Publ. Math. Debrecen
52 / 3-4 (1998), 429–468

On some subsets of the set of shifted primes

By KARL-HEINZ INDLEKOFER[1] (Paderborn)
and NIKOLAI M. TIMOFEEV[2] (Vladimir)

Dedicated to Professor Zoltán Daróczy and Professor Imre Kátai on their 60th birthday

Abstract. Let p run through the set of primes, and let a be a non-zero integer. For some subsets W, W^* of the set of natural numbers we prove (see Theorem 1) that

$$\frac{1}{\pi(x)}|\{p : p+a \le x, p+a \in W\}| \ll \frac{\log y}{x}|\{n : n \le x, n \in W^*, (n-a, P(y)) = 1\}| + y^{-\frac{1}{48}} + \frac{(\log\log 10x)^3}{\log x},$$

where $|a| < y \le (\log x)^A$, $P(y) = \prod_{p\le y} p$.

As an application of this main result we deduce corresponding estimates in the case (see Theorem 2, 3)

$$W = W^* = \{n : g_i(n) = b_i, i = 1, \dots, m\}$$

and in the case (see Theorem 5, 6)

$$W = \{n : g(n) \in [b, b+h)\}, \quad W^* = \{n : g(n) \in [b(1-\varepsilon),\ (b+h)\cdot(1+\varepsilon))\}$$

where g_i, g are additive or multiplicative functions.

In particular we prove

Mathematics Subject Classification: 11K65, 11N64.
Key words and phrases: additive and multiplicative functions, shifted primes.
[1] This work was partially supported by the DFG (Deutsche Forschungsgemeinschaft).
[2] This work was partially supported by the DFG (Deutsche Forschungsgemeinschaft) and by the Russian Foundation for Fundamental research (Grant 96-01-00502).

Theorem 4. *Let $f_1, \dots, f_k$ be multiplicative functions satisfying the following conditions:*

(i) *there are natural numbers m_i such that $f_i^{m_i} \equiv 1$, $i = 1, \dots, k$,*

(ii) *for any $1 \le i_1 < i_2 < \dots < i_s \le k$, $0 < r_i < m_i$, $i = 1, \dots, k$ and for any Dirichlet-character $\chi_d \pmod d$*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} f_{i_1}^{r_{i_1}} \dots f_{i_s}^{r_{i_s}} \chi_d(n) = 0.$$

Then

$$\limsup_{x\to\infty} \frac{1}{\pi(x)} \left|\{p : p \le x, f_1(p+1) = \xi_1, \dots, f_k(p+1) = \xi_k\}\right| \le \frac{4}{m_1 \dots m_k},$$

where $\xi_1^{m_1} = \dots = \xi_k^{m_k} = 1$.

1. Introduction and main results

Assume that g is a real-valued additive or multiplicative function. There are many results about the behaviour of the means

$$\frac{1}{x} \left|\{n : n \le x, g(n) = b\}\right|, \quad \frac{1}{x} \left|\{n : n \le x, b \le g(n) < b + h\}\right|$$

as $x \to \infty$ where $|A|$ denotes the number of elements in the set A.

Typical results in this connection are the following theorems of Halász and Ruzsa.

Halász's Theorem ([1]). *Let g be a real-valued additive function. Then*

$$\sup_b \frac{1}{x} \left|\{n : n \le x, g(n) = b\}\right| \ll \frac{1}{\sqrt{E(x)}},$$

where

$$E(x) = \sum_{p\le x, g(p)\ne 0} \frac{1}{p}.$$

We write $G(x) \ll F(x)$ if the functions F, G satisfy $|G(x)| \le C|F(x)|$ for some absolute constant C and all values x being considered.

Ruzsa's Theorem ([2]). *There exists a constant C such that for any real-valued additive function g the estimate*

$$\sup_b |\{n : n \leq x, b \leq g(n) < b+1\}| \leq C \frac{1}{\sqrt{D(x)}},$$

holds, where

$$D(x) = \min_\lambda \left(\lambda^2 + \sum_{p \leq x} \frac{1}{p} \min(1, (g(p) - \lambda \log p)^2) \right).$$

These results are general and convenient for studying the behaviour of additive and multiplicative functions. But for some functions and for some b there are more exact estimates.

Suppose that p ranges over the set of primes and a is a non-zero integer. The main purpose of the present paper is to prove an inequality of the kind

$$\frac{1}{\pi(x)} |\{p : p \leq x, g(p+a) \in A\}| \ll \frac{1}{x} |\{n : n \leq x, g(n) \in A^*\}| + R(x),$$

where g is additive or multiplicative, A and A^* are some subsets of $\mathbb{C}$ and $R(x) \to 0$ as $x \to \infty$ and where the right hand side can be estimated, for example, by wellknown methods.

Some results of this kind are already known. For example, in [3] N. M. Timofeev proved that for any real-valued additive function g

$$\sup_b \frac{1}{\pi(x)} |\{p : p \leq x, b \leq g(p+1) < b+1\}| \ll \frac{\log^2(D(x)+2)}{\sqrt{D(x)}}.$$

This result is an analogue of Ruzsa's Theorem. In [4] P.D.T.A. Elliott proved this inequality without the factor $\log^2(D(x)+2)$. In [3], [4] an analogue of Halasz's result has been proved, too (see also [6], [7]). Here we consider much more general situations and get the mentioned results as corollaries.

Let $P(t) = \prod_{p \leq t} p$ and $P(v,u) = \prod_{v < p \leq u} p$. For each natural number n let $p(n)$ and $q(n)$ denote the largest and the smallest prime divisor of n, respectively. If $n = 1$ put $p(1) = q(1) = 1$.

For the description of the above mentioned sets W we need the following definitions.

Definition 1. Let

$$\mathcal{C} \subset \{((U_i, V_i))_{i\in\mathbb{N}},\ U_i, V_i \subset \mathbb{N},\ i = 1, 2, \dots\}$$

denote the class of sequences $((U_i, V_i))_{i\in\mathbb{N}}$ of pairs of subsets of $\mathbb{N}$ such that

(i) $U_i \cap U_j = \emptyset$ if $i \neq j$,

(ii) for every $((U_i, V_i))_{i\in\mathbb{N}}$ of $\mathcal{C}$ there exists $s \in \mathbb{N}$ such that for any $i_1 < i_2 < \dots < i_s$ the condition $V_{i_1} \cap V_{i_2} \cap \dots\dots \cap V_{i_s} = \emptyset$ holds.

Next we define a class $\mathcal{W}$ of subsets of $\mathbb{N}$.

Definition 2. $W \in \mathcal{W}$ if the following holds:
For any $t > 2$ there exists a sequence $((U_i, V_i))_{i\in\mathbb{N}} \in \mathcal{C}$ such that

(i) for s in Definition 1 the condition $s = s(t) \leq c \log t$ holds with some constant c,

(ii) if $n \in U_i$, $m \in V_i$ then $p(n) \leq t$, $q(m) > t$ or $m = 1$,

(iii) for every $n \in W$ with $n = n'n''$, where $p(n') \leq t$ and $q(n'') > t$ or $n'' = 1$, there exists $i \in \mathbb{N}$ such that $n' \in U_i$ and $n'' \in V_i$.

