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On generalized q-multiplicative functions

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Dedicated to the 60th birthday
of Professors Zoltán Daróczy and Imre Kátai

Abstract. The R-multiplicative functions are defined as a generalization of q-multiplicative functions. Those $\mathcal{R}$-multiplicative functions are investigated for which a linear recurrence holds.

1. Introduction

The letters $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$, $\mathbb{C}$ denote the sets of the natural numbers, nonnegative integers, real numbers, complex numbers, respectively. For $A \subseteqq \mathbb{N}_0$ let $|A|$ be the number of the elements of A.

Definition 1.1. Let $\mathcal{R}_0, \mathcal{R}_1, \dots, \dots$ be a sequence of subsets of $\mathbb{N}_0$. We say that it is an $\mathcal{R}$-system, if the following conditions hold:

a) $0 \in \mathcal{R}_i$ and $1 < |\mathcal{R}_i| < \infty$ $(i = 0, 1, 2, \dots)$;
b) for $(0 \leqq) i < j$, the smallest positive element of $\mathcal{R}_i$ is smaller than the smallest positive element of $\mathcal{R}_j$;
c) each $n \in \mathbb{N}_0$ can be uniquely written as

$$n = \sum_{j=0}^{s} r_j \quad (r_j \in \mathcal{R}_j,\ s \geqq 0). \tag{1.1}$$

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We say that an $\mathcal{R}$-system is monotonic, if in addition (d), and that it is bounded if in addition (e) holds, where:

d) for each $(0 \leqq) i < j$, the largest element of $\mathcal{R}_i$ is smaller than the smallest positive element of $\mathcal{R}_j$,

e) $|\mathcal{R}_j|$ is bounded.

Examples 1.2. Let k_i $(i = 0, 1, \dots)$ be a sequence of integers, $k_i \geqq 2$, furthermore let $d_0 = 1$, $d_i = d_{i-1}k_{i-1}$ $(i > 0)$, $\mathcal{N}_i = \{0, 1, \dots, k_i - 1\}$, $\mathcal{R} = d_i\mathcal{N}_i = \{0, d_i, \dots, (k_i - 1)d_i\}$ $(i = 0, 1, \dots)$. Then $\mathcal{R}_i$ $(i = 0, 1, \dots)$ is an $\mathcal{R}$-system. Such $\mathcal{R}$-systems are called "britannic number systems" by N. G. De Bruijn [1].

Especially, if $k_i = q \geqq 2$ $(i = 0, 1, \dots)$ then we obtain the q-ary number system. It is easy to see that the monotonic $\mathcal{R}$-systems are exactly the "britannic number systems".

Definition 1.3. The function $f : \mathbb{N}_0 \to \mathbb{C}$ is called $\mathcal{R}$-multiplicative (wich respect to a given $\mathcal{R}$-system), if

$$f(0) = 1 \text{ and } f(n) = \prod_{j=0}^{s} f(r_j). \tag{1.2}$$

It is clear that $f(n) = c^n$ $(0 \neq c \in \mathbb{C})$ is $\mathcal{R}$-multiplicative for every $\mathcal{R}$-system.

2. $\mathcal{R}$-multiplicative functions with regular behaviour

Let an $\mathcal{R}$-sysytem be given, $f : \mathbb{N}_0 \to \mathbb{C}$ be an $\mathcal{R}$-multiplicative function, $P(z) = a_k z^k + \dots + a_1 z + a_0 \in \mathbb{C}[z]$, and

$$P(E)f(N) := a_k f(n+k) + \dots + a_1 f(n+1) + a_0 f(n).$$

Let us consider the following conditions:

