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Localizable composable measures of fuzziness

By BRUCE R. EBANKS (Huntington)

Dedicated to Professors Zoltán Daróczy and Imre Kátai on their 60th birthday

Abstract. We consider measures of intrinsic, not random, uncertainty. An earlier paper introduced composition laws governing the application of such measures to disjoint unions and direct products of fuzzy sets. The associativity and commutativity of these operations induce certain properties on the composition laws. Now, with the help of the additional axiom of *localization*, we determine explicit forms for measures of intrinsic uncertainty (i.e. fuzziness). In addition, justification is provided for choosing the product (rather than the minimum) operation in the definition of the fuzzy direct product.

1. Introduction

A program of axiomatic characterization of measures of intrinsic uncertainty (i.e. measures of "fuzziness") was initiated in [3]. That paper introduced composition laws governing the desired behavior of a measure with respect to the operations of disjoint union and direct product of fuzzy sets. Now we develop the theory further through the introduction of a localization axiom. With this additional axiom, we shall find explicit forms for measures of fuzziness.

Let (X, Σ, μ) be a measure space. A *fuzzy set* on X is a measurable function $f : X \to [0, 1]$. For each $x \in X$, the value $f(x)$ is interpreted as

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the degree to which x enjoys some property S. Such a map f may therefore be considered a generalized characteristic function on X. If $f(X) \subset \{0, 1\}$, then f is a classical characteristic function, which we call a *sharp* (or *crisp*) *set* on X. In the present paper, we deal with fuzzy sets on finite universal sets $X = \{x_1, x_2, \dots, x_n\}$, where n may be any positive integer.

Given two fuzzy sets f and g on the same X, we define their (fuzzy) *union* $f \vee g$ and *intersection* $f \wedge g$ by

$$(f \vee g)(x) = \max\{f(x), g(x)\}, \quad (f \wedge g)(x) = \min\{f(x), g(x)\}.$$

We say that f and g are *disjoint* if $f \wedge g = 0$; in this case we call $f \vee g$ the disjoint union.

Given fuzzy sets f (on X) and g (on Y), we define the *direct product* $f \times g$ (on $X \times Y$) by (cf. [1])

$$(f \times g)(x, y) = f(x)g(y), \quad \forall x \in X,\ y \in Y. \tag{1.1}$$

Note. Another definition of the direct product of fuzzy sets which is seen rather often is $(f \times g)(x, y) = \mathrm{Min}\{f(x), g(y)\}$. We shall examine this definition in Section 4 and provide evidence there of its inappropriateness.

A *measure of intrinsic uncertainty* (or *measure of fuzziness*) is a nonnegative-valued functional M defined on a collection $\mathcal{F}$ on fuzzy sets which is closed under the formation of finite direct products and unions. That is, $M : \mathcal{F} \to \mathbb{R}_+ = [0, +\infty)$.

Some examples of measures of fuzziness which have been proposed are (cf. [1])

$$M_1(f) = -\sum_{i=1}^{n} f(x_i) \log_2 f(x_i) \quad (0 \log_2 0 := 0),$$

which is formally analogous to the Shannon entropy, and (cf. [2])

$$M_2(f) = \sum_{i=1}^{n} f(x_i)[1 - f(x_i)].$$

These both have the property that

$$M(f) = 0 \text{ if } f(X) \subset \{0, 1\}, \tag{M1}$$

which is very natural for a measure of fuzziness. It means that a sharp set (i.e. a characteristic function) has no fuzziness.

Another property shared by M_1 and M_2 is the following *localization* property.

(M2) There exists a continuous map $\Delta : [0,1]^2 \to \mathbb{R}$ such that

$$M(f) = M(f') + \Delta[f(x_i), f'(x_i)]$$

whenever f and f' differ at only one point $x_i \in X$.

Intuitively, this means that the difference in the amounts of fuzziness of f and of f' depends only on their values at the single point where they differ. We can broaden this property by permitting a change of scale before the comparison of amounts of fuzziness is made. The *generalized localization* property holds that,

(M2′) There exists a continuous map $\Delta : [0,1]^2 \to \mathbb{R}$ and a continuous bijection Π of $\mathbb{R}_+$, with $\Pi(0) = 0$, for which

$$M(f) = \Pi^{-1}\{\Pi[M(f')] + \Delta[f(x_i), f'(x_i)]\}$$

whenever f and f' differ at exactly one $x_i \in X$.