Let $W \in \mathcal{W}$. For $t > 2$ let $((U_i, V_i))_{i\in\mathbb{N}}$ be a corresponding sequence according to the definition of $\mathcal{W}$. Then we define $W(t)$ by

$$W(t) := W_{((U_i,V_i))_{i\in\mathbb{N}}}(t) := \bigcup_{i\in\mathbb{N}} \{n = n'n'',\ n' \in U_i,\ n'' \in V_i\}.$$

Obviously $W \subset W(t)$ and $m'm'' \neq n'n''$ if $(n', n'') \in U_i \times V_i$ and $(m', m'') \in U_j \times V_j$ with $i \neq j$.

In the following we restrict our attention to a subclass $\mathcal{W}_0$ of $\mathcal{W}$.

Definition 3. $W \in \mathcal{W}_0$ in case for any $t > 2$, together with a corresponding $((U_i, V_i))_{i\in\mathbb{N}}$ from Definition 2, the set $W(t) = W_{((U_i,V_i))_{i\in\mathbb{N}}}(t)$ belongs to $\mathcal{W}_0$, too.

We use the notations

$$W(t_1, t) := (W(t_1))_{((U_i,V_i))_{i\in\mathbb{N}}}(t)$$

and $W(t_1, t_2, \dots, t_k) := W(t_1, \dots, t_{k-1})(t_k)$.

Remark 1. Let $g : \mathbb{N} \to \mathbb{C}$ be a complex-valued additive or multiplicative function. We put

$$W := \{n : g(n) = b\}$$

(if g is multiplicative we assume that $b \neq 0$). For $t > 2$ we choose

$$\mathbb{N}_1 : = \{n \in \mathbb{N} : p(n) \leq t\}$$
$$\mathbb{N}_2 : = \{n \in \mathbb{N} : q(n) > t\} \cup \{1\}.$$

Now, let $\{b_1, b_2, \dots\}$ be a set of complex numbers which contains all values $g(n)$ $(n \in \mathbb{N})$ of g, such that $b_i \neq b_j$ if $i \neq j$ (and all $b_i \neq 0$ if g is multiplicative). Then we define, for each $i = 1, 2, \dots,$

$$U_i := \{n \in \mathbb{N}_1 : g(n) = b_i\}$$
$$V_i := \{n \in \mathbb{N}_2 : g(n) = b * b_i\}$$

where $b * b_i = b - b_i$ if g is additive and $b * b_i = b/b_i$ if g is multiplicative. Then obviously $s = s(t) = 2$, $W(t) = W$ and thus $W \in \mathcal{W}_0$.

For sets of this type we prove the following theorem.

Theorem 1. *Let a be a non-zero integer such that $|a| < y \leq \log^A x$ with $A = 6100$. Then, if $W \in \mathcal{W}_0$*

$$\frac{1}{\pi(x)} \left|\{p : p \leq x, p + a \in W\}\right| \leq 4 \cdot \prod_{p \leq y} \left(1 - \frac{1}{p}\right)^{-1} \times M(x)$$
$$\times \frac{1}{x} \left|\{n : n \leq x, n \in W(t_0, t_1, \dots, t_k), (n - a, P(y)) = 1\}\right|$$
$$+ O\left(y^{-1/48} + (\log\log 10x)^3 (\log x)^{-1}\right),$$

where $t_0 = \exp\big((2\log\log 10x)^{-1} \cdot (\log x)\big)$,
$\log t_i = \min\left(u_i^{\lambda} \log^2 u_i^4, \frac{\log x}{4A \log\log x}\right)$, $M(x) = (1 + O(\log^{-\frac{1}{15}} x))$, $\lambda = \frac{1}{48}$ and $u_i = y^{2^{k-i}}$, $i = 1, \dots, k$ with $k = \left[\log_2 t \frac{\log(\log^A x)}{\log y}\right] + 1$.

Using Remark 1 we deduce from Theorem 1

Theorem 2. *For any additive or multiplicative function g we have*

$$\frac{1}{\pi(x)}\,|\{p : p\le x, g(p+a)=b\}|$$
$$\le 4\prod_{p\le y}\left(1-\frac{1}{p}\right)^{-1}\times M(x)\times\frac{1}{x}\,|\{n : n\le x,\ g(n)=b,\ ((n-a,P(y))=1\}|$$
$$+O\left(y^{-1/48}+\frac{(\log\log 10x)^3}{\log x}\right),$$

where $y>|a|$, $M(x)=1+O(\log^{-\frac{1}{15}}x)$ and b is a complex number and $b\ne 0$ if g is multiplicative.

In the same way we prove

Theorem 3. *Let $g_1,\dots,g_m$ be complex-valued additive or multiplicative functions and let $b_1,\dots,b_m$ be complex numbers where $b_i\ne 0$ if g_i is multiplicative. Then we have*

$$\frac{1}{\pi(x)}|\{p : p\le x,\ g_i(p+a)=b_i,\ i=1,\dots,m\}|\le 4\prod_{p\le y}\left(1-\frac{1}{p}\right)^{-1}$$
$$\times\frac{1}{x}\,|\{n : n\le x,\ g_i(n)=b_i,\ i=1,\dots,m,((n-a,P(y))=1\}|$$
$$+O\left(y^{-1/48}+(\log\log 10x)^3(\log x)^{-1}\right),$$

where $y>|a|$ and $M(x)=1+O(\log^{-\frac{1}{15}}x)$.

Remark 2. Observe that we obtain an estimate for any $\{b_1,\dots,b_m\}\in\mathbb{C}^m$ and not only for the supremum over $\{b_1,\dots,b_m\}$ as it was proved in [3], [4] (for $m=1$).

Put

$$y:=\min\left(\left(\frac{1}{x}|\{n : n\le x, g_i(n)=b_i, i=1,\dots,m\}|+\frac{1}{x}\right)^{-48}+2|a|,\log^A x\right).$$

Using the inequality

$$\prod_{p\le y}\left(1-\frac{1}{p}\right)^{-1}\le c_1\log y,$$

we obtain from Theorem 3 the following

Corollary 1. *Let $g_{,}\dots,b_m$ be complex-valued additive or multiplicative functions and $b_1,\dots,b_m\in\mathbb{C}$. Put*

$$E(x,b_1,\dots,b_m)=\frac{1}{x}\left|\{n\le x,\ g_i(n)=b_i,\ i=1,\dots,m\}\right|+(\log x)^{-1}.$$

Then

$$\frac{1}{\pi(x)}\left|\{p:p\le x,\ g_i(p+a)=b_i,\ i=1,\dots,m\}\right|$$
$$\ll\log\left(2E^{-1}(x,b_1,\dots,b_m)\right)\cdot E(x,b_1,\dots,b_m)+\frac{(\log\log 10x)^3}{\log x},$$

where $b_i\in\mathbb{C}$, and $b_i\ne 0$ if g_i is multiplicative.

2. Application to multiplicative functions I

Theorem 4. *Let $f_1,\dots,f_k$ be multiplicative functions satisfying the following conditions:*

(i) *there are natural numbers m_i such that $f_i^{m_i}\equiv 1$, $i=1,\dots,k$,*

(ii) *for any $1\le i_1<i_2<\dots<i_s\le k$, $0<r_i<m_i$, $i=1,\dots,k$ and for any Dirichlet-character $\chi_d \pmod d$*

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\le x}f_{i_1}^{r_{i_1}}\dots f_{i_s}^{r_{i_s}}\chi_d(n)=0.$$

Then

$$\limsup_{x\to\infty}\frac{1}{\pi(x)}\left|\{p:p\le x,f_1(p+1)=\xi_1,\dots,f_k(p+1)=\xi_k\}\right|$$
$$\le\frac{4}{m_1\dots m_k},$$

where $\xi_1,\dots,\xi_k\in\mathbb{C}$ with $\xi_1^{m_1}=\dots=\xi_k^{m_k}=1$.

