (Definition 2). Now we would like to introduce another definition; more restrictive but, perhaps, more adequate to the name *linear invariance*. Namely we propose:

For a vector space $(X, \mathbb{K}, +, \cdot)$ *over* $\mathbb{K}$ *(topological vector space / inner product space etc.) we call an ideal* $\mathcal{J} \subset 2^X$ linearly invariant *iff (besides of* (i), (ii) *from Definition 1) it satisfies*

(iv') *for every* $x \in X$ *and for every automorphism* $l : X \to X$

$$\textit{if } A \in \mathcal{J}, \quad \textit{then } x - l(A) \in \mathcal{J}.$$

Here by automorphism we mean a bijective and linear mapping preserving all additional structures imposed on X (linear homeomorphism for topological vector space, unitary mapping for inner product space etc.)

Notice, that all "classical" examples of ideals, especially those considered in Examples 2–6 (Lebesgue nullsets, bounded subsets, first category subsets, subsets with positive outer measure, at most countable subsets in suitable spaces) are still linearly invariant under the stronger condition (iv'). If one assumes (iv') in the definition of linear invariance, then – in particular – the condition (4) holds for all $k \in \mathbb{K}$ whence (Remark 1) the concepts of *conjugacy* and $\mathbb{K}$-*conjugacy* are equivalent. Thus in the assumptions of Theorems 1 and 2 we can reduce $\{0, 1, i\}$-conjugacy just to conjugacy.

Of course, the condition (iv') is strictly stronger than (iv). For let

$$\mathcal{J}_1 := \{A \subset \mathbb{R} : \mathrm{m}_{\mathcal{L}}(A) = 0\}$$

where $\mathrm{m}_{\mathcal{L}}$ denotes Lebesgue measure in $\mathbb{R}$ and let

$$\mathcal{J}_2 := \{B \subset \mathbb{R}^2 : \mathrm{m}^2_{\mathcal{L}}(B) = 0 \text{ and } \forall\, x \in \mathbb{R} \;\; B[x] \text{ is bounded in } \mathbb{R}\}$$

where $\mathrm{m}^2_{\mathcal{L}}$ denotes Lebesgue measure in $\mathbb{R}^2$. $\mathcal{J}_1$ is a proper linearly invariant (with (iv')) ideal in $\mathbb{R}$ and $\mathcal{J}_2$ is a proper ideal in $\mathbb{R}^2$ satisfying (iv) and $\mathbb{R}$-conjugate with $\mathcal{J}_1$. However, $\mathcal{J}_2$ does not satisfy (iv') Indeed, the set $B := \{(x, x) : x \in \mathbb{R}\}$ belongs to $\mathcal{J}_2$ while for the automorphism $\varphi : \mathbb{R}^2 \ni (x, y) \to (x - y, y) \in \mathbb{R}^2$ we have $\varphi(B) = \{(0, x) : x \in \mathbb{R}\}$ and $\varphi(B)[\mathbf{0}] = \mathbb{R}$ whence $\varphi(B) \notin \mathcal{J}_2$.

One can check that ideals $\Omega(\mathcal{J})$ and $\Pi(\mathcal{J})$ are not necessarily linearly invariant under the new definition, contrary to Definition 2.

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