$$\liminf_{x\to\infty} \frac{1}{x} \sum_{n \leqq x} |P(E)f(n)| = 0, \tag{2.1}$$

$$\sum_{n\leqq x}|P(E)f(n)|=o(x)\quad (x\to\infty), \tag{2.2}$$

$$P(E)f(n)=0\quad (\forall n\in\mathbb{N}_0), \tag{2.3}$$

$$\liminf_{x\to\infty}\frac{1}{x}\sum_{n\leqq x}|f(n)|=0, \tag{2.4}$$

$$\sum_{n\leqq x}|f(n)|=o(x)\quad (x\to\infty). \tag{2.5}$$

Theorem 2.1. *The following assertions are valid:*

a) *If* (2.1) *holds then either* (2.3) *or* (2.4) *hold*

b) (2.2) *is satisfied if and only if either* (2.3) *or* (2.5) *holds.*

c) *There exists such an* f *for which* (2.4) *holds, and* (2.5) *does not holds.*

d) *Assuming that the* $\mathcal{R}$*-system is monotonic and bounded, the fulfilment of* (2.4) *and that of* $|f(n)|\leqq 1$ $(n\in\mathbb{N}_0)$ *imply* (2.5).

Proof. Assume that the $\mathcal{R}$-system is given: $\mathcal{R}=\{\mathcal{R}_0,\mathcal{R}_1,\dots\}$. Let the sets $\mathcal{A}_s,\mathcal{T}_s$ be defined as follows: $\mathcal{A}_0=\emptyset$, $\mathcal{A}=\mathcal{R}_0\oplus\mathcal{R}_1\oplus\dots\oplus\mathcal{R}_{s-1}$ $(s\geqq 1)$, $\mathcal{T}_s=\bigcup_{\ell=0}^{\infty}(\mathcal{R}_s\oplus\dots\oplus\mathcal{R}_{s+\ell})$, i.e. $\mathcal{T}_s$ consists of those integers n which can be written as $n=r_s+\dots+r_{s+\ell}$ for some integer ℓ, and $r_j\in\mathcal{R}_j$ $(j=s,\dots,s+\ell)$.

It is clear that $f(n+m)=f(n)f(m)$ is satisfied if $n\in\mathcal{A}_s$, $m\in\mathcal{T}_s$ and f is $\mathcal{R}$-multiplicative.

a) Let f be $\mathcal{R}$-multiplicative, assume that (2.1) holds, and that for some $\alpha\in\mathbb{N}_0$ $P(E)f(\alpha)\neq 0$. Let i_0 be so large that $\alpha+j\in\mathcal{A}_{i_0}$ for $j=0,1,\dots,k$, where $k=\deg P$. Let $\mathcal{A}_{i_0}=\{\beta_1,\beta_2,\dots,\beta_A\}$. Then for every large x,

$$\frac{1}{x}\sum_{n\leqq x}|P(E)f(n)|\geqq\frac{x-\alpha}{x}|P(E)f(\alpha)|\frac{1}{x-\alpha}\sum_{\substack{n\leqq x-\alpha\\ n\in\mathcal{T}_{i_0}}}|f(n)|\geqq 0. \tag{2.6}$$

The relation (2.1) and (2.6) imply that for an appropriate sequence $y_t\to\infty (t\to\infty)$ we get

$$\frac{1}{y_t}\sum_{\substack{n\leq y_t\\ n\in\tau_{i_0}}}|f(n)|\to 0\quad (t\to\infty). \tag{2.7}$$

Furthermore, for large y,

$$\sum_{n\leqq y}|f(n)|=\sum_{j=1}^{A}|f(\beta_j)|\Big(\sum_{\substack{n+\beta_j\leqq y\\ n\in\tau_{i_0}}}|f(n)|\Big)\leqq A\cdot H\sum_{\substack{n\leqq y\\ n\in\tau_{i_0}}}|f(n)|,$$

where $H=\max\limits_{1\leq j\leq A}|f(\beta_j)|$. Hence, and from (2.7),

$$\frac{1}{y_t}\sum_{n\leq y_t}|f(n)|\to 0\quad (t\to\infty). \tag{2.8}$$

b) Assume that for some $\mathcal{R}$-multiplicative function f (2.2) holds, and that there exists an $\alpha\in\mathbb{N}_0$ such that $P(E)f(\alpha)\neq 0$. We can choose x_t to run over the whole set $\mathbb{N}_0$ and reason as above.

