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Further metric results on series expansions

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Dedicated to my old friends and fellow academicians
Professors Zoltán Daróczy and Imre Kátai on their 60th birthdays

Abstract. The paper investigates the speed of convergence for a large class of Oppenheim expansion. Both two sided estimates and asymptotic results are established, the latter being valid for almost all x. A new special case of Oppenheim series is introduced, which is termed the Daróczy–Kátai–Birthday (DKB) expansion, and it is shown that the speed of convergence of the new expansion is faster than the best known classical expansions.

Introduction

In the algorithm of expanding real numbers $0 < x \leq 1$ into infinite series of rationals we use the following notation and basic assumptions. Each of the sequences $h_n(j)$, $n \geq 1$, of functions of the positive integers $j \geq 1$ is assumed to be integer valued with $h_n(j) \geq 1$ for all n and j. We set

$$r_n(j) = \frac{h_n(j)}{j(j-1)}. \tag{1}$$

We use the language of probability theory, but probability P always stands for 'length' or Lebesgue measure.

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For a given sequence $h_n(j)$, $n \geq 1$, of functions, we define our algorithm by the integers, called digits,

$$d_k = d_k(x),\ k \geq 1 : x = x_1,\ \frac{1}{d_k} < x_k \leq \frac{1}{d_k - 1}, \tag{2}$$

and the remainder terms x_k, $k \geq 1$,

$$x_{k+1} = \left(x_k - \frac{1}{d_k}\right) \frac{1}{r_k(d_k)}. \tag{3}$$

Iteration of the algorithm at (2) and (3) leads to the equation

$$x = \frac{p_n}{q_n} + r_1(d_1) r_2(d_2) \cdots r_n(d_n) x_{n+1}, \tag{4}$$

where we put, with positive integers p_n and q_n,

$$\frac{p_n}{q_n} = \frac{1}{d_1} + \frac{r_1(d_1)}{d_2} + \cdots + \frac{r_1(d_1) r_2(d_2) \cdots r_{n-1}(d_{n-1})}{d_n}. \tag{5}$$

By the choice of the inequalities in (2), the expansion at (4) and (5) never terminates. The recursive formula (3) also entails the inequality

$$d_{k+1} > h_k(d_k), \tag{6}$$

which in turn, by the assumption on the sequence $h_n(j)$, yields that the infinite series resulting from (4) always converges to x. We thus have

$$x = \frac{1}{d_1} + \frac{r_1(d_1)}{d_2} + \cdots + \frac{r_1(d_1) r_2(d_2) \cdots r_n(d_n)}{d_{n+1}} + \cdots \tag{7}$$

The expansion leading to (7) is unique, and any series of the form, satisfying (6), is obtained by the algorithm (2) and (3). For easier reference we record four special cases:

(E) Engel series: $h_n(j) = j - 1$ for all n. We get

$$x = \frac{1}{d_1} + \frac{1}{d_1 d_2} + \cdots + \frac{1}{d_1 d_2 \cdots d_n} + \cdots$$

(S) Sylvester series: $h_n(j) = j(j-1)$ for all n. The corresponding series is of the form

$$x = \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_n} + \cdots$$

(L) Lüroth expansion or series: $h_n(j) = 1$ for all n. Hence,

$$x = \frac{1}{d_1} + \frac{1}{d_1(d_1-1)}\frac{1}{d_2} + \cdots + \left(\prod_{j=1}^{n-1} \frac{1}{d_j(d_j-1)}\right)\frac{1}{d_n} + \cdots$$

(DKB) Daróczy–Kátai-Birthday expansion: $h_n(j) = j^2(j-1)$. The resulting series expansion of x takes the form

$$x = \frac{1}{d_1} + \frac{d_1}{d_2} + \cdots + \frac{d_1 d_2 \cdots d_n}{d_{n+1}} + \cdots$$

The general algorithm (2) and (3), together with the series representation (7) under (6), is known as the Oppenheim expansion of real numbers. The special cases (E), (S), and (L) are classical expansions with extensive literature covering them; see J. Galambos [1] for details. The case (DKB) is introduced here for the first time, although quite a few of its properties can be deduced from the general theory of Oppenheim series, developed in the just cited monograph of the present author.

Notice that in each of the four special cases above $h_n(j)$ is independent of n and is a polynomial in j with degrees varying from zero to three. It is also to be observed that each numerator of the fractions in the series representing x is one in the three classical expansions, while these numerators are rapidly growing integers in the (DKB) expansion. Yet, we shall show in the sequel that the fastest approximation of x among the four recorded special cases is achieved by the (DKB) expansion. This, and other nice and superior properties of the (DKB) expansion served as a guide in the development of the results of the present paper.