We claim that (M1) and (M2) (or more generally (M2′)) are desirable properties for a measure of intrinsic uncertainty, for the intuitive reasons expressed above.

There are two other axioms for measures of fuzziness which we shall suppose. These were first proposed and studied in [3], and they deal with the operation of direct product of fuzzy sets. In order to state these axioms, we first define the *power* of a fuzzy set, which is gauged by the functional $P : \mathcal{F} \to \mathbb{R}_+$ given by

$$P(f) = \sum_{i=1}^{n} f(x_i),$$

whenever f is a fuzzy set on $X = \{x_1, x_2, \ldots, x_n\}$. This is a natural extension of the notion of cardinality for sharp sets. Now our third axiom is the following.

(M3) There is a *direct product composition law*

$$G : \mathbb{R}_+^4 \to \mathbb{R}_+ \text{ such that}$$

$$M(f \times g) = G[P(f), P(g), M(f), M(g)].$$

This axiom is a further generalization of some additivity and generalized additivity properties introduced in [1], [2] and [5], and it may be described briefly as follows. We assume that the amount of fuzziness of $f \times g$ is a function of the measures of fuzziness of f and g, and of the "sizes" of f and g (namely $P(f)$ and $P(g)$).

Our fourth and final axiom supposes a natural symmetry of $f \times g$ and $g \times f$:

$$\text{(M4)} \qquad M(f \times g) = M(g \times f).$$

This is intuitively clear.

Throughout this discussion, we assume that the set $D(\subset \mathbb{R}_+^2)$ of all pairs $(x, u) = (P(f), M(f))$, as f ranges over $\mathcal{F}$, is what we shall call a *power-measure-* (or *PM-*) *domain*. By that we mean that D is nonempty, open, connected, and $D + D$ contains D. Why is this plausible? It seems reasonable to assume that sets of different sizes may be equally fuzzy, and that sets of the same size may differ as to their degrees of fuzziness. In fact, if we consider for a moment just the subcollection of fuzzy sets on a singleton universe $X = \{x_1\}$, a seemingly natural shape for the graph of this subset of D would be the first quadrant region bounded by the positive x-axis and the graph of something like $u = kx(1 - x)$ for some positive constant k. This subset of D already satisfies the necessary criteria for a PM-domain. As the cardinality of the universal set X goes to infinity, it also seems plausible that D would possibly fill up the first quadrant of the (x, u)-plane.

The measure $M = 0$ trivially satisfies (M1)–(M4). But this trivial measure is excluded by our PM-domain hypothesis.

In this paper we find all measures of intrinsic uncertainty satisfying the axioms introduced above. Some properties of such measures are presented in Section 2, and their explicit forms are given in Section 3.

2. Some consequences of the axioms

We begin this section by observing that the postulated properties of M are invariant under changes of scale. That is, if M is a measure satisfying (M1), (M2′), (M3), (M4) and if Σ is a continuous bijection of $S = M(\mathcal{F})$ onto a subset of $\mathbb{R}_+$, with $\Sigma(0) = 0$, then the induced measure $M' = \Sigma \circ M$ is again a measure of fuzziness on $\mathcal{F}$ satisfying the same axioms. M' will have direct product composition law G' and scaling function Π' given by

$$G'(x, y, u, v) = \Sigma \circ G(x, y, \Sigma^{-1}(u), \Sigma^{-1}(v)),$$
$$\Pi' = \Pi \circ \Sigma^{-1}.$$

In this situation, we can consider M and M' to be isomorphic measures.

A nontrivial measure M satisfying (M1)–(M4) will be called a *canonical* measure of fuzziness. Since every measure of fuzziness satisfying (M1), (M2′), (M3), (M4) is isomorphic to a canonical measure, by taking $\Sigma = \Pi$, it is sufficient to discover the canonical measures of intrinsic uncertainty.

Our first result deals with the first two axioms only.