c) We shall give an $\mathcal{R}$-system and an $\mathcal{R}$-multiplicative function for which (2.4) holds and (2.5) does not hold. Let f be a q-multiplicative function taking on positive values. Let

$$H(s):=\frac{1}{q^s}\sum_{n=0}^{q^s-1}f(n)=\prod_{i=0}^{s-1}\left(\frac{1}{q}\sum_{j=0}^{q-1}f(jq^i)\right).$$

Let $s_1<s_2<\dots$ be a rare sequence of integers, let $f(jq^t)=\frac{1}{2q^2}$ if $j=1,2,\dots,q-1$; $t\in\mathbb{N}_0$ except for $j=1$, when $t\in\{s_1,s_2,\dots\}$. Let $f(1\cdot q^{s^\ell})=q^{s_\ell}+1$. Then obviously,

$$\frac{1}{q^{s_\ell}+1}\sum_{n=0}^{q^{s_\ell}}f(n)>1,$$

on the other hand

$$\frac{1}{q}\sum_{j=0}^{q-1}f(jq^\ell)=\begin{cases}\dfrac{1}{q}+\dfrac{q-1}{2q} & \text{if } \ell\notin\{s_1,s_2,\dots\}\\[2mm] q^\ell+\dfrac{2}{q}+\dfrac{q-1}{2q^2}<q^m+1 & \text{if } \ell\in\{s_1,s_2\dots\}.\end{cases}$$

Thus

$$H(s)\leqq\left(\frac{3}{2q}\right)^s\prod_{s_\nu\leqq s}(q^{s_\nu}+1)\left(\frac{2q}{3}\right).$$

If we choose $s_\nu = 2^{2^\nu}$, and $t_\nu = s_{\nu+1} - 1$, then

$$H(t_\nu) \leqq \left(\frac{3}{2q}\right)^{t_\nu - 1 - \nu} q^{2s_\nu + 1} \to 0 \quad (\nu \to \infty).$$

d) To prove it, we may assume that f takes on nonnegative values, $0 \leqq f(n) \leqq 1$. Let n_s be a strictly monotonic sequence of positive integers, and $n_s - 1 = Ad_{j_s} + n_s'$, where $1 \leqq A < k_{j_s}$, $0 \leqq n_s' < d_{j_s}$. Then

$$\frac{1}{n_s} \sum_{m=0}^{n_s - 1} f(m) \geqq \frac{d_{j_s}}{n_s} \cdot \frac{1}{d_{j_s}} \sum_{m=0}^{d_{j_s} - 1} f(m) \geqq 0. \tag{2.9}$$

Let n_s be such a sequence for which the right hand side of (2.9) tends to zero. Since the sequence k_i in the definition of the $\mathcal{R}$-system is assumed to be bounded, $k_i \leqq M$, therefore

$$\frac{d_{j_s}}{n_s} \geqq \frac{d_{j_s}}{d_{j_s+1}} = \frac{1}{k_{j_s}} \geqq \frac{1}{M},$$

consequently from (2.9)

$$\frac{1}{d_{j_s}} \sum_{m=0}^{d_{j_s} - 1} f(m) \to 0 \quad (s \to \infty). \tag{2.10}$$

It is obvious that for every $h \geqq 1$,

$$\frac{1}{d_h} \sum_{m=0}^{d_h - 1} f(m) = \prod_{i=0}^{h-1} \left(\frac{1}{k_i} \sum_{t \in \mathcal{R}_i} f(r) \right) =: A(h). \tag{2.11}$$

Since $0 < \frac{1}{k_i} \sum_{t \in \mathcal{R}_i} f(r) \leqq 1$, the sequence $A(h)$ is decreasing monotonically, and by (2.10) we obtain that $A(h) \to 0$ $(h \to \infty)$.

Let $n > 1$, $n - 1 = Ad_{i(n)} + n'$ $(0 \leqq A < k_{i(n)},\ 0 \leqq n' < d_{i(n)})$. Then

$$\begin{aligned} 0 &\leqq \frac{1}{n} \sum_{m=0}^{n-1} f(m) \leqq \frac{1}{n} \sum_{j=0}^{A} f(jd_{i(n)}) \sum_{m=0}^{d_{i(n)}-1} f(m) \leqq \\ &\leqq (A+1) \frac{d_{i(n)}}{n} A(i(n)) \leqq 2A(i(n)). \end{aligned} \tag{2.12}$$

Hence the assertion follows.