Speed of convergence: estimates

After having computed n digits $d_1, d_2, \ldots, d_n$, we approximate x by the rational number p_n/q_n of (5) and the error of the approximation is given by the last term of (4). Hence, by (2),

(8) $$g_n < x - \frac{p_n}{q_n} \le g_n \frac{d_{n+1}}{d_{n+1}-1}$$

where

$$g_n = \frac{r_1(d_1)r_2(d_2)\cdots r_n(d_n)}{d_{n+1}} \tag{9}$$

Consequently, if the sequence d_j, $j \geq 1$, is known to grow rapidly, then g_n of (9) essentially expresses the speed of convergence in (8). In some cases, such as the Lüroth series, we can only say that $d_j \geq 2$; in such cases no better general statement can be made than having the lower and upper bounds g_n and $2g_n$, respectively, at (8).

The following estimate of g_n seems to be uniform for all expansions. By definition,

$$g_{n+1} = \frac{g_n r_{n+1}(d_{n+1})d_{n+1}}{d_{n+2}} = \frac{g_n h_{n+1}(d_{n+1})}{d_{n+2}(d_{n+1}-1)} < \frac{g_n}{d_{n+1}-1}, \tag{10}$$

where we appealed to (1) and (6). By iteration we get

$$g_n < \frac{1}{(d_1-1)(d_2-1)\cdots(d_n-1)}. \tag{11}$$

Note that in our estimate (11) we eliminated numerators when they were present, and we also got rid of d_{n+1}. In addition, (11) is applicable at every x. Hence, if the expansion of x is done through a computer program, (11) is a very valuable estimator. One can also see without computers that for the (DKB) expansion very few steps are needed for a given accuracy. For example, for any $0 < x < \frac{1}{2}$, $d_1 \geq 3$, and then by (6), $d_2 \geq 19$, $d_3 \geq 6499$, and $d_4 \geq 6499^2 \times 6498 + 1 > 2.7 \times 10^{11}$. Hence, no x in $(0, \frac{1}{2})$ requires more than 4 iterations of the algorithm in order to get an approximation accurate up to 14 decimal digits.

While much improvement on (11) cannot be made if we insist on estimates for all x, considerable improvement can be made in the metric sense, that is, when we consider estimates valid for almost all x or for a large proportion of the interval $(0,1)$. For metric results, we utilize a recent representation of the digits d_j, $j \geq 1$. The present author, Galambos [2], showed that the probability distribution of the sequence d_j, $j \geq 1$, is identical to the following sequence D_j, $j \geq 1$. Let $X_1, X_2, \ldots$ be independent unit exponential variables, that is, $F(x) = P(X_j \leq x) = 1 - e^{-x}$, $x \geq 0$. Let $D_1 = [\exp(X_1)] + 1$, where $[y]$ signifies the integer part of y, and define

$$D_{j+1} = [h_j(D_j)\exp(X_{j+1}] + 1, \quad j \geq 1. \tag{12}$$

Hence

$$h_j(D_j)\exp(X_{j+1}) < D_{j+1} \le h_j(D_j)\exp(X_{j+1}) + 1, \tag{13}$$

which we use in (10) in the form

$$\frac{h_{n+1}(D_{n+1})}{D_{n+2}} < e^{-X_{n+2}}.$$

Upon denoting by G_n the expression at (9) with D_j in place of d_j, $1 \le j \le n+1$, (10) becomes

$$G_{n+1} < \frac{G_n \exp(-X_{n+2})}{D_{n+1} - 1}. \tag{14}$$

Applying (14) repeatedly we have

$$G_n \le \frac{\exp\left(-\sum_{j=1}^{n} X_{j+1}\right)}{(D_1 - 1)(D_2 - 1)\cdots(D_n - 1)}. \tag{15}$$

The combination of the strong law of large numbers and the iterated logarithm theorem (see [3], Sections 2.5 and 6.6) imply that, for almost all x, as $n \longrightarrow +\infty$,

$$\exp\left(-\sum_{j=1}^{n} X_{j+1}\right) \le \exp\left\{-n + (1+\varepsilon)(2n\log\log n)^{\frac{1}{2}}\right\}$$

with an arbitrary $\varepsilon > 0$, which, in particular, implies that with an arbitrary $\varepsilon_1 > 0$,

$$\exp\left(-\sum_{j=1}^{n} X_{j+1}\right) \le \exp\left\{-(1-\varepsilon_1)n\right\}.$$

Hence, as $n \to +\infty$, for almost all x,

$$G_n < \frac{\exp\{-(1-\varepsilon_1)n\}}{(D_1 - 1)(D_2 - 1)\cdots(D_n - 1)}. \tag{16}$$

This is the improvement over (11) when we drop the requirement that our estimate be valid for all x. For statements involving distributions only, one can write back d_j for D_j and g_n for G_n.

