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Transfer functions and spectral projections

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Dedicated to Professors Zoltán Daróczy and Imre Kátai
on the occasion of their 60th birthday

Abstract. Assuming that a 2×2 matrix of bounded linear operators between Banach spaces represents a weak coupling between the factor spaces, the spectral integral for the corresponding transfer function is defined (in the spirit of the Riesz projection in the linear case). It is proved that if the spectral set for the corresponding operator entry is nontrivial, then the spectral integral is a projection reducing the transfer function, and the structure of the spectral projection of the matrix operator is described.

1. Introduction

The interest in the spectral theory of 2×2 operator matrices (or matrix operators) in Banach spaces has increased considerably in recent years. For a good introduction to the subject see, e.g., [Hal], [Nag] or [Heng], where further references and motivation can be found. From our point of view it is significant in the two latter papers that the connection between the (parts of the) spectrum of the operator matrix and the (corresponding parts of the) spectrum of the so called (Frobenius-) Schur-complements (in different terminology: transfer functions) is emphasized. Their setting is more general than ours in the sense that they consider unbounded matrix

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entries with applications to operator matrix semigroups, but we shall be able to make use of some of their basic results.

For further important results in the spectral theory of operator matrices and their applications we refer to the recent works of Atkinson, Langer, Mennicken and Shkalikov [ALMS] and Mennicken and Shkalikov [MS], both in the unbounded setting. The main result of the first paper is that under a list of conditions the essential spectra of the matrix operator and of the corresponding transfer function are identical, and outside this set the corresponding Fredholm indices coincide. In the second paper certain spectral subspaces of the closure of a symmetric operator matrix and the restrictions to these subspaces are characterized in a Hilbert space situation.

In this note we consider the case of bounded operator entries in the product of two complex Banach spaces under certain standing assumptions which will be formulated exactly in Section 2. These assumptions mean essentially that the operator matrix

$$V = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

represents a weak coupling between the factor spaces, i.e. it is, in a sense, not very far from a direct sum, and also that the union of the spectra of the operators V and A is the disjoint union of two closed sets, which may naturally be regarded as (generalizations of) spectral sets in the sense of F. Riesz and Dunford not only for V and A, but also for the so called transfer function $z \mapsto t(z) := D + C(z-A)^{-1}B$. The main result of this note is that, if we define the spectral integral $P(S, s_k)$ for the function $S(z) := z - t(z)$ and the spectral set s_k in the spirit of the classical situation (see the Definition in Section 2), then under the standing assumptions it will be equal to the corresponding spectral projection $P(D, s_k)$ for the operator D. Hence, if the spectral set s_k is nontrivial for D, then $P(S, s_k)$ will reduce (each operator of) the transfer function t. We shall also describe the structure of the spectral projections $P(V, s_k)$ for the matrix operator V, and the weakness of the the coupling will be mirrored in this structure. We believe that the results are new even if the spaces are finite-dimensional.

Our notations will be mostly traditional. We shall leave out the identity operator I in formulae like $R(z, A) = (zI - A)^{-1}$, write everywhere $r(A)$ for the resolvent set and $s(A)$ for the spectrum of an operator A. In a similar spirit we shall write $S(z)^{-1}$, $r(S)$ and $s(S)$ for the suitably defined objects in connection with the transfer function t (see Section 2). Finally, we wish to emphasize that if s_k is a compact set in the complex plane, then in all formulae in the paper containing

$$\int_{s_k}$$

this symbol will denote integration along a suitably chosen smooth path surrounding s_k. However, in some cases the specification of the path is an essential part of a proof, therefore the notation above seems to be adequate.

2. The results

Let X and Y be complex Banach spaces, let $W := X \times Y$, and let

$$V := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be an operator matrix in W consisting of bounded linear operators between the corresponding spaces. For some general results on the spectra of such (and of more general) operator matrices see, e.g., [Nag] or [Heng]. In particular, we shall need [Heng, Proposition 1.1] in this situation.