Proposition 2.1. *A measure M of intrinsic uncertainty satisfies* (M1) *and* (M2) *if and only if there exists a continuous map $\varphi : [0,1] \to \mathbb{R}_+$ satisfying $\varphi(0) = \varphi(1) = 0$ such that*

$$M(f) = \sum_{i=1}^{n} \varphi[f(x_i)], \tag{2.1}$$

for any fuzzy set f on a universal set $X = \{x_1, \ldots, x_n\}$.

Proof. Let $X = \{x_1, \ldots, x_n\}$ and $f = \{(x_i, f(x_i)) \mid i = 1, \ldots, n\}$, and suppose that (M1) and (M2) hold. Applying (M2) n times in succession, we obtain

$$\begin{aligned} M(f) &= M(\{(x_1, f(x_1)), \ldots, (x_n, f(x_n))\}) \\ &= M(\{(x_1, f(x_1)), \ldots, (x_{n-1}, f(x_{n-1})), (x_n, 0)\}) + \Delta[f(x_n), 0] \\ &= \ldots \\ &= M(\{(x_1, f(x_1)), (x_2, 0), \ldots, (x_n, 0)\}) + \sum_{j=2}^{n} \Delta[f(x_j), 0] \\ &= M(\{(x_1, 0), \ldots, (x_n, 0)\}) + \sum_{j=1}^{n} \Delta[f(x_j), 0]. \end{aligned}$$

Observe that $\{(x_1, 0), \ldots, (x_n, 0)\}$ is a sharp set (namely, the empty set) on X, so its measure under M is zero by (M1). Defining $\varphi : [0,1] \to \mathbb{R}_+$ by

$$\varphi(p) := \Delta(p, 0), \quad p \in [0,1],$$

we have (2.1). Clearly, (M1) and (2.1) also lead to the conclusion that $\varphi(0) = \varphi(1) = 0$.

The converse is easily verified, with $\Delta(u, v) = \varphi(u) - \varphi(v)$, and this completes the proof.

Next, we begin to explore the consequences of the axioms on the form of the direct product composition law G.

Proposition 2.2. *If D is a PM-domain and M satisfies* (M1) *through* (M4)*, then the map G appearing in* (M3) *has the form*

$$G(x, y, u, v) = \gamma xy + \beta(xv + yu) + \delta uv, \quad (x, u), (y, v) \in D, \tag{2.2}$$

for some nonnegative constants γ, β, δ.

Proof. By Proposition 2.1 we have (2.1) with continuous $\varphi : [0, 1] \to \mathbb{R}_+$ such that $\varphi(0) = \varphi(1) = 0$. Let f and g be disjoint fuzzy sets on $X = \{x_1, \ldots, x_n\}$. Then $f(x_i)g(x_i) = 0$ for $i = 1, \ldots, n$, so in this case we have

$$M(f \vee g) = \sum_{i=1}^{n} \varphi[f \vee g(x_i)] = \sum_{i=1}^{n} \varphi[f(x_i)] + \sum_{i=1}^{n} \varphi[g(x_i)] = M(f) + M(g),$$

since $\varphi(0) = 0$. (In fact, it follows that M is a valuation on the lattice $\mathcal{F}$.)

Next, observe that $\times$ distributes over $\vee$, that is

$$(f \vee g) \times h = (f \times h) \vee (g \times h), \tag{2.3}$$

and that $f \times h$ and $g \times h$ are disjoint because f and g are. So, taking M of both sides of (2.3) and using (M3), we get

$$G[P(f \vee g), P(h), M(f \vee g), M(h)] = M(f \times h) + M(g \times h),$$

that is,

$$\begin{aligned} &G\{P(f) + P(g), P(h), M(f) + M(g), M(h)\} \\ &= G[P(f), P(h), M(f), M(h)] + G[P(g), P(h), M(g), M(h)], \end{aligned}$$

since $P(f \vee g) = P(f) + P(g)$ for disjoint f and g by definition of P. Hence we have

$$G(x + y, z, u + v, w) = G(x, z, u, w) + G(y, z, v, w), \tag{2.4}$$

for all $(x,u),(y,v),(z,w),(x+y,u+v)$ in D.

