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On the stability of the square-norm equation

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Dedicated to Professors Zoltán Daróczy and Imre Kátai on the occasion of their 60th birthday

Abstract. The main result of this paper is the following: if $\alpha \geq 0$, $\alpha \neq 2$ and a real function f satisfies

$$f(x+2y)-2f(x+y)+f(x)-2f(y)=o(y^{\alpha})$$
$$((x,y)\to(0,0),\ x\leq 0\leq x+2y),$$

then there exists a real function q such that

$$q(x+2y)-2q(x+y)+q(x)-2q(y)=0 \quad (x,y\in\mathbb{R})$$

and

$$f(x)-q(x)=o(|x|^{\alpha}) \quad (x\to 0).$$

1. Introduction

In the present paper we consider the square-norm functional equation

$$q(x+y)+q(x-y)-2q(x)-2q(y)=0 \qquad (x,y\in\mathbb{R})$$

for real functions q and we call its solutions quadratic functions. We write this equation in the form

$$q(x+2y)-2q(x+y)+q(x)-2q(y)=0 \qquad (x,y\in\mathbb{R})$$

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and we prove the following stability theorem: If, for $\alpha \geq 0$ $\alpha \neq 2$, a real function f satisfies

$$\lim_{\substack{(x,y)\to(0,0)\\ x\leq 0\leq x+2y}} \frac{f(x+2y)-2f(x+y)+f(x)-2f(y)}{y^\alpha} = 0,$$

that is

$$f(x+2y)-2f(x+y)+f(x)-2f(y) = o(y^\alpha)$$
$$((x,y)\to(0,0),\ x\leq 0\leq x+2y),$$

then there exists a quadratic function $q:\mathbb{R}\to\mathbb{R}$ for which

$$f(x)-q(x) = o(|x|^\alpha) \qquad (x\to 0).$$

By giving some counterexamples we also prove that the statement is not valid for $\alpha = 2$.

The study of the stability of functional equations in the sense above was inspired by some works of A. Dinghas, A. Simon and P. Volkmann on the Dinghas interval-derivative ([1], [6], [7], [9]). For some results concerning the stability of monomial and polynomial functional equations in this sense we refer to [2], [3], [4], [6] and [7], a similar consideration of the square-norm equation to what we have here is given in [8].

2. Stability

Lemma 1. *Let δ be a positive real number and $f:(-\delta,\delta)\to\mathbb{R}$ be a function. If, for a nonnegative real number $K \geq 0$,*

(1) $$|f(x+2y)-2f(x+y)+f(x)-2f(y)| \leq K$$
$$(x\in(-\delta,0],\ y, x+2y\in[0,\delta))$$

then there exist a $\bar{K}\geq 0$ and a quadratic function $q:\mathbb{R}\to\mathbb{R}$, for which

$$|f(x)-q(x)| \leq \bar{K} \qquad (x\in(-\delta,\delta)).$$

Proof. We prove that (1) implies the existence of a real number $K_1 \geq 0$, such that

(2) $$|f(x+2y)-2f(x+y)+f(x)-2f(y)| \leq K_1$$

for all $x,\ y,\ x+2y \in (-\delta, \delta)$. By Corollario 1 in [8] this property yields our statement.

Let $\delta > 0$, and $f : (-\delta, \delta) \to \mathbb{R}$ satisfy (1). If we write $x = y = 0$ in (1), we obtain $|2f(0)| \leq K$, furthermore, with $x = -y$ we get

$$|f(y) - f(-y)| \leq 2K \qquad (y \in (0, \delta)). \tag{3}$$

Let $\bar{x}$ and $\bar{y}$ be fixed real numbers with the property $\bar{x},\ \bar{y},\ \bar{x}+2\bar{y} \in (-\delta, \delta)$. Then we have one of the following relations:

(A) $\bar{x} \in (-\delta, 0]$, $\bar{y} \in [0, \delta)$, $\bar{x} + 2\bar{y} \in [0, \delta)$;

(B) $\bar{x} \in [0, \delta)$, $\bar{y} \in [0, \delta)$, $\bar{x} + 2\bar{y} \in [0, \delta)$;

(C) $\bar{x} \in (-\delta, 0]$, $\bar{y} \in [0, \delta)$, $\bar{x} + 2\bar{y} \in (-\delta, 0]$;

(D) $\bar{x} \in (-\delta, 0]$, $\bar{y} \in (-\delta, 0]$, $\bar{x} + 2\bar{y} \in (-\delta, 0]$;

(E) $\bar{x} \in [0, \delta)$, $\bar{y} \in (-\delta, 0]$, $\bar{x} + 2\bar{y} \in [0, \delta)$;

(F) $\bar{x} \in [0, \delta)$, $\bar{y} \in (-\delta, 0]$, $\bar{x} + 2\bar{y} \in (-\delta, 0]$.