Proof. An application of Theorem 3 gives

$$\frac{1}{\pi(x)}\left|\{p : p\le x, f_i(p+1)=\xi_i,\ i=1,\dots,k\}\right| \le 4\frac{1}{x}\prod_{p\le y}\left(1-\frac{1}{p}\right)^{-1}$$

$$\times\left|\{n : n\le x, f_i(n)=\xi_i, i=1,\dots,k,((n-a,P(y))=1\}\right|$$

$$+O\left(\frac{1}{y^{1/48}}\right) \tag{1}$$

where $1<y\le\log x$. In terms of the Möbius function we have

$$\frac{1}{x}\left|\{n : n\le x, f_i(n)=\xi_i, i=1,\dots,k,(n-1,P(y))=1\}\right| \tag{2}$$

$$=\sum_{\substack{d\le Q\\ d|P(y)}}\mu(d)\frac{1}{x}\left|\{n : n\le x, f_i(n)=\xi_i, i=1,\dots,k, n\equiv 1 \pmod d\}\right|$$

$$+O\Bigg(\sum_{\substack{d|P(y)\\ d>Q}}\frac{1}{d}\Bigg)$$

Next, we make use of the following result.

Lemma 1. *For any y, any $u>e^3$, and $2\le v<u$ we have*

$$\sum_{\substack{p|n\Rightarrow v<p\le u\\ n>y}}\frac{1}{n}\ll\left(\frac{\log u}{\log v}\right)^4\exp\left(-\frac{\log y}{\log u}\right).$$

Proof of Lemma 1. Applying Rankin's method gives

$$\sum_{\substack{p|n\Rightarrow v<p\le u\\ n>y}}\frac{1}{n}\le y^{-\frac{1}{\log u}}\cdot\sum_{p|n\Rightarrow v<p\le u}n^{-1+\frac{1}{\log u}}$$

$$\ll\exp\left(-\frac{\log y}{\log u}+\sum_{v<p\le u}\frac{1}{p}\exp\left(\frac{\log p}{\log u}\right)\right)\ll\exp\left(-\frac{\log y}{\log u}+e\log\frac{\log u}{\log v}\right)$$

which proves Lemma 1.

Returning to (2) we see that the second sum is

$$\ll \log^4 y \exp\left(-\frac{\log Q}{\log y}\right).$$

Let $Q = e^y$. Then the representation

$$\frac{1}{x}\left|\{n : n \le x, f_i(n) = \xi_i, i = 1, \dots, k, n \equiv 1 \pmod{d}\}\right|$$
$$= \frac{1}{m_1 \dots m_k}\frac{1}{x}\sum_{n\le x}\prod_{i=1}^{k}\left(1 + (\xi_i^{-1} f_i(n)) + \ldots + (\xi_i^{-1} f_i(n))^{m_i - 1}\right)$$
$$\times \frac{1}{\varphi(d)}\sum_{\chi_d}\chi_d(n)$$

holds and condition (ii) shows that this sum equals

$$\frac{1}{x\varphi(d)m_1 \dots m_k}\sum_{\substack{n\le x\\(n,d)=1}} 1 + o(1) = \frac{1}{dm_1\dots m_k} + o(1).$$

Thus for any fixed $y > 1$ we have

$$\frac{1}{x}\left|\{n : n \le x, f_i(n) = \xi_i, i = 1, \dots, k, (n-1, P(y)) = 1\}\right|$$
$$= \frac{1}{m_1\dots m_k}\prod_{p\le y}\left(1 - \frac{1}{p}\right) + o(1) + O\left(\frac{1}{y^{1/48}}\right).$$

By (1) we obtain Theorem 4.

Remark 3. Let us suppose that the functions $f_1, \dots, f_k$ satisfy condition (i) of Theorem 4 for any $1 \le i_1 < \ldots < i_s \le k$, $0 < r_i < m_i$ and that for any Dirichlet-character $\chi_d \pmod{d}$ such that $\chi_d^r(n) \equiv \chi_0(n)$ with $r = m_1 \dots m_k$, the series

$$\sum_p \left(1 - \operatorname{Re} f_{i_1}^{r_{i_1}}(p) \dots f_{i_s}^{r_{i_s}}(p)\chi_d(p)\right)\frac{1}{p}$$

diverges. Then, using the theorem of G. Halász [5] we conclude that for any $\chi_d \pmod d$

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\le x}\chi_d(n)f_{i_1}^{r_{i_1}}(n)\dots f_{i_s}^{r_{i_s}}(n)=0.$$

Hence the function $f_1,\dots,f_k$ satisfy condition (ii) of Theorem 4, too.

We need the following two results.

Lemma 2 (Theorem 3 [6]). *Assume that a is a non-zero integer and f is a multiplicative function with $|f(n)|\le 1$. Further, assume that there exists a primitive Dirichlet-character $\chi_d \pmod d$ and a real number t_0 such that*

$$\sum_{p\le x}\left|1-\chi_d(p)f(p)p^{-it_0}\right|\frac{\log p}{p}\le\varepsilon(x)\log x,$$

where $\varepsilon(x)$ decreases and tends to zero but $\varepsilon(x)\sqrt{\log x}$ tends to infinity as $x\to\infty$. Then

$$\frac{1}{\pi(x)}\sum_{p\le x}f(p+a)=\frac{\mu(d)}{\varphi(d)}\bar\chi_d(a)\frac{x^{it_0}}{1+it_0}\prod_{\substack{p|a\\ p\ne 2}}\left(1+\sum_{r=1}^{\infty}\frac{\chi_d(p^r)f(p^r)}{p^{r(1+it_0)}}\right)$$

$$\times\prod_{\substack{p\le x\\ p\nmid ad}}\left(1+\sum_{r=1}^{\infty}\frac{1}{\varphi(p^r)}\left(\chi_d(p^r)f(p^r)p^{-irt_0}-\chi_d(p^{r-1})f(p^{r-1})p^{-i(r-1)t_0}\right)\right)$$

$$+O\left(\varepsilon^{\beta}(y)\right),$$

where $\beta=\beta(d)>0$, $y=x^{\sqrt{\varepsilon(x)}}$, and the constant implied in $O(\dots)$ depends only on a.

