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On an equation of ring homomorphisms

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Dedicated to Professors Zoltán Daróczy and Imre Kátai on the occasion of their 60th birthdays

Abstract. We deal with a functional equation

$$f(x+y)+f(xy)=f(x)+f(y)+f(x)f(y) \tag{$*$}$$

considered by J. Dhombres (*Relations de dépendance entre les équations fonctionnelles de Cauchy*, Aequationes Mathematicae 35 (1988), 186–212) for functions f mapping a given ring into another one. Neither divisibility hypothesis nor the existence of the unit elements is assumed. Some alterations are also examined to answer the questions posed by Ludwig Reich and Jaime Garcia-Roig during the 32nd International Symposium on Functional Equations (Gargnano, Italy, June 12–19, 1994) in connection with author's results on $(*)$ presented overthere.

1. Introduction

Given a homomorphism between two rings X and Y we deal with a map $f : X \longrightarrow Y$ satisfying the system

$$\begin{cases} f(x+y)=f(x)+f(y) \\ f(xy)=f(x)f(y) \end{cases} \tag{H}$$

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of two Cauchy equations for every $x, y \in X$. Summing up these two equations side by side we derive the functional equation

$$f(x+y) + f(xy) = f(x) + f(y) + f(x)f(y) \tag{1}$$

valid for all $x, y \in X$. How far are the solutions of (1) from a ring homomorphism? Such a question was asked in 1988 by J. Dhombres [1] who proved, among others, the following

Theorem. *Let X and Y be rings with unit elements and let the division by 2 be performable in X. Then each solution $f : X \to Y$ of equation* (1) *such that $f(0) = 0$ yields a ring homomorphism between X and Y, i.e. f yields a solution of the system* (H).

The proof presented by J. Dhombres relies very heavily upon the existence of the units in both of the rings considered and the 2-divisibility of the domain. The crucial part of this proof is to get the oddness of a solution $f : X \to Y$ of equation (1). Except for the assumption that f vanishes at zero (which eliminates nonzero constant solutions $f(x) = c$, $x \in X$, with $c^2 = 0$ in the case where the ring Y admits such elements) the omission of any of the hypotheses mentioned above makes the investigation of (1) difficult indeed. On the other hand even in the very simple case of unitary rings $X = \mathbb{Z}$ (the integers) and $Y = \mathbb{R}$ (the reals) equation (1) admits non-odd (actually even) and hence nonhomomorphic solutions of the form

$$f(x) = \begin{cases} 0 & \text{for } x \in 2\mathbb{Z} \\ -1 & \text{for } x \in 2\mathbb{Z} + 1. \end{cases}$$

Last but not least numerous important rings are not unitary. Note that in such a case the usual embedding of that ring into a ring with a unit does not allow to extend the unknown function onto the new domain with the preservation of the equation.

In what follows we examine equation (1) under possibly mild assumptions upon the underlying rings. In the last section we answer some related questions asked by Ludwig Reich and Jaime Garcia-Roig during the 32nd International Symposium on Functional Equations (Gargnano, Italy, June 12–19, 1994).

2. Preliminary results

We begin with a purely technical

Lemma 1. *Let X and Y be any rings. If Y has no elements of order 2 and $f : X \to Y$ is a solution of equation* (1)*, then putting*

$$n(x) := f(x) - f(-x), \quad p(x) := f(x) + f(-x), \quad x \in X,$$

for every $x, y \in X$ one has:

(i) $p(x)n(y) = n(y)p(x)$;

(ii) $p(x+y) - p(x-y) = n(x)n(y) - 2n(xy)$;

(iii) $n(x+y) + n(x-y) = n(x)p(y) + 2n(x)$;

(iv) $2[n(x+y) - n(x) - n(y)] = p(x)n(y) + p(y)n(x)$;

(v) $p(x+y) + p(x-y) - 2p(x) - 2p(y) = p(x)p(y) - 2p(xy)$;

(vi) $p(xy)n(x) = p(yx)n(x) = 0$;

(vii) $n(x+xy) + n(x-xy) = 2n(x)$.

Proof. Ad (i). Fix arbitrarily $x, y \in X$ and put

$$C_f(x,y) := f(x+y) - f(x) - f(y).$$

Then, by means of equation (1), we have $f(x)f(y) = f(xy) + C_f(x,y)$, whence

$$p(x)n(y) = C_f(x,y) - C_f(x,-y) + C_f(-x,y) - C_f(-x,-y)$$

whereas

$$n(y)p(x) = C_f(y,x) + C_f(y,-x) - C_f(-y,x) - C_f(-y,-x)$$

and it suffices to make use of the symmetry of the function C_f.

