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Differences between polynomials and factorials

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Abstract. Let A_n, $n = 1, 2, 3, \dots$, be an increasing sequence of positive integers such that for each prime number p, there is an integer $s = s(p)$ for which $p|A_n$ for every $n \geq s$. Such are, for instance, the sequences of factorials $A_n = n!$, least common multiples of the first n positive integers $A_n = \mathrm{LCM}(1, 2, \dots, n)$, and the products of the first n primes $A_n = p_1 p_2 \cdots p_n$. For any of such sequences A_n, $n = 1, 2, 3, \dots$, and any polynomial $f \in \mathbb{Z}[x]$ of degree at least 2, we show that the set of positive integers that are not expressible as $f(x) - A_n$ for some $x, n \in \mathbb{N}$ has a positive lower density. We also investigate the case when $f \in \mathbb{Z}[x]$ is linear and a few related problems. It is shown, for instance, that $x^2 - ay^2 = n!$ has infinitely many integer solutions in $(x, y, n) \in \mathbb{N}^3$, where $n \geq 2$, for each integer a in the range $1 \leq a \leq 12$, but has no such solution for $a = 13$.

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