Lemma 3. *Let f be a multiplicative function with $f^m(n)\equiv 1$, and, for any $1\le j\le m-1$, let the series*

$$\sum_p\left(1-\operatorname{Re}f^j(p)\right)\frac{1}{p}\tag{3}$$

diverge. If there exists a primitive Dirichlet-character $\chi_\delta \pmod\delta$ such that the series

$$\sum_p\left(1-\operatorname{Re}f(p)\chi_\delta(p)\right)\frac{1}{p}\tag{4}$$

converges, then

$$\limsup_{x\to\infty} \frac{1}{\pi(x)} \left|\{p : p \le x, f(p+1) = \xi\}\right| \le \frac{m+1}{2m}.$$

Proof. For any z with $|z| = 1$ it follows that

$$\begin{aligned} 1 - \operatorname{Re} z^n &= \operatorname{Re}(1-z)(1+z+\ldots+z^{n-1}) \\ &= n \operatorname{Re}(1-z) + \operatorname{Re}(1-z)\big((z-1)+\ldots+(z^{n-1}-1)\big) \\ &\le n \operatorname{Re}(1-z) + |z-1|^2 \frac{n^2}{2} \le 2n^2(1-\operatorname{Re} z). \end{aligned}$$

Since the series (4) converges we conclude that the series

$$\sum_p \big(1 - \operatorname{Re}(f(p)\chi_\delta(p))^j\big) \frac{1}{p}$$

converges, too. Now, the series (3) diverge. Hence we obtain that for $1 \le j \le m-1$, χ_δ^j is not principal. Let $\chi_{\delta(j)}$ be a primitive character which generates $(\chi_\delta)^j$. We have

$$\sum_{p\le x} \left|1 - (f(p))^j \chi_{\delta_{(j)}}(p)\right| \frac{\log p}{p}$$

$$\le \left(4\sum_{p\le y}\frac{\log p}{p} + 2\log x \sum_{p>y}\big(1-\operatorname{Re}(f(p)\chi_\delta)^j\big)\frac{1}{p}\right)^{\frac12}\left(\sum_{p\le x}\frac{\log p}{p}\right)^{\frac12} \longrightarrow 0$$

as $x \to \infty$ if $y = y(x) \to \infty$ as $x \to \infty$ such that $\log y(x)/\log x \to 0$. We see that f^j satisfies the conditions of Lemma 2 and therefore

$$\frac{1}{\pi(x)}\left|\sum_{p\le x} f^j(p+1)\right| \le \frac{\mu^2(\delta(j))}{\varphi(\delta(j))}\prod_{p\le x}\left|1-\frac{1}{p-1}+\sum_{r=1}^{\infty}\frac{f(p^r)}{p^r}\right| + o(1) \le \frac{\mu^2(\delta_{(j)})}{\varphi(\delta_{(j)})} + o(1)$$

holds for any $1 \le j \le m-1$. Hence

$$\begin{aligned} \frac{1}{\pi(x)}\left|\{p : p \le x, f(p+1) = \xi\}\right| &= \frac{1}{m\pi(x)}\sum_{j=0}^{m-1}\xi^{-j}\sum_{p\le x} f^j(p+1) \\ &\le \frac{1}{m} + \frac{1}{m}\sum_{j=1}^{m-1}\frac{\mu^2(\delta(j))}{\varphi(\delta(j))} + o(1). \end{aligned}$$

Since $\delta(j) \geq 3$ this completes the proof of Lemma 3.

Corollary 2. *Suppose that $f_1, f_2, \dots, f_k$ are multiplicative functions with $f_i^{m_i}(n) \equiv 1$ and that, for any $1 \leq i_1 < \dots < i_s \leq k$, $1 \leq r_i \leq m_i - 1$, $i = 1, \dots, k$ the series*

$$\sum_p \left(1 - \operatorname{Re} f_{i_1}^{r_{i_1}}(p) \dots f_{i_s}^{r_{i_s}}(p)\right) \frac{1}{p} \tag{5}$$

diverges. Then

$$\begin{aligned} &\Delta(\xi_1, \dots, \xi_k) \\ &:= \limsup_{x\to\infty} \frac{1}{\pi(x)} \left|\{p : p \leq x, f_1(p+1) = \xi_1, \dots, f_k(p+1) = \xi_k\}\right| \\ &\leq \max\left(\frac{4}{m_1 \dots m_k}, \frac{3}{4}\right). \end{aligned}$$

Proof. Suppose that for any χ_δ and for any $1 \leq i_1 < \dots < i_s \leq k$, $1 \leq r_i \leq m_i - 1$, $i = 1, \dots, k$ the series

$$\sum_p (1 - \operatorname{Re} f(p)\chi_\delta(p)) \frac{1}{p}, \tag{6}$$

diverges where $f(p) = f_{i_1}^{r_{i_1}}(p) \dots f_{i_s}^{r_{i_s}}(p)$. We show that for any $t \neq 0$ the series

$$\sum_p \left(1 - \operatorname{Re} f(p)\chi_\delta(p) \cdot p^{-it}\right) \frac{1}{p} \tag{7}$$

diverges, too. For, if this series converges then the series with $1 - \operatorname{Re} f(p)\chi_\delta(p)p^{-it}$ replaced by $1 - \operatorname{Re}(f(p)\chi_\delta(p)p^{-it})^m$ converges, too. Suppose $m = m_1 \dots m_k$. Then, we obtain that the series

$$\sum_p \left(1 - \operatorname{Re} \chi_\delta^m(p)p^{-imt}\right) \frac{1}{p}$$

converges for $t \neq 0$. But this series diverges, and the contradiction shows that the series (7) diverges, too.

Using HALÁSZ's theorem [5] we conclude that the conditions of Theorem 4 are satisfied. Therefore we have $\Delta \le 4/m_1 \dots m_k$.

Now suppose that there exist $1 \le i_1 < \dots < i_s \le k$, $1 \le r_i \le m_i - 1$, $i = 1, \dots, k$ and a primitive character χ_δ such that the series (6) converges. Put $f = f_{i_1}^{r_{i_1}} \dots f_{i_s}^{r_{i_s}}$ and let m be the least common multiple of the numbers $m_{i_1} \cdot (m_{i_1}, r_{i_1})^{-1}, \dots, m_{i_s} \cdot (m_{i_s}, r_{i_s})^{-1}$. For any $1 \le l < m$ we have

$$f^l(n) = f_{i_1}^{lr_{i_1}}(n) \dots f_{i_s}^{lr_{i_s}}(n) = f_{i_1}^{j_1}(n) \dots f_{i_s}^{j_s}(n)$$

where $0 \le j_1 < m_{i_1}, \dots, 0 \le j_s < m_{i_s}$. If $j_1 = j_2 = \dots = j_s = 0$ then $m|l$ where $1 \le l < m$. Hence $j_i \ne 0$ and it follows that the hypothesis of Lemma 3 for f is satisfied. Using Lemma 3 we get

$$\begin{aligned}\Delta(\xi_1, \dots, \xi_k) &\le \limsup_{x\to\infty} \frac{1}{\pi(x)} \left|\{p : p \le x, f(p+1) = \xi_{i_1}^{r_{i_1}} \dots \xi_{i_s}^{r_{i_s}}\}\right| \\ &\le \frac{m+1}{2m} \le \frac{3}{4}.\end{aligned}$$

Thus Corollary 2 is proved.

In particular we have

Corollary 3. *Assume that* $f_1^3(n) \equiv 1$, $f_2^m(n) \equiv 1$, $m \ge 2$. *If for any* $1 \le i \le m-1$, $0 \le j \le 2$ *the series*

$$\sum_p (1 - \operatorname{Re} f_1(p)) \frac{1}{p}, \quad \sum_p \left(1 - \operatorname{Re} f_1^j(p) f_2^i(p)\right) \frac{1}{p}$$

diverges then

$$\limsup_{x\to\infty} \frac{1}{\pi(x)} \max_{\xi,\eta} |\{p : p \le x, f_1(p+1) = \xi, f_2(p+1) = \eta\}| \le \frac{3}{4}.$$

Remark 4. It is not difficult to show that under the conditions of Corollary 2 the inequality

$$\limsup_{x\to\infty} \frac{1}{\pi(x)} \max_{\xi_1,\dots,\xi_k} \Delta(\xi_1, \dots, \xi_k) \le \left(\frac{4}{m_1 m_2 \dots m_k}, \frac{3}{4}\right).$$

holds.