Proposition H. *Let $R(z, A) := (z - A)^{-1}$ denote the resolvent operator for any $z \in r(A)$, let*

$$S(z) := z - D - CR(z, A)B,$$

and let $r(S)$ denote the set of all $z \in \mathbb{C}$ such that $S(z)^{-1}$ is a bounded linear operator in Y. Then

$$r(A) \cap r(V) = r(A) \cap r(S) = r(S),$$

and for any z in this set

$$R(z, V) = \begin{pmatrix} R(z, A)(I + BS(z)^{-1}CR(z, A)) & R(z, A)BS(z)^{-1} \\ S(z)^{-1}CR(z, A) & S(z)^{-1} \end{pmatrix}.$$

Note that the operator function $z \mapsto D + CR(z, A)B$ is called a transfer function or, in another context, a Schur complement. Further, let us introduce the notation $s(S) := \mathbb{C} \setminus r(S)$.

In this note we shall always make the following

Standing assumptions.

1) The set $s(A) \cup s(V) = s(A) \cup s(S) = s(S)$ *is the disjoint union of the closed sets* s_1 *and* s_2.

2) $BC = BD = DC = 0$.

Remark 1. Assumptions 2) may be interpreted as a sort of weak coupling by V between the spaces X and Y: they are trivially satisfied if $B = C = 0$. Moreover, as a consequence of 2), the operator D commutes with every $S(z), z \in r(A)$. Further, in the proofs below, reasonings reminiscent of some methods of functional equations can and will be applied.

Remark 2. Assumptions 2) above will be satisfied exactly for those transfer functions, for which (writing ker for kernel and rg for range)

$$\mathrm{rg}(C) \cup \mathrm{rg}(D) \subset \ker(B), \qquad \mathrm{rg}(C) \subset \ker(D).$$

There is a wealth of such operator triples (B, C, D) even in the finite dimensional special case, when the operators can, e.g., be (parts of) a realization of a finite dimensional time-invariant linear system, and infinite dimensional such operator triples also abound. On the other hand, this admittedly restrictive assumption will allow us to express the Riesz projections $P(V, s_k)$ for the matrix operator with the help of the constituent operators in the Theorem below.

Lemma. *For any pair* $z, u \in r(A)$ *the operators* $S(z)$ *and* $S(u)$ *commute. Further,*

$$r(A) \cap r(D) \setminus \{0\} = r(S) \setminus \{0\},$$

and for any z *in this set*

$$S(z)^{-1} = R(z, D) + z^{-2} C R(z, A) B. \tag{1}$$

Moreover, $0 \in r(A) \cap r(D)$ *if and only if* $0 \in r(S)$. *If this is the case, then* $B = C = 0$, *and* $S(0)^{-1} = R(0, D)$. *Finally, we always have*

$$s(A) \cup s(V) = s(S) = s(A) \cup s(D) = s_1 \cup s_2.$$

Proof. Let $z, u \in r(A)$. By assumption 2), then

$$S(z)S(u) = zu - (z+u)D + D^2 - zCR(u, A)B - uCR(z, A)B = S(u)S(z).$$

Assume now that $z \in r(A) \cap r(D) \setminus \{0\}$, and let

$$T(z) := R(z, D) + z^{-2}CR(z, A)B.$$

Note that, by assumption 2),

$$R(z, D)C = z^{-1}R(z, D)(z - D)C = z^{-1}C, \tag{2}$$

hence

$$T(z)S(z) = I - R(z, D)CR(z, A)B + z^{-2}CR(z, A)B(z - D) = I.$$

Similarly, by assumption,

$$BR(z, D) = B(z - D)R(z, D)z^{-1} = z^{-1}B, \tag{3}$$

and therefore

$$S(z)T(z) = I - CR(z, A)BR(z, D) + (z - D)z^{-2}CR(z, A)B = I.$$

Hence $T(z) = S(z)^{-1}$ and $z \in r(S) \setminus \{0\}$. In the converse direction assume that z belongs to the latter set, and define

$$H(z) := S(z)^{-1} - z^{-2}CR(z, A)B.$$

By the standing assumptions, $S(z)CR(z, A)B = zCR(z, A)B$, hence

$$-z^{-1}CR(z, A)B = -S(z)^{-1}CR(z, A)B.$$

Therefore we obtain

$$H(z)(z - D) = S(z)^{-1}(z - D) - z^{-1}CR(z, A)B = S(z)^{-1}S(z) = I.$$

In a similar way we obtain that $CR(z, A)BS(z) = zCR(z, A)B$, hence

$$-z^{-1}CR(z, A)B = -CR(z, A)BS(z)^{-1}.$$

Therefore

$$(z - D)H(z) = (z - D)S(z)^{-1} - z^{-1}CR(z, A)B = S(z)S(z)^{-1} = I.$$

We have proved the equality of the two sets and the form of $S(z)^{-1}$ for any z from this set.