Ad (ii). Since $2f(x) = n(x) + p(x)$ for all $x \in X$, by (1) we get

$$2[f(x)f(y) - f(xy)] = 2C_f(x,y) = C_n(x,y) + C_p(x,y)$$

and hence, in view of the oddness of n and the evenness of p, we have also

$$2[f(-x)f(-y) - f(xy)] = -C_n(x,y) + C_p(x,y).$$

Summing up these two equalities side by side leads to

$$2[f(x)f(y) + f(-x)f(-y) - 2f(xy)] = 2C_p(x,y)$$

whence, by replacing here y by $-y$,

$$2[f(x)f(-y) + f(-x)f(y) - 2f(-xy)] = 2C_p(x,-y).$$

Now, subtract the latter two equalities to get

$$2[n(x)n(y) - 2n(xy)] = 2[C_p(x,y) - C_p(x,-y)] = 2[p(x+y) - p(x-y)]$$

whence our assertion follows due to the lack of elements of order 2 in the ring Y.

Ad (iii). Equation (1) says that for all $x, y \in X$ one has

$$\begin{gathered} n(x+y) + p(x+y) + n(xy) + p(xy) \\ = n(x) + p(x) + n(y) + p(y) + [n(x) + p(x)]f(y). \end{gathered}$$

Replace here y by $-y$ and subtract the resulting equation from the latter one to obtain

$$\begin{gathered} n(x+y) - n(x-y) + p(x+y) - p(x-y) + 2n(xy) \\ = 2n(y) + n(x)n(y) + p(x)n(y). \end{gathered}$$

Now, an appeal to (ii) proves that

$$n(x+y) - n(x-y) = 2n(y) + p(x)n(y) \tag{2}$$

whence, by interchanging the roles of x and y, we infer that

$$n(x+y) + n(x-y) = 2n(x) + p(y)n(x), \tag{3}$$

which is nothing else but (iii) because of (i).

Ad (iv). It sufficies to add the equalities (2) and (3) side by side.

Ad (v). Replace y by $-y$ in equation (1) and add the resulting equation to (1) side by side to get

$$f(x+y)+f(x-y)+p(xy)=2f(x)+p(y)+f(x)p(y).$$

Putting here $-x$ in place of x gives

$$f(-x+y)+f(-x-y)+p(xy)=2f(-x)+p(y)+f(-x)p(y).$$

Now (v) results by adding the latter equation to the former.

Ad (vi). On account of (iii) and (i), for every $x,y\in X$, we get

$$n(x+y)+n(x-y)-2n(x)=p(y)n(x)$$

whence, by setting here $y-x$ instead of y, we derive the equation

$$n(y)+n(2x-y)-2n(x)=p(x-y)n(x).$$

Replacing here y by $-y$ and adding the resulting equation to the former leads to

$$n(2x+y)+n(2x-y)-4n(x)=[p(x+y)+p(x-y)]n(x).$$

On the other hand, in view of (iii) and (i), one has

$$n(2x+y)+n(2x-y)-2n(2x)=p(y)n(2x),$$

whence by subtraction

$$2[n(2x)-2n(x)]=[p(x+y)+p(x-y)]n(x)-p(y)n(2x).$$

Now, equation (iv) applied for $y=x$ gives

$$2n(x)p(x)=[p(x+y)+p(x-y)]n(x)-p(y)n(2x)$$

and, consequently, by setting $y=x$ in (i),

$$[p(x+y)+p(x-y)-2p(x)]n(x)=p(y)n(2x)$$

or, equivalently, by (i)

$$(4)\quad \begin{aligned}[p(x+y)+p(x-y)-2p(x)-2p(y)]n(x) &= p(y)[n(2x)-2n(x)] \\ &= [n(2x)-2n(x)]p(y).\end{aligned}$$

Thus, on account of (v) and (iv), we get

$$[p(x)p(y)-2p(xy)]n(x) = n(x)p(x)p(y)$$

and finally, by a double application of (i), $2p(xy)n(x) = 0$ which was to be shown because Y has no elements of order 2.

The equality $p(yx)n(x) = 0$ results from (4) and the fact that due to the symmetry of the expression $p(x+y)+p(x-y)-2p(x)-2p(y)$ we have also (see (v))

$$p(x+y)+p(x-y)-2p(x)-2p(y) = p(y)p(x)-2p(yx)$$

for all $x, y \in X$.