Using Corollary 3 we prove

Corollary 4. *Suppose that the functions f_1, f_2 satisfy the conditions of Corollary 3 and*

$$\liminf_{x\to\infty} \frac{1}{\pi(x)}\left|\sum_{p\le x} f_1(p+1)\right| \ge \Theta > \frac{\sqrt{7}}{4}. \tag{8}$$

Then

$$\limsup_{x\to\infty} \frac{1}{\pi(x)}\left|\sum_{p\le x} f_2(p+1)\right| \le \sigma$$

where $\sigma = 1 - \left(\frac{1}{2}\sqrt{\frac{4}{3}\Theta^2 - \frac{1}{3}} - \frac{1}{4}\right)(1-\cos\frac{\pi}{m})$.

Proof. Let ξ be a complex third root of unity and

$$a_1(x,i) = \frac{1}{\pi(x)}\left|\{p : p\le x, f_1(p+1) = \xi^i\}\right|.$$

Then we have

$$\frac{1}{\pi(x)}\sum_{p\le x} f_1(p+1) = \sum_{i=0}^{2} \xi^i a_1(x,i).$$

Now suppose that $\alpha = u_0 + u_1\xi + u_2\xi^2$, where $u_i \in \mathbb{R}$. Then

$$|\alpha|^2 = u_0^2 + u_1^2 + u_2^2 - u_0u_1 - u_0u_2 - u_1u_2.$$

It is easy to prove that the maximum of $|\alpha|^2$ taken over all (u_0, u_1, u_2) such that $0 \le u_0 \le a$, $0 \le u_1 \le a$, $0 \le u_2 \le a$, $a > \frac{1}{2}$, $u_0 + u_1 + u_2 = 1$ equals $1 + 3a(a-1)$ and is achieved at the points: $(a, 1-a, 0)$, $(1-a, a, 0)$, $(a, 0, 1-a)$, $(0, a, 1-a)$, $(1-a, 0, a)$, $(0, 1-a, a)$. Let c be a solution of the equation $1 + 3c(c-1) = \Theta^2$, $c = \frac{1}{2}(1 + \sqrt{\frac{4}{3}\Theta^2 - \frac{1}{3}})$. From (8) it follows that

$$\liminf_{x\to\infty} \max_{i=0,1,2} a_1(x,i) \ge c. \tag{9}$$

Now suppose that

$$\frac{1}{\pi(x)}\sum_{p\le x} f_2(p+1) = \sum_{i=0}^{m-1} a_2(x,i)\eta^i,$$

where

$$a_2(x,i)=\frac{1}{\pi(x)}\left|\{p:p\le x, f_2(p+1)=\eta^i\}\right|.$$

Using Corollary 3 and (9) we obtain

$$(10)\qquad \begin{aligned} a_2(x,i)&=\sum_{j=0}^{2}\frac{1}{\pi(x)}\left|\{p:p\le x, f_1(p+1)=\xi^j, f_2(p+1)=\eta^i\}\right| \\ &\le\frac{3}{4}+(1-c)+o(1)=b+o(1), \end{aligned}$$

where $b=\frac{5}{4}-\frac{1}{2}\sqrt{\frac{4}{3}\Theta^2-\frac{1}{3}}$. If $\Theta>\frac{\sqrt{7}}{4}$ then $b<1$. Let w be a complex number of modulus 1 such that

$$w\cdot\sum_{i=0}^{m-1}a_2(x,i)\eta^i\ge 0.$$

Then

$$1-\frac{1}{\pi(x)}\left|\sum_{p\le x}f_2(p+1)\right|\ge\sum_{i=0}^{m-1}a_2(x,i)(1-\operatorname{Re}w\eta^i).$$

Since the inequality $1-\operatorname{Re}w\eta^i<1-\cos\frac{\pi}{m}$ holds for only one i among the values considered, the right-hand side is greater than or equal to

$$\sum_{i=0}^{m-1}a_2(x,i)\Big(1-\cos\frac{\pi}{m}\Big)-\max_i a_2(x,i)\Big(1-\cos\frac{\pi}{m}\Big)\ge(1-b)\Big(1-\cos\frac{\pi}{m}\Big).$$

Hence

$$\limsup_{x\to\infty}\frac{1}{\pi(x)}\left|\sum_{p\le x}f_2(p+1)\right|\le 1-(1-b)\left(1-\cos\frac{\pi}{m}\right),$$

and this proves Corollary 4.

From Corollary 2 we deduce

Corollary 5. *Let* $f_1^2(n) \equiv 1$, $f_2^2(n) \equiv 1$ *and* $f_3^m(n) \equiv 1$. *If for* $0 \le i \le 1$, $0 \le j \le 1$, $0 \le l \le m-1$, $i+j+l \ne 0$ *the series*

$$\sum_p \left(1 - f_1^i(p) f_2^j(p) \operatorname{Re} f_3^l(p)\right) \frac{1}{p}$$

diverges then

$$\limsup_{x\to\infty} \frac{1}{\pi(x)} \left|\{p : p \le x, f_i(p+1) = \xi_i, i = 1,2,3\}\right| \le \frac{3}{4}.$$

In the same way as before we prove

Corollary 6. *Suppose that the functions* f_1, f_2, f_3 *satisfy the conditions of Corollary 5. Further assume that for* $i = 1,2$

$$\liminf_{x\to\infty} \frac{1}{\pi(x)} \left| \sum_{p\le x} f_i(p+1) \right| \ge \Theta,$$

where $\Theta > \frac{3}{4}$. *Then*

$$\limsup_{x\to\infty} \frac{1}{\pi(x)} \left| \sum_{p\le x} f_3(p+1) \right| \le 1 - \left(\Theta - \frac{3}{4}\right)\left(1 - \cos\frac{\pi}{m}\right). \tag{11}$$

Proof. Assume, for example, that for $i = 1,2$

$$\liminf_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\le x} f_i(p+1) \ge \Theta.$$

Since $f_1 \cdot f_2 = (1-f_1)(1-f_2) + f_1 + f_2 - 1$ we get

$$\liminf_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\le x} f_1(p+1) f_2(p+1) \ge 2\Theta - 1.$$

Put

$$a_1(x,i,j) := \frac{1}{\pi(x)} \left|\{p : p \le x, f_1(p+1) = (-1)^i, f_2(p+1) = (-1)^j\}\right|.$$

We have

$$\liminf_{x\to\infty} \max_{i,j=0,1} a_1(x,i,j) \ge \liminf_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\le x} \frac{(1+f_1(p+1))(1+f_2(p+1))}{4}$$
$$\ge \frac{1}{4}(1+2\Theta+2\Theta-1)=\Theta.$$

In the same way as before (see (10)) we obtain

$$a_3(x,i) \le \frac{3}{4} + (1-\Theta) + o(1),$$

and this leads to the inequality (11).