For the moment, let us temporarily fix $(z,w)=(z_0,w_0)$ in D and define a map $F: D \to \mathbb{R}_+$ by

$$F(x,u) = G(x,z_0,u,w_0) \tag{2.5}$$

for all (x,u) in D. Then (2.4) shows that F is additive on a restricted domain, i.e.

$$F(x+y,u+v) = F(x,u) + F(y,v),$$

for all $(x,u),(y,v)$ in D with $(x+y,u+v)$ in D. Now F satisfies all the hypotheses of Corollary XIII.6.2 (p. 332) in [4], and therefore there exists a unique additive function $A: \mathbb{R}^2 \to \mathbb{R}$ for which $F = A$ on D. Moreover, since F (and hence also A) is nonnegative on the nonempty open set D, we conclude that F has the form

$$F(x,u) = ax + bu,$$

for some (nonnegative) constants a, b. Thus, releasing z and w, we find that G has the form (cf. (2.5))

$$G(x,z,u,w) = a(z,w)x + b(z,w)u, \tag{2.6}$$

for some functions $a, b: D \to \mathbb{R}_+$.

Finally let us recall (M4), which implies that

$$G(x,z,u,w) = G(z,x,w,u). \tag{2.7}$$

Applying this to (2.6), we arrive at

$$a(z,w)x + b(z,w)u = a(x,u)z + b(x,u)w.$$

Because D is a PM-domain, and since (x,u) and (z,w) are permitted to vary independently in D, we conclude that

$$a(x,u) = \gamma x + \beta u, \quad \text{and} \quad b(x,u) = \beta x + \delta u,$$

for some (nonnegative) constants γ, β, δ. Inserting this into (2.6), (2.2) is established and the proof is finished.

3. Main results: Forms of canonical measures

Next, we come to the main result of this paper.

Theorem 3.1. *Let D be a PM-domain. $M : \mathcal{F} \to \mathbb{R}_+$ is a canonical measure of intrinsic uncertainty if and only if M has one of the following forms. Either*

$$M(f) = b\sum_{i=1}^{n} f(x_i)\log_2 f(x_i), \quad (\text{where } 0\log_2 0 := 0), \tag{3.1}$$

for some $b < 0$, or else there is some $\alpha \in (0,1) \cup (1,+\infty)$ for which

$$M(f) = c\sum_{i=1}^{n}[f(x_i)^\alpha - f(x_i)] \tag{3.2}$$

with some constant c ($c < 0$ if $\alpha > 1$, $c > 0$ if $0 < \alpha < 1$).

Proof. First let us verify that (3.1) and (3.2) define canonical measures of fuzziness. In both cases M is of the form (2.1) with φ continuous and $\varphi(0) = \varphi(1) = 0$. Hence (M1) and (M2) are verified, by Proposition 2.1. (M4) is obviously satisfied in both cases. Finally, (M3) is satisfied in case of (3.1) with $G(x,y,u,v) = yu + xv$, and in case of (3.2) with $G(x,y,u,v) = yu + xv + \frac{1}{c}uv$.

Conversely, suppose D is a PM-domain and M is a canonical measure of intrinsic uncertainty. By Propositions 2.1 and 2.2, we have the representations (2.1) for M (with φ continuous and $\varphi(0) = \varphi(1) = 0$) and (2.2) for G. Let us take $X = \{x_1\}$, $Y = \{y_1\}$, $p = P(f) = f(x_1)$ and $q = P(g) = g(y_1)$. Then, with (2.1) and (2.2), the direct product composition law in (M3) becomes

$$\begin{aligned} \varphi(pq) &= M(f \times g) = G(p,q,\varphi(p),\varphi(q)] \\ &= \gamma pq + \beta[q\varphi(p) + p\varphi(q)] + \delta\varphi(p)\varphi(q), \ \ p,q \in [0,1]. \end{aligned} \tag{3.3}$$

With $p = 1$ we get

$$\varphi(q)(1-\beta) = \gamma q, \quad q \in [0,1]. \tag{3.4}$$

We consider two cases.

Case 1. Suppose $\beta \neq 1$. Then (3.4) yields the linearity of φ. But the only linear map satisfying $\varphi(0) = \varphi(1) = 0$ is the zero map. Since this would give $M = 0$, this case is impossible.