Ad (vii). Put xy in place of y in (iii) and apply (vi) jointly with (i).

This finishes the proof of the lemma.

Corollary 1. *Under the assumptions of Lemma 1 if, moreover,*

(e) *for every $x \in X$ there exists an $e_x \in X$ such that $xe_x = x$,*

then

$$n(2x) = 2n(x) \quad \textit{for all } x \in X.$$

Proof. Put $y = e_x$ in Lemma 1 (vii).

Lemma 2. *Under the assumptions of Lemma 1, if $f(0) = 0$ and condition* (e) *holds true then*

$$(5)\qquad p(4x) = 4p(2x) \quad \textit{for all } x \in X$$

and

$$(6)\qquad p(x)n(y) = p(2x)n(y) \quad \textit{for all } x, y \in X.$$

Proof. Setting $y = x$ in Lemma 1 (ii) and taking into account that the equality $f(0) = 0$ forces p to vanish at zero, we infer that

$$p(2x) = n(x)^2 - 2n(x^2) \quad \text{for all } x \in X. \tag{7}$$

Consequently, for every $x \in X$, in view of Corollary 1 one has

$$p(4x) = n(2x)^2 - 2n(4x^2) = 4n(x)^2 - 8n(x^2) = 4[n(x)^2 - 2n(x^2)] = 4p(2x),$$

i.e. relation (5) has been proved.

To show (6) replace x and y in formula (2) by $2x$ and $2y$, respectively, to get the equality

$$n(2x + 2y) - n(2x - 2y) = 2n(2y) + p(2x)n(2y)$$

valid for all $x, y \in X$. By Corollary 1 this means that

$$2[n(x + y) - n(x - y) - 2n(y)] = 2p(2x)n(y), \quad x, y \in X,$$

whence, using formula (2) once again, we obtain

$$2p(x)n(y) = 2p(2x)n(y), \quad x, y \in X,$$

which finishes the proof since the ring Y contains no elements of order 2.

Corollary 2. *Under the assumptions of Lemma 2 we have*

$$3p(x)n(y) = 0$$

for every x, y from X.

Proof. Fix arbitrarily $x, y \in X$. Equation (6) jointly with (5) gives

$$p(2x)n(y) = p(4x)n(y) = 4p(2x)n(y)$$

whence $3p(2x)n(y) = 0$. It remains to apply (6) once more to finish the proof.

Corollary 3. *Under the assumptions of Lemma 2 the function $3n$ is additive.*

Proof. “Multiply” both sides of equation (iv) in Lemma 1 by 3 and apply Corollary 2.

3. Main results

The assertions of Propositions 1 and 2 below look somewhat heavy. This is caused by the fact that we have tried to avoid additional assumptions upon the target ring. Once we suppose that it does not admit zero divisors or contains no elements of order 3, the results become much more agreeable (see Theorems 1 and 2 below).

Proposition 1. *Let X and Y be two rings such that*

(e) *for every $x \in X$ there exists an $e_x \in X$ with $xe_x = x$.*

If Y has no elements of order 2 and $f : X \to Y$ is a solution of the equation

$$f(x+y) + f(xy) = f(x) + f(y) + f(x)f(y) \tag{1}$$

such that $f(0) = 0$, then either $3f$ is even and $3f(2x) = 0$ for all $x \in X$, or there exists a $c \in Y \setminus \{0\}$ such that

$$\begin{cases} c[f(x+y) - f(x) - f(y)] = 0 \\ c[f(xy) - f(x)f(y)] = 0. \end{cases} \tag{H_c}$$

Proof. We preserve the notation used in Lemma 1. Corollary 3 states that the map $N := 3n$ is additive. In view of Corollary 2 we have also $p(x)N(y) = 0$ for all $x, y \in X$. Let us distinguish two cases:

– N vanishes on the whole of X;

– $N(y) \neq 0$ for some $y \in X$.

Since $6f = N + 3p$ the first case implies that $6f$ is even. Consequently, so is $3f$ because Y has no elements of order 2. Obviously, by (1), we have

$$3f(x+y) + 3f(xy) = 3f(x) + 3f(y) + 3f(x)f(y)$$

for every x, y in X and, due to the evenness of the map $3f$, we get also

$$3f(x-y) + 3f(xy) = 3f(x) + 3f(y) + 3f(x)f(y)$$

for all $x, y \in X$. Therefore

$$3f(x+y) = 3f(x-y) \quad \text{for every } x, y \in X.$$

In particular, by setting here $y = x$, one obtains

$$3f(2x) = 0 \quad \text{for all } x \in X,$$

because f is supposed to vanish at zero.

