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n-Complete standard wreath products

By J. PANAGOPOULOS (Athens), E. RAPTIS (Athens) and D. VARSOS (Athens)

Abstract. Let $\gamma_{n+1}(G)$ be the $n+1$ term of the lower central series of the group G and $f_n : \mathrm{Aut}(G) \longrightarrow \mathrm{Aut}(G/\gamma_{n+1}(G))$ the obvious homomorphism. If $\mathrm{Ker}\, f_n \leq I(G)$, the group of the inner automorphisms, then the group G is said to be n-Complete.

In this paper we examine the n-Completeness of a restricted standard wreath product $W = A\mathrm{wr}B$ in respect of the n-Completeness of the groups A and B.

1. Introduction

Let $W = A\mathrm{wr}B$ be the restricted standard wreath product of the groups A and B. In [3] necessary and sufficient conditions are given, under which the group W is semicomplete. In this paper we study the more general problem of the n-completeness of W in conection with the n-completeness of the groups A and B. In section 3 it is proved that if W is n-complete then A is at most n-complete and B is nilpotent of class at most n. We shall see in the section 4 that the above conditions are not sufficient. Also, in the section 4, we give examples of non n-complete standard wreath products constructing outer automorphisms of these groups.

2. Definitions and notations

The restricted standard wreath product $W = A\mathrm{wr}B$ of two groups A and B is the splitting extension of the direct power A^B by the group B, with B acting on A^B according to the rule: if $b \in B$ then $f^b(x) = f(xb^{-1})$ for all $f \in A^B$, $x \in B$. The base group A^B is characteristic in W in all cases exept when A is of order 2 or is a dihedrall group of order $4k+2$ and B is of order 2. In the following it is assumed that A^B is characteristic in W.

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If G is a group and $G=\gamma_1(G), \gamma_2(G), \dots, \gamma_n(G), \dots$ the lower central series of G we define the series $K_1, K_2, \dots, K_n, \dots$, where K_n consists of the automorphisms of G which induce the identity on the group $G/_{\gamma_{n+1}(G)}$. Now we give the definition of a n-complete group which is a generalization of the definition of a semicomplete group.

Definition 2.1. A group G is called n-complete if n is the least positive integer such that $K_n \le I(G)$, where $I(G)$ is the group of the inner automorphisms of G.

If a group is nilpotent of class m, then clearly is n-complete for some n with $n \le m$. We have by definition 2.1 that a group is 1-complete if and only if it is semicomplete.

In the following we need the next extensions:

i) If $a \in \mathrm{Aut}(A)$ then $a^* \in \mathrm{Aut}(W)$, where $(bf)^{a^*} = bf^{a^*}$ for all $b \in B$, $f \in A^B$ and $f^{a^*}(x) = (f(x))^a$ for all $x \in B$.

ii) If $\beta \in \mathrm{Aut}(B)$ then $\beta^* \in \mathrm{Aut}(W)$, where $(bf)^{\beta^*} = b^{\beta} f^{\beta^*}$ for all $b \in B$, $f \in A^B$ and $f^{\beta^*}(x) = f(x^{\beta^{-1}})$ for all $x \in B$.

3. Characterizations of A and B when $W = A\mathrm{wr}B$ is n-complete

Proposition 3.1. *If $W = A\mathrm{wr}B$ is n-complete then A is at most n-complete.*

Proof. If $a \in K_n(A)$ and $f \in A^B$, then $f^{a^*}(x) = (f(x))^a = f(x)u_x$ for all $x \in B$ and $u_x \in \gamma_{n+1}(A)$. If $g \in A^B$ such that $g(x) = u_x$ for all $x \in B$, then $f^{a^*}(x) = (fg)(x)$ for all $x \in B$. Hence $f^{a^*} = fg$, where $g \in \gamma_{n+1}(W)$. Thus we conclude that $a^* \in K_n(W)$. Since W is n-complete we have that $K_n(W) \le I(W)$ and so $a^* \in I(W)$. But $a^* \in I(W)$ if and only if $a \in I(A)$ (cf. J. Panagopoulos [3], Prop. 4.3). Hence, $K_n(A) \le I(A)$. □

Proposition 3.2. *If $W = A\mathrm{wr}B$ is n-complete then B is nilpotent of class at most n.*