In case (A) the statement is trivial. In case (B) writing $x = -\bar{x} - 2\bar{y}$ and $y = \bar{x} + \bar{y}$ in (1) we obtain

$$|f(\bar{x}) - 2f(-\bar{y}) + f(-\bar{x} - 2\bar{y}) - 2f(\bar{x} + \bar{y})| \leq K.$$

The addition of this inequality to $|f(\bar{x} + 2\bar{y}) - f(-\bar{x} - 2\bar{y})| \leq 2K$ and $|2f(-\bar{y}) - 2f(\bar{y})| \leq 4K$ gives

$$|f(\bar{x}+2\bar{y}) - 2f(\bar{x}+\bar{y}) + f(\bar{x}) - 2f(\bar{y})| \leq 7K \qquad (\bar{x}, \bar{y}, \bar{x}+\bar{y} \in [0, \delta)). \tag{4}$$

In case (C) from (3) we get

$$\begin{aligned} |f(\bar{x} + 2\bar{y}) - f(-(\bar{x} + 2\bar{y})) - 2f(\bar{x} + \bar{y}) + 2f(-(\bar{x} + \bar{y})) \\ + f(\bar{x}) - f(-\bar{x}) - 2f(\bar{y}) + 2f(\bar{y})| \leq 8K, \end{aligned}$$

which together with (4) yields (2). In case (D) inequalities (3) and (4) similarly imply (2). In case (E) we get (2) by writing $x = \bar{x}+2\bar{y}$ and $y = \bar{y}$ in (4). Finally (1) and (3) give (2) in case (F).

Theorem 1. *Let $\alpha \geq 0$, $\alpha \neq 2$ be a real number. If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies*

$$f(x+2y) - 2f(x+y) + f(x) - 2f(y) = o(y^\alpha) \qquad ((x,y) \to (0,0),\ x \leq 0 \leq x+2y) \tag{5}$$

then there exists a quadratic function $q : \mathbb{R} \to \mathbb{R}$, such that

$$f(x) - q(x) = o(|x|^\alpha) \qquad (x \to 0).$$

Proof. For $\alpha > 2$ the statement was proved in Theorem 4 in [4].

Now let $\alpha \in [0,2)$. By (5) there exist positive real numbers δ and K, such that

$$|f(x+2y) - 2f(x+y) + f(x) - 2f(y)| \leq K \qquad (x \in (-\delta, 0],\ y, x+2y \in [0,\delta)),$$

therefore, by Lemma 1 there exist a $\bar{K} \geq 0$ and a quadratic function $q : \mathbb{R} \to \mathbb{R}$ with the property

$$|f(x) - q(x)| \leq \bar{K} \qquad (x \in (-\delta, \delta)).$$

For the function $\varepsilon : (-\delta,\delta) \to \mathbb{R}$, $\varepsilon(x) = f(x) - q(x)$ Theorem 1 in [4] implies $\varepsilon(0) = 0$ and

$$\varepsilon(2z) - 2^2\varepsilon(z) = o(|z|^\alpha) \qquad (z \to 0). \tag{6}$$

Using these results our proof is similar to some reasoning in the proof of Théorème 2 in [7]. It is easy to see that (6) is equivalent to the following property: there exist a positive real number δ_1 and a continuous, increasing function $h : [0, \delta_1] \to \mathbb{R}$, such that $\lim_{z \searrow 0} h(z) = 0$ and

$$|\varepsilon(2z) - 4\varepsilon(z)| \leq |z|^\alpha h(|z|) \qquad (z \in [-\delta_1, \delta_1]).$$

For

$$\bar{\varepsilon}(z) = \begin{cases} \frac{\varepsilon(z)}{|z|^\alpha}, & \text{if } z \in [-\delta_1, \delta_1],\ z \neq 0, \\ 0, & \text{if } z = 0 \end{cases}$$

we have

$$\Big|\,|z|^\alpha\bar\varepsilon(z)-2^{\alpha-2}|z|^\alpha\bar\varepsilon(2z)\,\Big| \le \frac{1}{4}|z|^\alpha h(|z|) \qquad (z\in[-\delta_1,\delta_1]),$$

that is

$$|\bar\varepsilon(z)-2^{\alpha-2}\bar\varepsilon(2z)|\le \frac{1}{4}h(|z|) \qquad (z\in[-\delta_1,\delta_1]).$$