Assume now that $0 \in r(A) \cap r(D)$. By assumption 2), we obtain then $B = C = 0$, hence

$$V = A \oplus D, \qquad S(z) = z - D.$$

Therefore $0 \in r(S)$ and $S(0)^{-1} = R(0, D)$. Conversely, assume that $0 \in r(S)$. Then there exists the operator $S(0)^{-1}$. By definition, $S(0) = -D + CA^{-1}B$. By the standing assumptions, $S(0)^2 = D^2$. Hence there exists $D^{-2} = S(0)^{-2}$ and, by spectral mapping, $0 \in r(D) \cap r(A)$. Hence $s(S) = s(A) \cup s(D)$, and the final statement of the Lemma follows. The proof is complete. □

Note that, by assumption 1), the sets

$$s_k \cap s(T) \qquad (k = 1 \text{ or } 2,\ T = V, A, D \text{ or } S)$$

contain all the singularities of the locally holomorphic function $R(z, T)$ (for $T = S$ we understand here naturally $R(z, S) := S(z)^{-1}$). It follows that for the purposes of the following Riesz-type contour integrals the sets s_1, s_2 can and will be considered as spectral sets (in the sense of F. Riesz–Dunford) for any of the operator functions $R(z, T)$.

Definition. The spectral integrals corresponding to T and the spectral set s_k will be denoted by $P(T, s_k)$. For example,

$$P(V, s_k) := \frac{1}{2\pi i} \int_{s_k} R(z, V) dz =: \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix},$$

where the integral is along a smooth curve surrounding s_k (see the Introduction), and the right-hand side equality is the definition of the operator entries in the matrix. Similarly,

$$P(S, s_k) := \frac{1}{2\pi i} \int_{s_k} S(z)^{-1} dz = D_k.$$

Theorem. *Under the standing assumptions we obtain that*

$$P(V, s_k) = \begin{pmatrix} P(A, s_k) & B_k \\ C_k & P(D, s_k) \end{pmatrix}, \qquad B_k C_k = C_k B_k = 0 \quad (k = 1, 2).$$

Further, for $0 \neq z \in r(A) \cap r(V) = r(S)$ *we have*

$$R(z,V) = \begin{pmatrix} R(z,A) & z^{-1}R(z,A)B \\ z^{-1}CR(z,A) & S(z)^{-1} \end{pmatrix}.$$

If $0 \notin s_k$*, then*

$$B_k = [A|P(A,s_k)X]^{-1}B, \qquad C_k = C[A|P(A,s_k)X]^{-1},$$
$$B_j = -B_k, \qquad C_j = -C_k \quad (j \neq k),$$

where $A|\cdot$ *denotes the corresponding restriction of the operator.*

Finally, if $0 \in r(A) \cap r(V) = r(S)$*, then the nondiagonal entries above vanish, hence* $B_k = C_k = 0$*.*

Proof. By definition, for $k = 1,2$ we have

$$\begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix} = P(V,s_k) = P(V,s_k)^2 = \begin{pmatrix} A_k^2 + B_kC_k & * \\ * & C_kB_k + D_k^2 \end{pmatrix}.$$

From the matrix representation of $R(z,V)$ we obtain

$$B_k = \frac{1}{2\pi i}\int_{s_k} R(z,A)BS(z)^{-1}dz, \quad C_k = \frac{1}{2\pi i}\int_{s_k} S(z)^{-1}CR(z,A)dz,$$

where the contour integrals above (and later) are to be understood as explained at the end of the Introduction. By standing assumption 1), the integration paths lie in $r(S)$, and we may and will assume that they avoid the point $z = 0$. Since $B = BS(z)S(z)^{-1} = zBS(z)^{-1}$ (by assumption 2), or due to (1) and (3), we obtain that

$$BS(z)^{-1} = z^{-1}B.$$

Similarly, $C = S(z)^{-1}S(z)C = S(z)^{-1}zC$, or (1) and (2), imply

$$S(z)^{-1}C = z^{-1}C.$$

Hence we obtain that

$$(+) \qquad B_k = \frac{1}{2\pi i}\int_{s_k} z^{-1}R(z,A)dzB, \quad C_k = C\frac{1}{2\pi i}\int_{s_k} z^{-1}R(z,A)dz.$$