Speed of convergence: asymptotic results

In order to allow us to compare the speed of convergence in different expansions we have to obtain asymptotic values not just one sided inequalities. The tools have already been developed in the preceding section. In particular, we shall freely turn to the sequence D_j instead of d_j since our development is for 'almost all x'.

We shall impose a growth requirement on the sequence $h_n(j)$, $n \geq 1$, in order to guarantee that $d_n \to +\infty$ with n for almost all x. It has been demonstrated by the present author (Galambos [2]) that the assumption $h_n(j) \geq j-1$ does imply the divergence of d_n, so we make this assumption. This eliminates only the expansion (L) from our special cases. We shall return to this case, however, at the end of the present section.

Now, when $d_n \to +\infty$, we have from (8) that the asymptotic speed of convergence in approximating x by p_n/q_n of (5) is exactly g_n. Arguing as at (10) we have from (12) that, for metric results,

$$\frac{g_{n+1}}{g_n} = \frac{h_{n+1}(d_{n+1})}{d_{n+2}(d_{n+1}-1)} = \frac{\exp(-X_{n+2})}{d_{n+1}-1}\left\{1 + O\left(\frac{1}{h_{n+1}(d_{n+1})}\right)\right\}.$$

Let us take the logarithm. Since $\log(1+z) = O(z)$,

$$\begin{aligned}\log\left(\frac{g_{n+1}}{g_n}\right) &= -X_{n+2} - \log(d_{n+1}-1) + O\left(\frac{1}{h_{n+1}(d_{n+1})}\right)\\ &= -X_{n+2} - \log(d_{n+1}) + O\left(\frac{1}{d_{n+1}}\right)\end{aligned}$$

where we utilized the assumption $h_{n+1}(d_{n+1}) \geq d_{n+1} - 1$. Here, $\log(d_{n+1})$ dominates the fixed random variable X_{n+2}, and thus, by Theorem 6.13 of Galambos [1], we have the limit theorem:

If $h_n(j)$ is a polynomial of degree $t > 1$ for all n, then, for almost all x, as $n \longrightarrow +\infty$,

$$\lim\left(t^{-n}\log\left(\frac{g_{n+1}}{g_n}\right)\right) = -G(x) < 0$$

exists and is finite.

We can now make comparisons among the four special Oppenheim expansions of the Introduction. The limit theorem above is applicable to expansions (S) with $t = 2$ and (DKB) with $t = 3$. Clearly, the latter has a much faster speed of convergence than the first one does. For the expansion (E), Corollary 6.19 of Galambos [1] states that $(1/n)\log(d_n) \to 1$ for almost all x. Our preceding computations yield:

For (E), for almost all x,

$$\lim\left(\frac{1}{n}\right)\log\left(\frac{g_{n+1}}{g_n}\right) = -1.$$

The speed of convergence is therefore far smaller in magnitude than for either (S) or (DKB).

Because for expansion (L) our assumption on $h_n(j)$ is not satisfied, a direct computation is required. Note that since $h_n(j) = 1$ for all n and j, the distributional properties of the sequence d_j, $j \geq 1$, via (12), are identical to the independent sequence

$$U_j = [\exp(X_j)] + 1, \quad j \geq 1$$

Going back to (4) and writing $g_n^* = x - p_n/q_n$, then

$$\log\left(\frac{g_{n+1}^*}{g_n^*}\right) = \log(x_{n+1}) - \log(d_{n+1}) - \log(d_{n+1} - 1) - \log(x_{n+2}), \tag{17}$$

where x_{n+1} and x_{n+2} are uniform variables (Theorem 6.1 in Galambos [1]) and d_{n+1} is distributed as U_n. Since these random variables are independent and their distributions do not depend on n, we have that the left hand side of (17) is a stationary sequence. The implied speed of convergence is considerably slower than the previously analyzed three expansions. We add that the statement at (17) could have been deduced from a recent result of Galambos [4].

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