Remark 5. Let f be a multiplicative function such that $f^m(n) \equiv 1$, and let the series

$$\sum_p \frac{1-\operatorname{Re} f(p)\chi_d(p)}{p}$$

diverge for any Dirichlet-character $\chi_d \pmod d$. We conjecture that

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\le x} f(p+1) = 0.$$

For example, it has been conjectured that

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\le x} \mu(p+1) = 0$$

where μ is the Möbius function. At the present time it has even not been proved that

$$\limsup_{x\to\infty} \frac{1}{\pi(x)} \left| \sum_{p\le x} (-1)^{\Omega(p+1)} \right| < 1$$

where $\Omega(n)$ is the number of all prime divisors of n.

The best results in this direction has been proved by P.D.T.A. Elliott (see [7]). He can prove the nontrivial estimate in the case $f^4(n) \equiv 1$.

Remark 6. Let $\mathcal{P}_k$ be the set of integers $n \ge 2$ the number of distinct prime factors of which is $\le k$, i.e. $\mathcal{P}_1 = \mathcal{P}$, and let $\mathcal{P}_{k,+1} := \mathcal{P}_k + 1$. Let furthermore $\mathcal{M}_0$ be the class of those complex-valued (completely) multiplicative functions which are nowhere zero.

In [13] Kátai conjectured that $\lambda(p+1)$ takes on both the values 1 and -1 infinitely often, if p runs over $\mathcal{P}$. Here λ is the Liouville-function. If the equation $p-2q=1$ has infinitely many solutions in primes, then the same is true for $p+1=2(q+1)$. Since $\lambda(p+1)=\lambda(2)\lambda(q+1)=-\lambda(q+1)$, the conjecture hence would follow.

By using this simple observation and Chen's method (see [9]) one can prove the following assertion: For every $a\in\mathbb{N}$ there exists an infinite sequence of pairs of integers $P_2^{(\nu)}+1, Q_2^{(\nu)}+1\in\mathcal{P}_{2,+1}$ ($\nu=1,2,\dots$), such that

$$P_2^{(\nu)}+1=a(Q_2^{(\nu)}+1)$$

holds true.

This implies the following assertion, evidently.

If $f\in\mathcal{M}_0$, then either $f(a)=1$ identically, or f takes on at least two distinct values on the set $\mathcal{P}_{2,+1}\cap[t,\infty)$ for every $t>0$.

Conjecture (Kátai [14]). *If $f\in\mathcal{M}_0$ and f is not indentically 1, then $f(p+1)$ $(p\in\mathcal{P})$ takes on at least two distinct values.*

3. Application to additive functions

We return to Theorem 1 and shall prove results of a different type. Let g be a real-valued additive function and $W=\{n : b\le g(n)<b+h\}$. For any $t>2$ consider $U_i=\{n_1 : p(n_1)\le t, ih(t)\le g(n_1)<(i+1)h(t)\}$, $V_i=\{n_2 : (n_2,\ P(t))=1,\ b-(i+1)h(t)\le g(n_2)<b-ih(t)+h\}$, $i=0,\pm1,\dots$. Then $U_i\cap U_j=\emptyset$ for $i\ne j$, and if $s\ge s(t)=\dfrac{h}{h(t)}$ we have $V_{i_1}\cap\dots\cap V_{i_s}=\emptyset$ for any $i_1<\dots<i_s$. If $n_1\in U_i$, $n_2\in V_i$ then $n=n_1n_2\in W(t)=\{n : b-h(t)\le g(n)<b+h(t)+h\}$. Hence

$$W(t_0,\dots,t_k)=\{n : b-h(t_0)-\dots-h(t_k)\le g(n)<b+h+h(t_0)+\dots+h(t_k)\}$$

and $s(t_k)=\dfrac{h+2h(t_1)+\dots+2h(t_{k-1})}{h(t_k)}$. Let $h(t)=\dfrac{h}{\log t}$. By construction $t_0=\exp\left(\frac{\log x}{12\log\log 10x}\right)$ and $\log t_i=\min\left(u_i^{\lambda}\log^2 u_i^4, \frac{\log x}{4A\log\log x}\right)$ where $\lambda=\dfrac{1}{48}$, $u_i=y^{2^{k-i}}$, $i=1,\dots,k$. Hence

$$s(t_j)\le\log t_j\left(1+2\sum_{i=0}^{k}\frac{1}{\log t_i}\right)\ll\left(k\frac{4A\log\log x}{\log x}+\sum_{i\ge1}\frac{1}{u_i^{\lambda}}\right)\log t_j\ll\log t_j$$

and, for $x \geq x_0$, $y \geq 2|a|$,

$$h(t_0) + \ldots + h(t_k) \leq h\left(\sum_{i=0}^{k} \frac{1}{\log t_i}\right) \leq \frac{h}{\log y}.$$

Thus by Theorem 1 we obtain

Theorem 5. *Let g be a real-valued additive function. Then*

$$\frac{1}{\pi(x)} \left|\{p : p \leq x, b \leq g(p+a) < b+h\}\right|$$

$$\leq 4 \prod_{p \leq y}\left(1 - \frac{1}{p}\right)^{-1} \frac{1}{x}\Bigg|\Bigg\{n : n \leq x, ((n-a), P(y)) = 1,$$

$$-h\frac{1}{\log y} \leq g(n) < b+h\left(1 + \frac{1}{\log y}\right)\Bigg\}\Bigg| M(x) + O\left(\frac{1}{y^{1/48}} + \frac{(\log\log 10x)^3}{\log x}\right),$$

where $M(x) = 1 + O(\log^{-1/15} x)$ and $2|a| \leq y \leq \log^A x$.

Put

$$y = \left(\frac{1}{x}|\{n : n \leq x, b - h\varepsilon \leq g(n) < b + h(1+\varepsilon)\}| + \frac{1}{\log x}\right)^{-48} + \exp\left(\frac{1}{\varepsilon}\right).$$

Then we have

Corollary 7. *For any real-valued additive function g the estimate*

$$\frac{1}{\pi(x)}|\{p : p \leq x, b \leq g(p+a) < b+h\}| \ll_\varepsilon$$

$$\ll S(b, h, \varepsilon, x) \cdot \log 2S^{-1}(b, h, \varepsilon, x) + \frac{(\log\log 10x)^3}{\log x},$$

holds, where

$$S(b, h, \varepsilon, x) := \frac{1}{x}\left|\{n : n \leq x, b - h\varepsilon \leq g(n) < b + h(1+\varepsilon)\}\right| + \frac{1}{x}.$$

Suppose that $A \leq g(p+a) < B$. We divide the interval $\big[A, B\big)$ into intervals $\big[b, b+1\big)$. For each interval for which

$$R(x, b) := \frac{1}{x}|\{n : n \leq x, b - 1 \leq g(n) < b+2\}| \neq 0$$

we have

$$\frac{1}{\pi(x)}|\{p : p \le x, b \le g(p+a) < b+1\}|$$
$$\ll R(x,b)\log 2\bigl(R(x,b)\bigr)^{-1} + \frac{(\log\log 10x)^3}{\log x}.$$

Let $x_1, \dots, x_n$ be positive numbers such that $x_1 + \ldots + x_n = y$. Then, by Jensen's inequality, we get

$$x_1 \log\frac{1}{x_1} + \ldots + x_n \log\frac{1}{x_n} \le y \log\frac{n}{y}.$$

Using this inequality we obtain the following result.