The proof of Theorem 2.1 is complete.

Theorem 2.2. *Assume that the $\mathcal{R}$-system is monotonic, and that $F : \mathbb{N}_0 \to \mathbb{R}$ is such an $\mathcal{R}$-multiplicative function for which $0 \leqq F(n) \leqq 1$ $(\forall n \in \mathbb{N}_0)$. Then*

$$\sum_{n \leqq x} F(n) = o(x) \quad (x \to \infty) \tag{2.13}$$

holds if and only if

$$\sum_{j=0}^{\infty} \frac{1}{k_j} \left(k_j - \sum_{r \in \mathcal{R}_j} F(r) \right) = \infty. \tag{2.14}$$

Proof. From (2.11), (2.12) it follows that (2.13) holds if and only if

$$\prod_{j=0}^{\infty} \left(\frac{1}{k_j} \sum_{r \in \mathcal{R}_j} F(r) \right) \to 0. \tag{2.15}$$

Since

$$\frac{1}{k_j} \sum_{r \in \mathcal{R}_j} F(r) = 1 - \frac{1}{k_j} \left(k_j - \sum_{r \in \mathcal{R}_j} F(r) \right),$$

and

$$k_j - \sum_{r \in \mathcal{R}_j} F(r) \geqq 0,$$

(2.15) holds if and only if (2.14) is satisfied.

3. The $\mathcal{R}$-multiplicative solutions of the recursion $P(E)f(n) = 0$

Theorem 3.1. *Let $\mathcal{R}_0, \mathcal{R}_1, \dots$ be such an $\mathcal{R}$-system for which $\mathcal{R}_0 = \{0, 1, \dots, d-1\}$, $d \geqq 2$. Assume that $P(z) \in \mathbb{C}[z]$ is of degree k, $1 \leqq k \leqq d$,* *and that $P(0) \neq 0$. Then the recursion*

$$P(E)f(n) = 0 \quad (n \in \mathbb{N}_0) \tag{3.1}$$

holds for an $\mathcal{R}$-multiplicative function if and only if

$$f(n) = \sum_{j=1}^{s} \alpha_j \rho_j^n, \tag{3.2}$$

where $\sum_{j=1}^{s} \alpha_j = 1$, and $\rho_j (j = 1, 2, \dots, s)$ are distinct roots for P for which

$$\rho_1^d = \rho_2^d = \cdots = \rho_s^d \quad (= c). \tag{3.3}$$

Proof. First we prove that the function f defined by (3.2) is $\mathcal{R}$-multiplicative. $f(0) = \alpha_1 + \cdots + \alpha_s = 1$. Let N, $r \in \mathbb{N}_0$. From (3.2), (3.3) we obtain that $f(Nd+r) = C^N(\alpha_1\rho_1^r + \cdots + \alpha_s\rho_s^r) = C^N f(r) = f(Nd)f(r)$, from which the $\mathcal{R}$-multiplicativity of f is clear. The fulfilment of (3.1) is obious.

Assume now that f is an R-multiplicative function and that (3.1) holds. Then

$$\begin{aligned} 0 &= P(E)f(Nd + d - k) \\ &= a_k(f((N+1)d) - f(Nd)f(d)) + f(Nd)P(E)f(d-k). \end{aligned}$$

Hence, by (3.2) and from $a_k \neq 0$ we have that

$$f((N+1)d) = f(Nd)f(d). \tag{3.4}$$

From $f(d) = 0$ it would follow that $f(0) = 0$, which cannot be. Thus

$$f(Nd) = C^N \quad (\forall N \in \mathbb{N}_0), \quad 0 \neq C \in \mathbb{C}. \tag{3.5}$$

The general solution of (3.1) can be written as

$$f(n) = p_1(n)\rho_1^n + \cdots + p_h(n)\rho_h^n, \tag{3.6}$$

where ρ_j are the roots of $P(z)$ and p_j are polynomials.