In the second case, for a fixed $y \in X$ such that $N(y) \neq 0$, we have also $c := 2N(y) = 6n(y) \neq 0$ whence, by means of Corollary 2 and Lemma 1 (i),

$$cf(x) = n(y)N(x) + 3n(y)p(x) = n(y)N(x) \quad \text{for all } x \in X,$$

which proves that cf is additive because, clearly, so is the map $n(y)N$. Consequently, the first equation of system (H_c) is satisfied. To see that the other one is also fulfilled it sufficies to multiply both sides of (1) by c from the left and to apply the additivity of cf. This completes the proof.

Proposition 2. *Under the assumptions of Proposition 1 the map f yields a solution of the system*

$$\begin{cases} 81[f(2x+2y) - f(2x) - f(2y)] = 0 \\ 81[f(2x \cdot 2y) - f(2x)f(2y)] = 0 \end{cases} \tag{8}$$

for every $x, y \in X$.

Proof. Using the notation stated in Lemma 1 put, as previously, $N := 3n$ and define

$$q(x) := p(2x), \quad x \in X.$$

Since $p(0) = 2f(0) = 0$, setting $y = x$ in Lemma 1 (ii) we get (7). Consequently, a simple calculation shows that, because of the additivity of N (see Corollary 3), both N^2 and $N((\cdot)^2)$ are quadratic maps, i.e. both of them are solutions to the functional equation

$$F(x+y) + F(x-y) = 2F(x) + 2F(y) \quad \text{for all } x, y \in X. \tag{9}$$

This implies that the map $Q := 9q$ is quadratic as well, because of (7) and the fact that

$$Q(x) = 9p(2x) = 9[n(x)^2 - 2n(x^2)] = N(x)^2 - 6N(x^2)$$

for all $x \in X$.

On the other hand, on account of Lemma 1 (ii), one has

$$
\begin{aligned}
&Q(x+y)-Q(x-y)=9[p(2x+2y)-p(2x-2y)]\\
&=9[n(2x)n(2y)-2n(4xy)]=N(2x)N(2y)-6N(4xy)\\
&=4[N(x)N(y)-6N(xy)]=36[n(x)n(y)-2n(xy)]
\end{aligned}
$$

for every $x, y \in X$. Thus, since Q yields also a solution to (9), we infer that

$$2Q(x+y)=2Q(x)+2Q(y)+36[n(x)n(y)-2n(xy)], \qquad x, y \in X,$$

whence, in view of the assumption that Y possesses no elements of order 2,

$$Q(x+y)-Q(x)-Q(y)=18[n(x)n(y)-2n(xy)], \qquad x, y \in X.$$

Now, bearing Corollary 2 in mind, we derive the validity of the equation

$$[Q(x+y)-Q(x)-Q(y)]Q(z)=0$$

for all triples (x, y, z) from the set X^3. In particular, $[Q(2x)-2Q(x)]$ $Q(z)=0$ holds true for all $x, z \in X$. Consequently, since every quadratic map $F : X \to Y$ has the property $F(2x)=4F(x), x \in X$, we conclude that $2Q(x)Q(z)=0$ for every $x, z \in X$, and, subsequently,

$$Q(x)Q(z)=0 \quad \text{for every } x, z \in X.$$

An appeal to property (v) from Lemma 1 gives now the relationship

$$
\begin{aligned}
0&=9[Q(x+y)+Q(x-y)-2Q(x)-2Q(y)]\\
&=Q(x)Q(y)-18Q(2xy)=-72Q(xy)
\end{aligned}
$$

for all $x, y \in X$, showing that $81q(xy)=0$, $x, y \in X$. This applied for $y=e_x$ (recall that condition (e) has been assumed) leads to the equality

$$81p(2x)=81q(x)=0$$

valid for every $x \in X$.

Finally, since

$$81 \cdot 2f(2x)=81n(2x)+81p(2x)=27N(2x)=2 \cdot 27N(x), \quad x \in X,$$

yielding

$$81f(2x) = 27N(x), \quad x \in X,$$

we derive the additivity of the map $X \ni x \mapsto 81f(2x) \in Y$, i.e. the first equation of system (8). The other one results now from the first with the aid of equation (1) "multiplied" side by side by the factor 81 and applied for $2x$ and $2y$ instead of x and y, respectively. This completes the proof.

The following two theorems are simple consequences of the propositions that have just been proved.