For the real numbers

$$s_k=\sup\left\{|\bar\varepsilon(z)|\;\Big|\;\frac{\delta_1}{2^k}\le|z|\le\frac{\delta_1}{2^{k-1}}\right\} \qquad (k\in\mathbb{N})$$

we get

$$s_{k+1}\le 2^{\alpha-2}s_k+\frac{1}{4}h\left(\frac{\delta_1}{2^k}\right) \qquad (k\in\mathbb{N}),$$

therefore, $\lim_{k\to\infty}s_k=0$, which implies

$$\varepsilon(z)=o(|z|^\alpha) \qquad (z\to 0).$$

3. Instability

Lemma 2. *Let δ be a positive real number and $f:[-\delta,\delta]\to\mathbb{R}$ be a continuous, odd function, which is two times differentiable on the interval $(0,\delta)$ and $f''(x)\le 0$, $(x\in(0,\delta))$. Then we have*

$$f(x+2y)-2f(x+y)+f(x)\ge 0 \tag{7}$$

for $y\in[0,\frac{\delta}{2}]$, $x\in[-2y,-y]$ and

$$f(x+2y)-2f(x+y)+f(x)\le 0 \tag{8}$$

for $y\in[0,\frac{\delta}{2}]$, $x\in[-y,0]$.

Proof. Let $\delta>0$ be given and $f:[-\delta,\delta]$ satisfy the properties in the Lemma. By the well-known mean value theorem for divided differences (s. a. o. [5], p. 168), for all pairwise different $x_1,x_2,x_3\in[0,\delta]$ there exists a $\xi\in(0,\delta)$, such that

$$\begin{aligned}\frac{f(x_1)}{(x_1-x_2)(x_1-x_3)}+\frac{f(x_2)}{(x_2-x_1)(x_2-x_3)}+\frac{f(x_3)}{(x_3-x_1)(x_3-x_2)}\\ =\frac{f''(\xi)}{2!}\le 0.\end{aligned} \tag{9}$$

For $y = 0$ inequalities (7) and (8) are trivial.

Let $y \in (0, \frac{\delta}{2}]$, $x \in (-2y, -y)$ be fixed and $2x + 3y \neq 0$. If we write $x_1 = x + 2y$, $x_2 = -x - y$ and $x_3 = -x$ in (9), we obtain

$$\text{(I)} \qquad \frac{1}{(2x+3y)(2x+2y)y}\big(yf(x+2y) + (2x+2y)f(-x-y) - (2x+3y)f(-x)\big) \leq 0;$$

putting $x_1 = 0$, $x_2 = x + 2y$ and $x_3 = -x - y$ we get

$$\text{(II)} \qquad \frac{1}{(x+2y)(x+y)(2x+3y)}\big((-2x-3y)f(0) + (x+y)f(x+2y) + (x+2y)f(-x-y)\big) \leq 0;$$

$x_1 = 0$, $x_2 = x + 2y$ and $x_3 = -x$ gives

$$\text{(III)} \qquad \frac{1}{x(x+2y)(2x+2y)}\big((-2x-2y)f(0) + xf(x+2y) + (x+2y)f(-x)\big) \leq 0;$$

$x_1 = 0$, $x_2 = -x - y$ and $x_3 = -x$ implies

$$\text{(IV)} \qquad \frac{1}{x(x+y)y}\big(yf(0) + xf(-x-y) + (-x-y)f(-x)\big) \leq 0.$$

In the case when $2x + 3y < 0$ we have

$$\text{(I')} \qquad yf(x+2y) + (2x+2y)f(-x-y) - (2x+3y)f(-x) \leq 0;$$

$$\text{(II')} \qquad (-2x-3y)f(0) + (x+y)f(x+2y) + (x+2y)f(-x-y) \leq 0;$$

$$\text{(III')} \qquad (-2x-2y)f(0) + xf(x+2y) + (x+2y)f(-x) \leq 0;$$

$$\text{(IV')} \qquad yf(0) + xf(-x-y) + (-x-y)f(-x) \leq 0.$$

The addition of these inequalities yields

$$2(x+y)(f(x+2y) + 2f(-x-y) - f(-x)) \leq 0,$$

thus $x + y < 0$ gives

$$f(x+2y) + 2f(-x-y) - f(-x) \geq 0,$$

and f being odd (7) is implied. If $2x+3y>0$ then instead of (II') we get

(II") $\qquad (2x+3y)f(0)-(x+y)f(x+2y)-(x+2y)f(-x-y)\le 0,$

and the addition of (II"), (III') and (IV') yields (7). Finally, since f is continuous, we have (7) also for $x=-y$, $x=-2y$ and $x=-\frac{3}{2}y$.