(3.5), (3.6) and the $\mathcal{R}$-multiplicativity of f imply that

$$\sum_{i=1}^{h} p_i(Nd+r)\sigma_i^N \rho_i^r - f(r)c^N = 0 \tag{3.7}$$

holds for every $N \in \mathbb{N}_0$ and $r = 0, 1, \dots, d-1$. Here $\sigma_j := \rho_j^d$ $(j = 1, \dots, h)$.

Since $f(0) = 1$, from (3.7) by $r = 0$ we obtain that $\sigma_i = C$ holds for at least one i.

Assume that $\sigma_1 = \cdots = \sigma_{i_1}$, $\sigma_{i_1+1} = \cdots = \sigma_{i_2}, \dots, \sigma_{i_{t-1}+1} = \cdots = \sigma_{i_t}$, $\sigma_{i_t+1} = \cdots = \sigma_h = C$, and that $\sigma_{i_\nu} \neq \sigma_{i_\mu}$ if $\nu \neq \mu$. Assume that $i_1 < h$.

Let

$$Q_{i_j}^{(r)}(N) := p_{i_{j-1}+1}(Nd+r)\rho_{i_{j-1}+1}^r + \cdots + p_{i_j}(Nd+r)\rho_{i_j}^r \qquad (j = 1, \dots, t; i_0 := 0),$$

$$Q^{(r)}(N) := p_{i_t+1}(Nd+r)\rho_{i_t+1}^r + \cdots + p_h(Nd+r)\rho_h^r - f(r).$$

By using these notations (3.7) can be rewritten as

$$Q_{i_1}^{(r)}(N)\sigma_{i_1}^N + \cdots + Q_{i_t}^{(r)}(N)\sigma_{i_t}^N + Q^{(r)}(N)C^N = 0 \quad (N \in \mathbb{N}_0). \tag{3.8}$$

Consequently the polynomials $Q_{i_\nu}^{(r)}(x)$, $Q^{(r)}(x)$ are zero identically, i.e.

$$\begin{gathered} p_{i_{j-1}+1}(z)\rho_{i_{j-1}+1}^r + \cdots + p_{i_j}(z)\rho_{i_j}^r = 0 \\ (r = 0, 1, \dots, i_j - i_{j-1} - 1 < d) \end{gathered} \tag{3.9}$$

for $z \in \mathbb{C}$, since $\rho_{i_{j-1}+1}, \dots, \rho_{i_j}$ are distinct complex numbers, the determinant of the system, being a Vandermonde determinant, is nonzero, hence $p_\nu(z) = 0$ identically, and consequently

$$f(n) = q_1(n)\rho_1^n + \cdots + q_s(n)\rho_s^n, \tag{3.10}$$

where $1 \leqq s \leqq h$, $\rho_1^d = \cdots = \rho_s^d = C$, $q_1, \dots, q_s \in \mathbb{C}[x]$.

From (3.5) and from the $\mathcal{R}$-multiplicativity of f,

$$f(r) = q_1(Nd+r)\rho_1^r + \cdots + q_s(Nd+r)\rho_s^r \quad (N \in \mathbb{N}_0,\ 0 \leqq r < d).$$

Hence we obtain easily that the coeffitients $q_j(Nd+r)$ do not depend on N, consequently $q_j(Nd+r) = \alpha_j$ $(j = 1, \dots, s)$.

Finally, from $f(0) = 1$ we obtain that $\alpha_1 + \cdots + \alpha_s = 1$. The proof of Theorem 3.1 is complete.

4. Remark to the Theorem 3.1

To prove (3.1) the assumption of the condition $\deg P \leqq d$ is inportant. If $\deg P > d$, then in general there exists such an $\mathcal{R}$-multiplicative function for which (3.1), (3.2) are satisfied, but (3.3) does not hold.

References

[1] N. G. De Bruijn, On number systems, *Niew Arch. Wisk.* (3) 4 (1956), 15–17.

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