Case 2. Suppose $\beta = 1$. Then (3.4) yields $\gamma = 0$, and (3.3) reduces to

$$\varphi(pq) = p\varphi(q) + q\varphi(p) + \delta\varphi(p)\varphi(p).$$

Defining a new map ψ on $[0, 1]$ by

$$\psi(p) = p + \delta\varphi(p), \tag{3.5}$$

we find that ψ is multiplcative, i.e.

$$\psi(pq) = \psi(p)\psi(q), \quad p, q \in [0, 1].$$

As we see also from (3.5) that $\psi(0) = 0$, $\psi(1) = 1$, and ψ is continuous, it follows that

$$\psi(p) = p^\alpha, \quad p \in [0, 1],$$

for some constant $\alpha > 0$.

As long as $\delta \neq 0$, (3.5) gives

$$\varphi(p) = \frac{1}{\delta}(p^\alpha - p), \quad p \in [0, 1],$$

which, through (2.1), gives us solution (3.2) with $c = \frac{1}{\delta}$. Moreover, $\alpha \neq 1$ since $\varphi \neq 0$, and the sign of c in (3.1) is chosen so that $M(f) \geq 0$.

On the other hand, if $\delta = 0$, then (3.5) provides no information about φ. Returning to equation (3.3), however, and recalling that $\beta = 1$ and $\gamma = 0$, we find that now φ satisfies

$$\varphi(pq) = q\varphi(p) + p\varphi(q), \quad p, q \in [0, 1].$$

That is, $p \mapsto \varphi(p)/p$ is a continuous solution of the logarithmic functional equation. Thus we arrive at the conclusion that

$$\varphi(p) = \begin{cases} bp\log p, & p \in (0, 1], \\ 0, & p = 0, \end{cases}$$

for some (negative) constant b. With (2.1) this leads to (3.1), and this concludes the proof of the main theorem.

4. Fuzzy direct products

In this section we provide an argument against the usage of the frequently seen definition

$$(f \times_{\min} g)(x,y) = \mathrm{Min}\{f(x), g(y)\}, \quad \forall x \in X,\ y \in Y, \tag{4.1}$$

as the fuzzy direct product. Indeed, we show now that there is no localizable measure of fuzziness which is composable with respect to the direct product (4.1). Combined with Theorem 3.1, this provides justification for choosing the product definition (1.1) over (4.1).

Theorem 4.1. *Let D be a PM-domain. There is no measure $M : \mathcal{F} \to \mathbb{R}_+$ of intrinsic uncertainty satisfying* (M1)–(M4) *with respect to the fuzzy direct product* (4.1).

Proof. Suppose D is a PM-domain and M is a measure of intrinsic uncertainty satisfying (M1)–(M4), where the direct product is defined by (4.1). The only spot in the proofs of Propositions 2.1 and 2.2 where the definition of direct product is used is at equation (2.3) and the sentence containing it. Straightforward calculations show that both the equation and the statement are valid also for $\times_{\min}$. Thus, as before, we again have the representations (2.1) for M (with φ continuous and $\varphi(0) = \varphi(1) = 0$) and (2.2) for G. As in the proof of Theorem 3.1 we take $X = \{x_1\}$, $Y = \{y_1\}$, $p = P(f) = f(x_1)$ and $q = P(g) = g(y_1)$. Now, with (2.1) and (2.2), the direct product composition law in (M3) becomes (instead of (3.3))

$$\begin{aligned} \varphi(\mathrm{Min}\{p,q\}) &= M(f \times_{\min} g) = G[p,q,\varphi(p),\varphi(q)] \\ &= \gamma pq + \beta[q\varphi(p) + p\varphi(q)] + \delta\varphi(p)\varphi(q), \quad p,q \in [0,1]. \end{aligned} \tag{4.1}$$

With $p = 1$, again we get

$$\varphi(q)(1-\beta) = \gamma q, \quad q \in [0,1]. \tag{3.4}$$

As before, it follows that $\beta = 1$ and $\gamma = 0$, so (4.1) reduces to

$$\varphi(\mathrm{Min}\{p,q\}) = p\varphi(q) + q\varphi(p) + \delta\varphi(p)\varphi(q). \tag{4.2}$$

We claim that there is no nontrivial (i.e., nonzero) solution of this equation.