Theorem 1. *Let X and Y be two rings such that*

(e) *for every $x \in X$ there exists an $e_x \in X$ with $xe_x = x$.*

Assume that Y has no elements of order 2 and does not admit zero divisors. If $f : X \to Y$ is a solution of the equation

$$f(x+y) + f(xy) = f(x) + f(y) + f(x)f(y) \tag{1}$$

such that $f(0) = 0$, then either $3f$ is even and $3f(2x) = 0$ for all $x \in X$, or f yields a homomorphism between X and Y.

Proof. The factor $c \neq 0$ occurring in the assertion of Proposition 1 may be cancelled.

Corollary 4. *Under the assumptions of Theorem 1 if, moreover, Y has no elements of order 3, then either f is even or f yields a homomorphism between X and Y.*

Our next result explains the role of the 2-divisibility hypothesis assumed by J. Dhombres (cf. the Introduction) upon the ring being the domain of the underlying map.

Theorem 2. *Let X and Y be two rings. Assume* (e) *and suppose that Y neither has elements of order 2 nor elements of order 3. If $f : X \to Y$ is a solution of equation* (1) *such that $f(0) = 0$, then $f|_{2X}$ yields a homomorphism between the rings $2X$ and Y.*

Proof. The "factor" $81 = 3^4$ occurring in the assertion of Proposition 2 may be cancelled. In the case where $3f$, and hence f itself, is even, Proposition 2 now says that $f(2x) = 0$ for all $x \in X$; in other words, the restriction $f|_{2X}$ yields a (trivial) homomorphism as well.

Corollary 5. *Under the assumptions of Theorem 2 if, moreover, the division by 2 is always performable (not necessarily uniquely) in the ring X, then f yields a homomorphism between X and Y.*

4. Even solutions

As we have seen (see e.g. Corollary 4) nonhomomorphic solutions (if any) of equation (1) may occur only among even mappings. Therefore, in the present section, we examine the behaviour of solutions of that kind.

Theorem 3. *Let X and Y be any rings. If $f : X \to Y$ is an even solution of the equation*

$$f(x+y)+f(xy)=f(x)+f(y)+f(x)f(y) \tag{1}$$

such that $f(0)=0$, then each element of the ring $2X$ yields a period of f; in particular, $f(2x)=0$ for all $x \in X$.

If, moreover, Y contains no elements of order 2 and

$$x+x^2 \in 2X \qquad \text{for all } x \in X, \tag{q}$$

then:

(a) *$-f$ is multiplicative;*

(b) *both f and $-f$ satisfy the functional equation of Mikusiński*

$$f(x+y)\,[f(x+y)-f(x)-f(y)]=0, \quad x,y \in X;$$

(c) *$f(x^2)=f(x)$ for all $x \in X$;*

(d) *$f(x)+f(x)^2=0$ for all $x \in X$;*

(e) *the set $Z := \{x \in X : f(x)=0\}$ yields a two-sided ideal of the ring X and $2X \subset Z$;*

(f) *$f(x)f(y)=f(y)f(x)$ for all $x,y \in X$.*

Conversely, for any $2X$-periodic and multiplicative solution $g : X \to Y$ of the Mikusiński's equation such that $g(x^2)=g(x)$, $x \in X$, and $g(x)g(y)=g(y)g(x)$ for $x,y \in X$, the function $f := -g$ is a solution to equation (1).

Proof. Replace y by $-y$ in (1) and subtract the resulting equation from (1) to get the equality

$$f(x+y)=f(x-y),$$

valid for all $x, y \in X$. Setting here $x+y$ in place of x gives

$$f(x+2y) = f(x) \qquad \text{for every } x, y \in X,$$

i.e. for each $y \in X$ the element $2y$ turns out to be a period for the map f. In particular, since $f(0)=0$, function f has to vanish on the whole of the ring $2X$, as claimed.