Proof. Let $L(B)$ be the left regular representation of the group B. To each element $\ell_b \in L(B)$, $(b \in B)$ there corresponds an automorphism ℓ_b^* of W defined by $(cf)^{\ell_b^*} = cf^{\ell_b^*}$, for all $c \in B$, $f \in A^B$, where $f^{\ell_b^*}(x) = f(bx)$ for all $x \in B$. (cf. J. Panagopoulos [3], Lemma 5.1). If $f_1 \in A^B$ such that $f_1(1) = a$, $f_1(x) = 1$ for all $x \in B$, $x \ne 1$ and $b \in B$, $b \ne 1$ then $f_1^{\ell_b^*}(b^{-1}) = f_1(1) = a$ and $f_1^{\ell_b^*}(x) = f_1(bx) = 1$ for all $x \ne b^{-1}$. It is easy to see that $f_1^{\ell_b^*} = f_1 g$, where $g(1) = a^{-1}$, $g(b^{-1}) = a$ and $g(x) = 1$ for all

$x \in B$, $x \neq 1, b^{-1}$. For the element $g \in A^B$ we have that $g = [b^{-1}, \varphi]$, where $\varphi \in A^B$ with $\varphi(1) = g(1)$ and $\varphi(x) = 1$ for all $x \neq 1$. (cf. P. Neumann [2], Lemma 4.2).

Also, if $x_i \in B$ we define the element $f_{x_i} \in A^B$ by $f_{x_i}(x_i) = a$ and $f_{x_i}(c) = 1$ for all $c \in B$, $c \neq x_i$, then $f_{x_i}^{\ell_b^*} = f_{x_i} g^{x_i}$. If we choose an element $b \in \gamma_n(B)$, then the automorphism ℓ_b^* belongs to the group $K_n(W) \leq I(W)$. But $\ell_b^* \in I(W)$ if and only if $b \in Z(B)$ which means that the group B is nilpotent of class at most n. □

Proposition 3.3. *If $W = A\mathrm{wr}B$ is n-complete and B is nilpotent of class n, then A is directly indecomposable.*

Proof. Let $A = U \times V$ a non trivial direct decomposition of A. If $f \in A^B$ then $f(x) = u_x v_x$ for all $x \in B$, where $u_x \in U$, $v_x \in V$. If $g_f \in A^B$ such that $g_f(x) = u_x$ for all $x \in B$ and $z \in \gamma_n(B) \leq Z(B)$, $z \neq 1$, we define a mapping $\gamma : W \to W$ by $(bf)^\gamma = bf[g_f, z]$. Since $g_{fh} = g_f g_h$ and $g_f^y = g_{f^y}$ for all $f, h \in A^B$ and $y \in B$, we can see that γ is an outer automorphism of W with $\gamma \in K_n(W)$. This is a contradiction. □

Proposition 3.4. *Let $W = A\mathrm{wr}B$, where A is finite nilpotent and B is nilpotent of class n. If W is n-complete, then A is a p-group (p prime), and if A is abelian then it is cyclic of order p.*

Proof. Proposition 3.3 gives that A is a p-group. Moreover since A is abelian we have that A is cyclic of order p^r for some positive integer r. If $r \neq 1$ we choose an element $z \in \gamma_n(B)$, $z \neq 1$ and define a mapping $\gamma : W \to W$ by $(bf)^\gamma = bf[f, z]^p$, which is an automorphism of W belonging to the group $K_n(W)$. (cf. J. Panagopoulos [3], Proposition 4.6). Since $r > 1$ it is easy to see that γ is an outher automorphism, so W is not n-complete. Hence, $r = 1$. □

The proof of the next proposition is the same as the proof of Proposition 4.7 in J. Panagopoulos [3]. The only change concerns the choice of the element $z \in \gamma_n(B)$, $z \neq 1$.

Proposition 3.5. *Let $W = A\mathrm{wr}B$ where A is non abelian with A/A', $Z(A)$ of finite order and B nilpotent of class n. If W is n-complete then $\big(|A/A'|, |Z(A)|\big) = 1$.* □

Now, we see that if A is finite nilpotent, B nilpotent of class n and $W = A\mathrm{wr}B$ is n-complete, then A will be abelian. Because, if A is not abelian then $\big(|A/A'|, |Z(A)|\big) \neq 1$ which contradicts Proposition 3.5. Hence, we have the following:

Proposition 3.6. *If $W = A\mathrm{wr}B$ is n-complete, A finite nilpotent and B nilpotent of class n, then A is cyclic of prime order.* □

4. n-Complete wreath products

In this section we give examples of non n-complete standard wreath products. Constructing outer automorphisms of $W = A\mathrm{wr}B$ we use the following conclusion:

An automorphism γ of A^B is extended to an automorphism of W leaving B elementwise fixed if and only if γ is commuted with the inner automorphisms induced by elements of B. (cf. C. Houghton [1]. §3.4).