Corollary 8. *For any real-valued additive function g we have*

$$\frac{1}{\pi(x)}|\{p : p \le x, A \le g(p+a) < B\}| \ll \frac{1}{x}|\{n : n \le x, A-1 \le g(n) < B+1\}|$$
$$\times \log\frac{2\cdot|B-A+1|}{\frac{1}{x}\cdot|\{n : n \le x, A-1 \le g(n) < B+1\}| + \frac{1}{x}} + |B-A+1|\cdot\frac{(\log\log 10x)^3}{\log x}.$$

4. Application to multiplicative functions II

In the same way as before we can prove similar results for multiplicative functions.

Let f be a real-valued multiplicative function and $b > 0$. Then

$$\frac{1}{\pi(x)}|\{p : p \le x, b \le f(p+a) < b\cdot H\}|$$
$$= \frac{1}{\pi(x)}|\{p : p \le x, \log b \le \log|f(p+a)| < \log b + \log H, f(p+a) > 0\}|.$$

Let, for any $t > 2$, $U_i^+ = \{n = n_1 : n_1 \in U_i, f(n_1) > 0\}$, $V_i^+ = \{n = n_2 : n_2 \in V_i, f(n_2) > 0\}$, $U_i^- = \{n = n_1 : n_1 \in U_i, f(n_1) < 0\}$, $V_i^- = \{n = n_2 : n_2 \in V_i, f(n_2) < 0\}$ where as before $U_i = \{n_1 : p(n_1) \le t, ih(t) \le \log|f(n_1)| < (i+1)\cdot h(t)\}$, $V_i = \{n_2 : (n_2, P(t)) = 1, \log b - (i+1)h(t) \le$

$\log|f(n_2)| < \log b + \log H - ih(t)\}$. If $b < 0$ and $H > 1$ we have

$$\frac{1}{\pi(x)}|\{p : p \le x, f(p+a) \in \big[b, bH\big)\}|$$
$$= \frac{1}{\pi(x)}|\{p : p \le x, |f(p+a)| \in \big[|b|, |b|H\big), f(p+a) < 0\}|.$$

Here we define U_i^+, V_i^+, U_i^-, V_i^- in the same way as before but $f(n_1) \cdot f(n_2) < 0$ for $n_1 \in U_i^{\pm}, n_2 \in V_i^{\pm}$.

Thus we have

Theorem 6. *Let f be a real-valued multiplicative function, $b \ne 0$ and $H > 1$. Then*

$$\frac{1}{\pi(x)}\left|\{p : p \le x, f(p+a) \in \big[b, bH\big)\}\right| \le 4\prod_{p\le y}\left(1-\frac{1}{p}\right)^{-1}$$
$$\times \frac{1}{x}\left|\left\{n : n \le x, \big(n-a, P(y)\big) = 1, f(n) \in \big[bH^{-\sigma(y)}, bH^{1+\sigma(y)}\big)\right\}\right|$$
$$\times M(x) + O\left(\frac{1}{y^{1/48}} + \frac{(\log\log 10x)^3}{\log x}\right),$$

where $\sigma(y) = \dfrac{1}{\log y}$, $M(x) = 1 + O(\log^{-1/15} x)$ and $2|a| \le y \le \log^A x$.

Corollary 9. *For any real-valued multiplicative function f we have*

$$\frac{1}{\pi(x)}\left|\{p : p \le x, f(p+a) \in \big[b, b\cdot H\big)\}\right| \ll_\varepsilon$$
$$\ll T(b, H, \varepsilon, x) \cdot \log\left(2\left(T(b, H, \varepsilon, x)\right)^{-1}\right) + \frac{(\log\log 10x)^3}{\log x},$$

where $b \ne 0$, $H > 1$ and

$$T(b, H, \varepsilon, x) = \frac{1}{x}\left|\{n : n \le x, f(n) \in \big[bH^{-\varepsilon}, bH^{1+\varepsilon}\big)\}\right| + \frac{1}{\log x}.$$

Corollary 10. *For any real-valued multiplicative function f and $B > A > 0$ we have*

$$\frac{1}{\pi(x)}|\{p : p \le x, f(p+a) \in \big[A, B\big)\}|$$
$$\ll \frac{1}{x}|\{n : n \le x, f(n) \in \big[Ae^{-1}, Be\big)\}|$$
$$\times \log \frac{2 \cdot |\log \frac{eB}{A}|}{\frac{1}{x}\left|\{n : n \le x, f(n) \in \big[Ae^{-1}, Be\big)\}\right| + \frac{1}{x}}$$
$$+ \left|\log \frac{eB}{A}\right| \cdot \frac{(\log\log 10x)^3}{\log x}.$$

5. Examples and remarks

Example 1. Let E be an arbitrary nonempty set of primes. Put

$$E(x) = \sum_{p \le x,\ p \in E} \frac{1}{p}$$

and let $\omega(n, E), \Omega(n, E)$ be the number of different prime divisors of n from E and the number of all prime divisors of n (counted with multiplicities) from E, respectively. Halász [10] and Norton [11] proved that the inequality

$$|\{n : n \le x, g(n, E) = m\}| \le \frac{c_1(\delta) x E^m(x)}{m! \exp(E(x))}$$

holds for $m \le (2-\delta)E(x)$, where $\delta > 0$ and $g(n, E) = \Omega(n, E)$ (see [10]) or $g(n, E) = \omega(n, E)$ (see [11]).

If E is the set of all primes then the corresponding functions are either equal to ω or Ω and

$$|\{n : n \le x, g(n) = m\}| \le c_2 \frac{x}{\log x} \frac{(\log\log x)^{m-1}}{(m-1)!}$$

for $m \le (2-\delta)\log\log 10x$, $\delta > 0$.

By Corollary 1 we obtain

$$|\{p : p \le x, g(p+a, E) = m\}| \ll \frac{x}{\log x} \frac{E^{m+1}(x)}{m! \exp(E(x))} + \frac{(\log\log 10x)^3}{\log^2 x}$$

for $m \le (2-\delta)E(x)$ and

$$|\{p : p \le x, g(p+a) = m\}| \ll \frac{x}{\log^2 x} \frac{(\log\log 10x)^m}{(m-1)!} + \frac{(\log\log 10x)^3}{\log^2 x}$$

for $m \le (2-\delta)\log\log 10x, \delta > 0$.

Let us remark that in this case there are more exact results (see [12]).

Example 2. Norton [11] proved

$$|\{n : n \le x, \tau(n) \ge \log x\}| \ll x(\log x)^{-\sigma}(\log\log x)^{-1/2}$$

where $\sigma = 1 - (1 + \log\log 2/\log 2) = 0.086\ldots$.