Ad (a). Fix arbitrarily elements x, y and z from X. Equation (1) gives then

$$f(xy+z) + f(xyz) = f(xy) + f(z) + f(xy)f(z)$$

as well as

$$f(x+yz) + f(xyz) = f(x) + f(yz) + f(x)f(yz).$$

Subtracting these two equalities leads now to the relationship

$$\begin{aligned} &f(xy+z) - f(x+yz) \\ &= f(xy) + f(z) + f(xy)f(z) - f(x) - f(yz) - f(x)f(yz). \end{aligned} \tag{10}$$

Similarly, applying the associativity of the addition in the ring X,

$$f(x+y+z) + f((x+y)z) = f(x+y) + f(z) + f(x+y)f(z)$$

and

$$f(x+y+z) + f(x(y+z)) = f(x) + f(y+z) + f(x)f(y+z)$$

whence, by subtraction,

$$\begin{aligned} &f((x+y)z) - f(x(y+z)) \\ &= f(x+y) + f(z) + f(x+y)f(z) - f(x) - f(y+z) - f(x)f(y+z). \end{aligned}$$

Expanding the terms $f(x+y)$ and $f(y+z)$ in the latter equality according to (1) proves that

$$f((x+y)z) - f(x(y+z)) = f(x)f(yz) + f(yz) - f(xy)f(z) - f(xy),$$

which, jointly with (10), implies that

$$f(xz+yz)-f(xy+xz))=f(x+yz)-f(xy+z)+f(z)-f(x).$$

In particular, for $z=y$ one gets

$$f(xy+y^2)-f(2xy)=f(x+y^2)-f(xy+y)+f(y)-f(x),$$

for every $x,y\in X$. Recalling that f vanishes on $2X$ and setting $x+y$ instead of x in the latter equation we infer that

$$f(xy+2y^2)=f(x+y+y^2)-f(xy+y^2+y)+f(y)-f(x+y).$$

Since both y^2+y and $2y^2$ are periods for f we arrive at

$$2f(xy)=f(x)+f(y)-f(x+y). \tag{11}$$

Finally, on account of (1), the right hand side of this equality may also be expressed in the form $f(xy)-f(x)f(y)$, whence

$$f(xy)=-f(x)f(y),$$

which states nothing else but the multiplicativity of the map $-f$.

Ad (b). Equation (11) jointly with the multiplicativity of $-f$ implies that

$$f(x+y)-f(x)-f(y)=2f(x)f(y) \qquad \text{for all } x,y\in X.$$

Fix arbitrarily elements x,y and z from X. The latter equation and the associativity of addition in X give immediately the equality

$$f(x)+f(y+z)-f(x+y)-f(z)=2f(x+y)f(z)-2f(x)f(y+z)$$

and, consequently, by expanding the terms $f(y+z)$ and $f(x+y)$ at the left hand side with the aid of equation (1) and by the multiplicativity of $-f$ we arrive at

$$2f(y)f(z)-2f(x)f(y)=2f(x+y)f(z)-2f(x)f(y+z).$$

Put here $y=x$ and make use of the fact that Y contains no elements of order 2 to get

$$f(x)[f(z)-f(x)]=f(2x)f(z)-f(x)f(x+z)=-f(x)f(x+z),$$

because f vanishes on $2X$. Thus

$$f(x)[f(z) - f(x) + f(x+z)] = 0,$$

whence, by setting here $x - z$ in place of x and bearing in mind that $f(x-z) = f(x+z)$, we get finally

$$f(x+z)\,[f(z) - f(x+z) + f(x)] = 0, \quad x, z \in X,$$

which was to be proved. Clearly, $-f$ satisfies Mikusiński's equation as well.

Ad (c). Results directly from (11) by putting $y = x$, applying the property that $f|_{2X} = 0$ and "dividing" by 2.

Ad (d). Results directly from the multiplicativity of $-f$ and the property (c) just proved.

Ad (e). Obviously, $2X \subset Z$; in particular, the set Z is nonempty. Fix arbitrarily points x, y from Z. Then, on account of (b) and the fact that $f(x+y) = f(x-y)$, one has

$$0 = f(x-y)[f(x-y) - f(x) - f(y)] = f(x-y)^2 = -f(x-y)$$

because of (d). Thus $x - y \in Z$, as required.

The inclusions: $Z \cdot X \subset Z$ and $X \cdot Z \subset Z$ result immediately from the multiplicativity of $-f$.

Ad (f). By (11),

$$\begin{aligned} -2f(x)f(y) = 2f(xy) &= f(x) + f(y) - f(x+y) \\ &= f(y) + f(x) - f(y+x) = 2f(yx) = -2f(y)f(x) \end{aligned}$$

as claimed, because Y contains no elements of order 2.