Proposition 4.1. *The wreath product $W = C_p\mathrm{wr}C_2$, where p is a prime with $p > 3$, is not n-complete.*

Proof. It is known that if $W = A\mathrm{wr}B$ then $W' = B'M$, where $M = \{f \in A^B \mid \pi(f) \in A'\}$ (cf. P. Newmann [2]. Theorem 4.1). Since $B = C_2$ and $|M| \mid |A|^{|B|} = p^2$ it follows that $|M| = p$. Thus, $\gamma_n(W) = M$ for all $n \in \mathbf{Z}^+$, $n \geq 2$, because W is not nilpotent. If $A = C_p = \langle a\rangle$, $B = C_2 = \langle b\rangle$ we choose the elements: $f_1 = (a^{p-1}, a^2)$, $f_2 = (a^2, a^{p-1})$, $g_1 = (a, 1)$, $g_2 = (1, a)$. Since $A^B = \langle f_1, f_2\rangle = \langle g_1, g_2\rangle$ and A^B is elementary abelian of rank 2 and $p \neq 3$ the mapping $g_1 \to f_1$, $g_2 \to f_2$ can be extended to an automorphism γ of A^B, which commutes with the automorphism of A^B induced by the element $b \in B$. Thus, the automorphism γ can be extended to an automorphism of W, which fixes B elementwise (cf. C. Houghton [1]).

Since $g_1^\gamma = (a, 1)^\gamma = (a^{p-1}, a^2) = (a, 1)(a^{p-2}, a^2)$, $g_2^\gamma = (1, a)^\gamma = (a^2, a^{p-1}) = (1, a)(a^2, a^{p-2})$ with (a^{p-2}, a^2), $(a^2, a^{p-2}) \in M = \gamma_n(W)$, $n \geq 2$, we have that $\gamma \in K_n(W)$, $n \geq 2$ and γ is an outher automorphism. Hence, $W = C_p\mathrm{wr}C_2$ is not n-complete. $\square$

Proposition 4.2. *The wreath product $W = C_p\mathrm{wr}B$, where p is a prime with $p > 3$ and B is finite nilpotent of class n with $k = |B| \geq 3$, is not n-complete.*

Proof. If we put $A = C_p = \langle a\rangle$ then the group A^B will be an elementary abelian p-group. Clearly, the set $g_{x_i} \in A^B$ for all $x_i \in B = \{x_1, x_2, \dots, x_k\}$ with $g_{x_i}(x_i) = a$, $g_{x_i}(x_j) = 1$, $x_j \neq x_i$ is a basis of A^B. We consider the mapping: $g_{x_i} \to f_{x_i} = g_{x_i}[b, g_{x_i}] = g_{x_i}^2 (g_{x_i}^{-1})^b$ for all $x_i \in B$, where $b \in \gamma_n(B)$. This mapping is extended to an automorphism $\bar\gamma$ of A^B because the set f_{x_i}, $x_i \in B$ is a basis of A^B. In fact, since $C_p \cong \mathbf{Z}_p$ ($\mathbf{Z}_p$ is the ring of integers mod p) and $p > 3$, the determinant

$$D = \begin{vmatrix} 2 & 0 & \dots & 0 & -1 & 0 & \dots & 0 \\ 0 & 2 & \dots & -1 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -1 & 0 & 0 & \dots & 2 \end{vmatrix} \in \mathbf{Z}_p,$$

is not zero, where the element 2 is in the main diagonal and in each row and column we have once the element -1. Since the automorphism $\bar{\gamma}$ of A^B commutes with the automorphisms of A^B which are induced by the elements of the group B, $\bar{\gamma}$ can be extended to an automorphism γ of the group W, which fixes B elementwise. Clearly, the automorphism γ is an outer automorphism with $\gamma \in K_n(W)$. Hence, the group W is not n-complete. $\square$

In the following we study the n-completeness of $W = A\mathrm{wr}B$ with $A = C_2$ and B finite. First we need the following:

Lemma 4.3. *If $W = A\mathrm{wr}B$ then $f \cdot f^{b^{2^{n-1}k}} \in \gamma_{n+1}(W)$ for all $f \in A^B$, $b \in B$, $k \in \mathbf{Z}$.*

Proof. By induction. If $n = 1$ then $[f, b^k] = f \cdot f^{b^k} \in W$ for all $f \in A^B$, $b \in B$. Let $f \cdot f^{b^{2^{n-1}k}} \in \gamma_{n+1}(W)$. Then $[f \cdot f^{b^{2^{n-1}k}}, b^{2^{n-1}k}] = f \cdot f^{b^{2^n}k} \in \gamma_{n+2}(W)$, for all $f \in A^B$, $b \in B$. $\square$

Proposition 4.4. *The wreath product $W = C_2\mathrm{wr}B$ is not n-complete, where B is finite abelian with $m = |B| \geq 4$ and m is an odd number.*

Proof. We distinguish the next cases.