Indeed, putting $q = 1/2$ in (4.2), we obtain

$$\varphi(p)\left[\frac{1}{2} - \delta\varphi\left(\frac{1}{2}\right)\right] = p\varphi\left(\frac{1}{2}\right), \quad p \le \frac{1}{2}. \tag{4.3}$$

Observe that $\left[\frac{1}{2} - \delta\varphi\left(\frac{1}{2}\right)\right] = 0$ is impossible, for then (4.3) would yield also $\varphi\left(\frac{1}{2}\right) = 0$, leading to the contradiction $1/2 = 0$. Therefore it follows from (4.3) that

$$\varphi(p) = kp, \quad p \le \frac{1}{2}, \tag{4.4}$$

for some constant k. Substituting this into (4.2), we find that

$$kp = k(2 + \delta k)pq, \quad p < q \le \frac{1}{2},$$

from which it follows that $k = 0$. With (4.4), this means that $\varphi = 0$ on $[0, 1/2]$. Returning now to (4.2), restricting p to $[0, 1/2]$ and q to $(1/2, 1]$, we deduce that $0 = p\varphi(q)$. Hence $\varphi = 0$. But then (2.1) shows that $M = 0$, contradicting the assumption that D is a PM-domain. This contradiction concludes the proof of the theorem.

5. Epilogue

For non-canonical measures, we easily derive the following from Theorem 3.1.

Corollary 5.1. *Let D be a PM-domain. A nontrivial measure $M : \mathcal{F} \to \mathbb{R}_+$ satisfies* (M1), (M2′), (M3) *and* (M4), *if and only if M has the form*

$$M = \Pi^{-1}(M'),$$

where M' is a canonical measure (and hence has one of the forms (3.1) *or* (3.2)*).*

Other axioms which have been introduced and may be desirable for a measure of fuzziness in some applications are *symmetry*:

$$M(1 - f) = M(f), \quad f \in \mathcal{F}; \tag{M5}$$

maximality:

(M6) $$M(f) \text{ is maximized on } X \text{ when } f(X) = \left\{\frac{1}{2}\right\};$$

and *sharpening*:

(M7) $$M(f) \geq M(g) \text{ if } g \text{ is "sharper" than } f.$$

We say that g is *sharper* than f if $\left|g - \frac{1}{2}\right| \geq \left|f - \frac{1}{2}\right|$, i.e. if $g(x) \geq f(x)$ whenever $f(x) \geq \frac{1}{2}$, and $g(x) \leq f(x)$ whenever $f(x) \leq \frac{1}{2}$. With respect to these properties, we have the following result.

Corollary 5.2. *Let D be a PM-domain. The only canonical measures of fuzziness which satisfy any one (and therefore all) of the properties* (M5)–(M7) *are those of the form*

(5.1) $$M(f) = a\sum_{i=1}^{n}[f(x_i) - f(x_i)^2] = a\sum_{i=1}^{n} f(x_i)[1 - f(x_i)],$$

for some constant $a > 0$. (That is, M is a constant multiple of M_2).

Proof. By Theorem 3.1, we have either (3.1) or (3.2). But (3.1) can never satisfy (M5), (M6) or (M7) if $b \neq 0$. Moreover (3.2) can satisfy (M5), (M6) or (M7) only if $2^{\alpha-1} = \alpha$. Since $\alpha \neq 1$ in (3.2), this is possible only if $\alpha = 2$. Defining $a := -c$ we have (5.1). Conversely, any M of the form (5.1) satisfies all three of (M5)–(M7).

We conclude with the remark that the constant a in (5.1) serves only to fix a unit of fuzziness. For instance, if we declare that one unit of fuzziness is the amount of fuzziness of the fuzzy set $\left\{\frac{1}{2}\right\}$ on a singleton universal set $X = \{x_1\}$, then this fixes $a = 4$. Then (5.1) takes the form

$$M(f) = 4\sum_{i=1}^{n} f(x_i)[1 - f(x_i)],$$

which could be called the *normalized* symmetric canonical measure of intrinsic uncertainty.

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BRUCE R. EBANKS
DEPARTMENT OF MATHEMATICS
MARSHALL UNIVERSITY
HUNTINGTON, WV 25755
U.S.A.

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