Take any $2X$-periodic multiplicative map $g : X \to Y$ satisfying the functional equation of Mikusiński and having the properties: $g(x)g(y) = g(y)g(x)$ for all $x, y \in X$ and $g(x^2) = g(x)$ for all $x \in X$. Obviously,

$$g(x+y) = g(x-y) \qquad \text{for all } x, y \in X,$$

and, therefore, $g(x)[g(x) - g(x+y) - g(y)] = 0$ whence

$$g(x) = g(x^2) = g(x)^2 = g(x)g(x+y) + g(x)g(y).$$

Interchanging here the roles of x and y, we get also

$$g(y) = g(y)g(y+x) + g(y)g(x) = g(y)g(x+y) + g(x)g(y)$$

and, consequently,

$$g(x) + g(y) = g(x)g(x+y) + g(y)g(x+y) + 2g(x)g(y).$$

On the other hand, for every $x, y \in X$, one has

$$g(x+y) = g((x+y)^2) = g(x+y)^2 = g(x+y)g(x) + g(x+y)g(y),$$

whence

$$g(x) + g(y) = g(x+y) + 2g(x)g(y) = g(x+y) + g(x)g(y) + g(xy).$$

In other words,

$$-f(x) - f(y) = -f(x+y) + f(x)f(y) - f(xy) \qquad \text{for all } x, y \in X,$$

which is nothing else but (1).

Thus the proof has been finished.

Corollary 6. *Let X be an arbitrary ring having the property* (q) *and let Y be a ring without zero divisors, with a unit element e and with no elements of order* 2. *If $f : X \to Y$ is an even solution of equation* (1), *then either $f = 0$ or there exists a two-sided ideal Z of the ring X, of index* 2, *such that*

$$f(x) = \begin{cases} 0 & \text{for } x \in Z \\ -e & \text{for } x \in X \setminus Z. \end{cases}$$

Proof. Theorem 3 states, among others, that f has to satisfy the functional equation of Mikusiński. Since there are no zero divisors in Y this means that for every $x, y \in X$ we have

$$f(x+y) = f(x) + f(y) \quad \text{provided that } f(x+y) \neq 0.$$

Therefore (see e.g. M. Kuczma [4]) either f is additive or there exists a subgroup $(Z,+)$ of the additive group of the ring X with index 2 and a constant $c \in Y \setminus \{0\}$ such that

$$f(x) = \begin{cases} 0 & \text{for } x \in Z \\ c & \text{for } x \in X \setminus Z. \end{cases} \tag{12}$$

Any even additive function with values in a ring with no elements of order 2 has to vanish identically. Therefore, the only nontrivial solutions are of the form (12). An appeal to Theorem 3 again shows that, actually, Z has to be a two-sided ideal in X and that $-f$ is multiplicative. In particular, for every $x, y \notin Z$ we have $-f(xy) = f(x)f(y) = c^2 \neq 0$, and so $-c = c^2$, i.e. $c(c+e) = 0$ showing that we have to have $c = -e$.

Conversely, it is not hard to check that every function of the form (12) with $c = -e$ yields a solution to (1).

4. Some alterations

During the 32nd International Symposium on Functional Equations (Gargnano, Italy, June 12–19, 1994) where some of author's results on equation (1) were presented, the following two related questions were asked by:

- *Ludwig Reich*: given two unitary rings X and Y denote by e_X and e_Y the unit elements in X and Y, respectively. Assume that $f : X \to Y$ is a solution of equation (1) satisfying the conditions $f(0) = 0$ and $f(e_X) = e_Y$. Under what circumstances has f to be a homomorphism?
- *Jaime Garcia-Roig*: let X and Y be two rings; what are the solutions $f : X \to Y$ of a slightly modified version of equation (1), namely

$$f(x+y+xy) = f(x) + f(y) + f(x)f(y), \quad x, y \in X? \tag{1'}$$

We give the following answers (announced already during the Symposium, see [3]) to these questions.

Theorem 4. *Under the assumptions and denotations occurring in the statement of L. Reich's question the lack of elements of order 2 in the* *ring Y forces a solution $f : X \to Y$ of equation* (1) *to be a homomorphism between X an Y*

Proof. Setting $y = e_X$ in (1) shows that $f(x + e_X) = f(x) + e_Y$ for all $x \in X$. Put $y + e_X$ instead of y in (1) and apply the relationship just derived to the resulting equation to obtain

$$f(xy + x) = f(xy) + f(x) \quad \text{for all } x, y \in X.$$

In particular, by taking here $y = e_X$, we infer that $f(2x) = 2f(x)$ for all $x \in X$. Replace now x and y in (1) by $2x$ and $2y$, respectively, to get

$$2f(x + y) + 4f(xy) = 2f(x) + 2f(y) + 4f(x)f(y),$$

or, equivalently,

$$f(x + y) + 2f(xy) = f(x) + f(y) + 2f(x)f(y),$$

for every $x, y \in X$ (recall that, by assumption, Y contains no elements elements of order 2). To finish the proof it remains to subtract equation (1) from the latter one side by side to obtain the multiplicativity of f; equation (1) jointly with the multiplicativity of the map f implies its additivity.