Therefore

$$(*) \qquad C_k B_k = \frac{1}{2\pi i} C \int_{s_k} z^{-1} R(z,A) dz \frac{1}{2\pi i} \int_{s_k} R(u,A) u^{-1} du B.$$

By the Lemma, we have

$$(**) \quad D_k = \frac{1}{2\pi i} \int_{s_k} S(z)^{-1} dz = P(D, s_k) + \frac{1}{2\pi i} C \int_{s_k} z^{-2} R(z,A) dz B.$$

We shall show that $(*)$ is equal to the second term on the right-hand side of $(**)$. We shall distinguish two cases.

Case (1): $0 \notin s_k$. Then the function $z \mapsto z^{-1}$ is holomorphic in a neighbourhood of s_k. Define the function f to be the above function in a neighbourhood of s_k and the 0 function in a neighbourhood of s_j $(j \neq k)$. Then f is locally holomorphic in a neighbourhood of $s(A)$, hence the Riesz–Dunford holomorphic calculus defines the bounded linear operator $f(A)$ in the space X. $(*)$ is then the operator $Cf(A)^2 B$. The multiplicative property of the calculus shows that the second term on the right-hand side of $(**)$ is the same operator.

Case (2): $0 \in s_k$. Consider now the product of the integrals in $(*)$. Compute the left-hand side integral along a contour (= boundary of a bounded Cauchy domain) $g(z)$ lying inside the contour $g(u)$, along which we compute the right-hand side integral, and apply the first resolvent equation. Since the function

$$(z,u) \mapsto \frac{1}{zu(u-z)} R(u,A)$$

is continuous on $g(z) \times g(u)$, we may also change the order of the integrals if needed, and we obtain with the notation $c := (2\pi i)^{-1}$

$$\begin{aligned} (*) = Cc \int_{g(z)} \left\{ c \int_{g(u)} \frac{1}{u(u-z)} du \right\} z^{-1} R(z,A) dz B \\ - Cc \int_{g(u)} \left\{ c \int_{g(z)} \frac{1}{z(u-z)} dz \right\} u^{-1} R(u,A) du B. \end{aligned}$$

Recalling that 0 and z lie inside the contour $g(u)$, calculating residues shows that the inner integral in the first line is equal to $-z^{-1} + z^{-1} = 0$.

Similarly, since 0 lies inside and u lies outside the contour $g(z)$, calculating residues for the inner integral in the second line gives $-u^{-1}$ as its value. Hence we obtain

$$(*) = cC \int_{g(u)} u^{-2} R(u, A) du B.$$

We see that the second term on the right-hand side of $(**)$ is $C_k B_k$. We have obtained that

$$D_k^2 + C_k B_k = D_k = P(D, s_k) + C_k B_k, \quad \text{hence } D_k^2 = P(D, s_k).$$

Denote the other spectral set by s_j $(j \neq k)$. By standing assumption 1) and the Lemma, then

$$P(D, s_k) + P(D, s_j) = I, \quad D_k + D_j = I.$$

Hence we obtain that

$$I - P(D, s_j) = P(D, s_k) = D_k^2 = (I - D_j)^2 = I - 2D_j + P(D, s_j).$$

Therefore

$$D_j = P(D, s_j) \qquad (j = 1, 2).$$

From the preceding paragraph it follows that $C_k B_k = 0$ for $k = 1, 2$.

By standing assumption 2), from the formulae at the beginning of this proof it follows that $B_k C_k = 0$ for $k = 1, 2$. Moreover, for $0 \neq z \in r(A) \cap r(V) = r(S)$ the general formula for $R(z, V)$ simplifies to the formula in the statement. Hence for $k = 1, 2$

$$A_k = c \int_{s_k} R(z, A) dz = P(A, s_k),$$

as stated.

Further, from $(+)$ and from standard results of the holomorphic functional calculus for bounded operators we obtain that the stated formulae for B_k and C_k hold if $0 \notin s_k$, consequently the operator A restricted to the range space $P(A, s_k)X$ has a bounded inverse operator. The expressions for B_j and C_j for $j \neq k$ follow from the fact that

$$P(V, s_k) + P(V, s_j) = I,$$

the identity in the space W.

The last statement follows from the Lemma and the form and the continuity of the resolvent function $z \mapsto R(z, V)$. $\square$

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