For $y\in(0,\frac{\delta}{2}]$, $x\in(-y,0)$ and $2x+y\neq 0$ we prove (8) in a similar way by replacing

(I) $x_1=x+2y$, $x_2=x+y$, $x_3=-x$;

(II) $x_1=0$, $x_2=x+2y$, $x_3=x+y$;

(III) $x_1=0$, $x_2=x+2y$, $x_3=-x$;

(IV) $x_1=0$, $x_2=x+y$, $x_3=-x$

in (9). The continuity of f gives (8) for $x=0$ and $x=-\frac{y}{2}$.

Theorem 2. *Let $\delta\in(0,1)$ and $\beta>0$ be real numbers. For the function $f_\beta:[-\delta,\delta]\to\mathbb{R}$*

$$f_\beta(x)=\begin{cases} x^2\ln(-\ln(|x|^\beta)), & \text{if } x\neq 0\\ 0, & \text{if } x=0.\end{cases} \tag{10}$$

we have

$$f_\beta(x+2y)-2f_\beta(x+y)+f_\beta(x)-2f_\beta(y)=o(y^2) \tag{11}$$
$$((x,y)\to(0,0),\ x\le 0\le x+2y),$$

but there exists no quadratic function $q:\mathbb{R}\to\mathbb{R}$, such that

$$f_\beta(x)-q(x)=o(|x|^2)\qquad (x\to 0).$$

Proof. Let $\delta\in(0,1)$ and $\beta>0$ be given and we define the function $f=f_\beta:[-\delta,\delta]\to\mathbb{R}$ by (10).

We prove that (11) holds for this function. Let

$$F(x,y)=\frac{f(x+2y)-2f(x+y)+f(x)-2f(y)}{y^2}$$
$$\left(y\in\left(0,\frac{\delta}{2}\right],\ x\in[-2y,0]\right).$$

For $y \in (0, \frac{\delta}{2}]$, $x \in (-2y, 0)$, $x \neq -y$ we have

$$\frac{\partial F}{\partial x}(x,y) = \frac{1}{y^2}\biggl(2(x+2y)\ln(-\ln(|x+2y|^\beta)) + \frac{\beta(x+2y)}{\ln(|x+2y|^\beta)} - 4(x+y)\ln(-\ln(|x+y|^\beta)) - 2\frac{\beta(x+y)}{\ln(|x+y|^\beta)} + 2x\ln(-\ln(|x|^\beta)) + \frac{\beta x}{\ln(|x|^\beta)}\biggr),$$

that is

$$\frac{\partial F}{\partial x}(x,y) = \frac{2}{y^2}(g(x+2y) - 2g(x+y) + g(x)),$$

where

$$g(x) = \begin{cases} x\left(\ln(-\ln(|x|^\beta)) + \frac{\beta}{2\ln(|x|^\beta)}\right), & \text{if } x \in [-\delta,\delta], x \neq 0,\\ 0, & \text{if } x = 0.\end{cases}$$

This function is continuous, odd, two times differentiable on $(0,\delta)$ and $g''(x) < 0$, $(x \in (0,\delta))$. Lemma 2 gives

$$\frac{\partial F}{\partial x}(x,y) \geq 0 \qquad (y \in (0, \frac{\delta}{2}],\ x \in (-2y,-y))$$

and

$$\frac{\partial F}{\partial x}(x,y) \leq 0 \qquad (y \in (0, \frac{\delta}{2}],\ x \in (-y,0)),$$

therefore, for a fixed $y \in (0, \frac{\delta}{2}]$, F is increasing in its first variable on the interval $(-2y,-y)$ and decreasing in its first variable on $(-y,0)$. Furthermore, F is continuous in its first variable and

$$\lim_{y\to 0} F(-2y,y) = \lim_{y\to 0} F(-y,y) = \lim_{y\to 0} F(0,y) = 0,$$

which implies (11) for f.

Now we suppose that $q : \mathbb{R} \to \mathbb{R}$ is a quadratic function with the property

$$f(x) - q(x) = o(|x|^2) \qquad (x \to 0).$$

The function f is continuous, therefore, there exists a real number $\varepsilon > 0$, such that q is bounded on the interval $(0,\varepsilon)$, thus it has the form $q(x) =$

cx^2, $(x \in \mathbb{R})$ with a $c \in \mathbb{R}$. However, there does not exist a $c \in \mathbb{R}$ for which

$$\lim_{x \to 0} \big(\ln\big(-\ln(|x|^\beta)\big) - c\big) = 0.$$

Remark. The construction of the counterexamples in Theorem 2 is based on [7], where the function $f : (0,1) \to \mathbb{R},\ f(x) = x\ln(-\ln(x))$ was given.

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