a) We assume that there exist two elements $b, c \in B$ of order three with $b \notin \langle c \rangle$. Since $A^B = \prod_{i=1}^{m} \langle a_i \rangle$, where $A = C_2 = \langle a \rangle$ and $\langle a_i \rangle \cong \langle a \rangle$ for all $i = 1, 2, \ldots, m$ we define an automorphism $\bar{\gamma}$ of A^B by $a_i^{\bar{\gamma}} = a_i a_i^b a_i^c a_i^{b^2c} a_i^{bc^2}$ for all $i = 1, 2, \ldots, m$. The automorphism $\bar{\gamma}$ is extended to an outer automorphism γ of W, which fixes B elementwise. (cf. J. Panagopoulos [3], Lemma 6.3). For this automorphism γ we have that $\gamma \in K_n(W)$. In fact:

i) If $2^{n-1} \equiv 1\,(\mathrm{mod} 3)$ then $b^{2^{n-1}} = b$ and $b^{2^n} = b^2$. Thus $a_i^{\gamma} = a_i a_i^b a_i^c a_i^{b^{2^n}c} a_i^{bc^{2^n}}$ with $a_i^b a_i^c a_i^{b^{2^n}c} a_i^{bc^{2^n}} \in \gamma_{n+2}(W) \leq \gamma_{n+1}(W)$.

ii) If $2^{n-1} \equiv 2\,(\mathrm{mod} 3)$ then $b^{2^{n-1}} = b^2$. Thus $a_i^{\gamma} = a_i a_i^b a_i^c a_i^{b^{2^n}c} a_i^{bc^{2^n}}$ with $a_i^b a_i^c a_i^{b^{2^n}c} a_i^{bc^{2^n}} \in \gamma_{n+1}(W)$.

b) We assume that B contains an element b of order r not divisible by three. We define an automorphism $\bar{\gamma}$ of A^B by $a_i^{\bar{\gamma}} = a_i a_i^b a_i^{b^2}$ for all $i = 1, \ldots, m$. The automorphism $\bar{\gamma}$ is extended to an outher automorphism γ of W. Now, let $k \in Z$ a solution of the congruence $2^{n-1}x \equiv 1\,(\mathrm{mod} r)$. Then, $a_i^{\gamma} = a_i a_i^b a_i^{bb^{2^{n-1}k}}$ with $a_i^b a_i^{bb^{2^{n-1}k}} \in \gamma_{n+1}(W)$.

c) Let the orders of the elements of B are divisible by 3. If all elements of B have order 3 we are in the case a). On the other hand there is an element $b \in B$ of order 9. We define an automorphism $\bar{\gamma}$ of A^B by $a_i^{\bar{\gamma}} =$

$a_i a_i^b a_i^{b^3}$ for all $i = 1, 2, \dots, m$. The automorphism $\bar{\gamma}$ is extended to an outer automorphism γ of W. If $k \in Z$ is a solution of the congruence $2^{n-1}x \equiv 2 \,(\mathrm{mod} 9)$, then $b^2 = b^{2^{n-1}k}$. Thus $a_i^\gamma = a_i a_i^b a_i^{bb^{2^{n-1}k}}$ with $a_i^b a_i^{bb^{2^{n-1}k}} \in \gamma_{n+1}(W)$.

Finally, we see that in all cases there exists an outer automorphism in $K_n(W)$, so that the wreath product $W = C_2 \mathrm{wr} B$ is not n-complete. □

We have assumed up to this point that the subgroup A^B is characteristic in $W = A\mathrm{wr}B$. Now, we study the case of W in which A is a special dihedral group and B is of order 2. At this case A^B is not characteristic in W.

Proposition 4.5. *Let $W = D_m \mathrm{wr} C_2$, where $D_m = \langle a, b \mid a^m = 1, b^2 = 1, (ab)^2 = 1\rangle$, $m = 2k+1$, $k \in \mathbf{N}$ and C_2 is the cyclic group of order 2. Then the group W is n-complete if and only if $m = 3$.*

PROOF. We have for the lower central series of the group D_m : $\gamma_{i+1}(D_m) = \langle a^{2^i} \rangle$, for all $i = 1, 2, \dots$. Since m is an odd number, it follows that

$$(1) \qquad \gamma_2(D_m) = \gamma_3(D_m) = \dots = \gamma_i(D_m) = \gamma_{i+1}(D_m) = \dots$$

If the group $W = D_m \mathrm{wr} C_2$ is n-complete, then it follows by Proposition 3.1, that the group D_m is at most n-complete. This means by (1), that D_m is semicomplete and this is true if and only if $m = 3$. Since the group $W = D_3 \mathrm{wr} C_2$ is semicomplete, clearly it will be n-complete. (cf. J. Panagopoulos [4]). □

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J. PANAGOPOULOS, E. RAPTIS AND D. VARSOS
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ATHENS
PANPISTEMIOPOLIS, 15784 ATHENS
GREECE

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