Theorem 5. *Let X and Y be two unitary rings and let e_X and e_Y denote the unit elements in X and Y, respectively. Assume that $f : X \to Y$ is a solution of equation* (1'). *Then there exists a unique multiplicative map $h : X \to Y$ such that $f(x) = h(x + e_X) - e_Y$ for all $x \in X$.*

Conversely, for every multiplicative map $h : X \to Y$ the function $f : X \to Y$ defined by the formula $f(x) = h(x + e_X) - e_Y$, $x \in X$, yields a solution to equation (1').

Proof. Assume that f is a solution to (1') and put $g(x) := f(x) + e_Y$, $x \in X$. Then, by (1'),

$$g(x + y + xy) = g(x)g(y), \quad x, y \in X,$$

whence, on replacing here x and y by $x - e_X$ and $y - e_X$, respectively, one obtains the equation

$$g(xy - e_X) = g(x - e_X)g(y - e_X), \quad x, y \in X,$$

stating that the function $h : X \to Y$ defined by the formula $h(x) := g(x - e_X) = f(x - e_X) + e_Y$, $x \in X$, is multiplicative. The uniqueness is obvious.

To prove the converse it suffices to put $x + e_X$ and $y + e_X$ in place of x and y, respectively, in the multiplicativity equation

$$h(xy) = h(x)h(y), \quad x, y \in X,$$

and to apply the definition of f.

5. Remarks and examples

We terminate this paper with a brief discussion of the assumptions that have been adopted and with some general remarks.

Remark 1. Equation (11) written in an equivalent form

$$f(x + y) - f(x) - f(y) = -2f(xy)$$

is of the type examined by B.R. Ebanks, P.L. Kannappan and P.K. Sahoo in [2]: the Cauchy difference depends only on the product of arguments. However, the results presented in [2] cannot be applied because fields are the only possible domains for f that are admissible by the authors.

Remark 2. Mikusiński's equation happened to play an essential role while dealing with even solutions of equation (1). Although much has been said about this and even more general equations of that type (see e.g. M. Kuczma [4, Chapter XIII, Section 8] and the references therein) it seems desirable to investigate Mikusiński's equation for mappins with ranges in rings with zero divisors. As far as I know, such a question remains open (no results in this direction till now).

Remark 3. The assumption that the rings considered are unitary yields a qualitative difference in the scale of difficulties while dealing with equation (1). This becomes evident when one compares the proofs presented in Section 4 with those occurring in the previous ones. In the majority of cases we were adopting assumption (e) for the domain ring. Clearly, (e) is trivially satisfied in unitary rings. Assumption (e) is, however, essentially weaker as can easily be seen from the following

Example 1. Let R be a unitary ring admitting no zero divisors. Consider the ring $X(R)$ of all almost everywhere vanishing sequences with values in R, with the usual (i.e. pointwise defined) addition and multiplication. Then $X(R)$ is not unitary whereas condition (e) is satisfied. Indeed, denoting by e the unit in R, we see that a constant sequence $u(n) = e$, $n \in \mathbb{N}$, would be the only possible candidate for a unit in $X(R)$ but, obviously, $u \notin X(R)$. On the other hand, for an arbitarily fixed element $x = (x_n)_{n\in\mathbb{N}}$ from $X(R)$ the sequence e_x defined by the formula

$$e_x(n) = \begin{cases} e & \text{whenever } x_n \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

belongs to $X(R)$ and satisfies the equality $xe_x = x$.

Noteworthy is the fact that, in general, $X(R)$ is not 2-divisible.

Actually, we have got not only one but a family of examples indexed by suitable rings R. This family may be enlarged considerably on replacing sequences by almost everywhere vanishing mappings defined on a set D with a distinguished set-theoretical proper ideal of "small" sets.

Example 2. Since for every $x \in \mathbb{Z}$ the number $x + x^2 = x(1 + x)$ is even, assumption (q) is satisfied in the ring of all integers although it is not 2-divisible. Similarly, preserving the notation of Example 1, the ring $X(\mathbb{Z})$ is not 2-divisible but condition (q) is obviously satisfied.

Plainly, in any ring in which the division by 2 is always performable (not necessarily uniquely), assumption (q) may simply be disregarded.

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