By Corollary 10 we have

$$\frac{1}{\pi(x)}\,|\{p : p \le x, \tau(p+a) \ge \log x\}|$$
$$\le \frac{1}{\pi(x)}\left|\left\{p : p \le x, \tau(p+a) \in [\log x, \log^2 x]\right\}\right| + \frac{1}{\pi(x)}\frac{1}{\log^2 x}\sum_{p \le x}\tau(p+a)$$
$$\ll \frac{\sqrt{\log\log x}}{\log^\sigma x}.$$

Remark 7. In the papers [3] and [4] the set

$$W = \left\{p : g(p+a) \in \left[h, h+1\right)\right\}$$

has been considered where g is an additive function. The main idea was the following. For every real u we have

$$\int_{-1}^{1}(1-|t|)e^{itu}\,dt = \left(\frac{\sin\frac{1}{2}u}{\frac{1}{2}u}\right)^2$$

and for $|u| \le 1$ this integral is larger than $4/\pi^2$. Hence

$$Q_h := \frac{1}{\pi(x)} \left|\left\{p : g(p+a) \in \left[h, h+1\right)\right\}\right|$$
$$\le \frac{3}{\pi(x)} \sum_{\substack{n\le x\\ (n,P(u,z))=1}} \int_{-1}^{1} (1-|t|)e^{-ith} f(n+a)\,dt + \frac{z}{\pi(x)}$$

where $f(n) = \exp\left(itg(n)\right)$. Timofeev [3] treats the sum

$$\sum_{\substack{n\le x\\ (n,\,P(y,z))=1}} f(n+a)$$

by the large sieve and by the dispersion method of Linnik. This was possible when $\log z/\log x \to 0$ and $\log y \to \infty$, as $x \to \infty$. Elliott [4] used the fact that the non-negativity of

$$\int_{-1}^{1} (1-|t|)e^{itu}\,du$$

allows the application of Selberg's sieve. Hence

$$Q_h \le \frac{3}{\pi(x)} \int_{-1}^{1} (1-|t|)e^{-ith} \sum_{\substack{d_j|P(\omega,z)\\ d_j\le z\\ j=1,2}} \lambda_{d_1}\lambda_{d_2} \sum_{\substack{n\le x\\ n\equiv 0 \pmod{[d_1,d_2]},\\ (n,\,P(y,z))=1}} f(n+a)\,dt + \frac{z}{\pi(x)}.$$

This idea and the new estimates of multiplicative functions on arithmetic progressions with small moduli $d = [d_1, d_2]$ allows Elliott [4] to improve the bound of Timofeev [3], and he proved results which are best possible. This method allows to estimate $\sup_h Q_h$.

We do not use here estimates for multiplicative functions. Using only the properties of the set W we have proved in particular (see Theorem 5 and Corollaries 7, 8) *for any* h

$$\frac{1}{\pi(x)}\left|\left\{p: p \le x, g(p+a) \in \big[h, h+1\big)\right\}\right| \le 4 \prod_{p \le y}\left(1-\frac{1}{p}\right)\frac{1}{x}$$
$$\times\left|\left\{n: n \le x, ((n-a), P(y)) = 1, g(n) \in \left[h-\frac{100}{\log y}, h+1+\frac{100}{\log y}\right)\right\}\right|$$
$$\times\left(1+O\left(\frac{1}{\log^{1/15} x}\right)\right)+O\left(y^{-1/48}+\frac{(\log\log 10x)^3}{\log x}\right).$$

Using this we obtain

$$\sup_h Q_h \ll \frac{\log(D^{-1}(x)+2)}{\sqrt{D(x)}}.$$

The result is not as good as the estimate of Elliott. The condition $((n-a), P(y)) = 1$ may be incorporated into sums of the Möbius function. Then the problem is reduced to the study of multiplicative functions on arithmetic progressions with small moduli. Here we have investigated this problem only in one case (see Corollaries 2–6).

6. Proof of Theorem 1

We observe that the main property of W is the possibility of the representation

$$\sum_{n \in W} 1 = \sum_i \sum_{n_1 \in U_i} \sum_{n_2 \in V_i} 1$$

with a small error term.

For the proof of Theorem 1 we need the following preliminary result.

Lemma 4. *Let* a *be a non-zero integer. Let* $\{a_n\}$, $\{b_m\}$ *be sequences of complex numbers, such that, if* $n \le z_1$ *or* $(n,a) \ne 1$ *then* $a_n = 0$ *and if* $m \le z_2$ *or* $(m,a) \ne 1$ *then* $b_m = 0$. *Put* $r(x,v,u) := \{n : n \le x,$

$(n-a, P(v,u)) = 1\}$, *and*

$$M_1(x) := \max_{y\le x}\left(y^{-1}\sum_{n\le y}\left(\sum_{km=n}|a_k||b_m|\right)^2\right)^{\frac{1}{2}},$$

$$M_2(x) := \max_{ty\le x}\left(\frac{1}{ty}\sum_{n\le t}|a_n|^2\sum_{n\le y}|b_n|^2\right)^{\frac{1}{2}},$$

$$\Delta(x,d) := \sum_{\substack{nm\in r(x,v,u),\\ nm\equiv a \pmod d}} a_n b_m - \frac{1}{d}\sum_{nm\in r(x,v,u)} a_n b_m.$$

Then

$$R(Q,v,u,z) := \sum_{\substack{d\le Q,\\ d|P(u,z)}} 3^{\omega(d)}|\Delta(x,d)|$$

$$\ll \frac{x}{\sqrt[8]{u}}M_1(x)\left(\frac{\log z}{\log v}\right)^{126} + x\sqrt{M_1(x)M_2(x)}\left(\frac{\log z}{\log v}\right)^{123}\sqrt{\log Q}(\log u)^6$$

$$\times\left(\log\frac{2x}{z_1z_2}\right)\left(\frac{uQ\exp(\log^2 u)}{\sqrt{x}} + u^{\frac{3}{4}}\left(\frac{1}{\sqrt{z_1}}+\frac{1}{\sqrt{z_2}}\right)+\frac{1}{\sqrt{u}}\right)^{\frac{1}{2}}.$$

uniformly for $1 < v \le u \le z$ *where* $16 + |a| \le u \le \exp\left(\sqrt[3]{\log x}\right)$, $Qu\exp(\log^2 u) \le \sqrt{x}$.

Remark 8. The same estimate holds for the sum

$$\sum_{\substack{d\le Q\\ p|d\Rightarrow v<p\le u}} 3^{\omega(d)}|\Delta(x,d)|.$$

Proof. Applying Cauchy's inequality gives

$$R(Q,v,u,z) \le \left(\sum_{\substack{d\le Q,\\ d|P(u,z)}} 9^{\omega(d)}|\Delta(x,d)|\right)^{\frac{1}{2}}\cdot (R_1(Q,v,u,z))^{\frac{1}{2}},$$

where

$$R_1(Q,v,u,z) := \sum_{\substack{d\le Q,\\ d|P(u,z)}} |\Delta(x,d)|.$$

The first sum does not exceed

$$\sum_{n\le x}\left(\sum_{km=n}|a_k||b_m|\right)\cdot\left(\sum_{\substack{d\le Q,\\ d|(P(u,z),n-a)}} 9^{\omega(d)} + \sum_{d|P(u,z)}\frac{9^{\omega(d)}}{d}\right)$$
$$\ll \sqrt{x}M_1(x)\left(\left(\sum_{n\le x}\left(\sum_{\substack{d\le Q\\ d|(P(u,z),n-a)}} 9^{\omega(d)}\right)\right)^{\frac{1}{2}} + \sqrt{x}\left(\frac{\log z}{\log u}\right)^9\right)$$

The number of representations of d as $[d_1,d_2]$ is less than $\tau_3(d)$. Thus we obtain that the first sum on the right-hand side is

$$\ll 2x\sum_{\substack{d_1,d_2|P(u,z)\\ [d_1,d_2]\le x}}\frac{9^{\omega(d_1)}9^{\omega(d_2)}}{[d_1,d_2]} \ll 2x\sum_{d\le x}\frac{81^{\omega(d)}}{d}\tau_3(d) \ll 2x\left(\frac{\log z}{\log u}\right)^{243}$$

Therefore we have

$$R(Q,v,u,z) \ll \sqrt{x}\sqrt{M_1(x)}\left(\frac{\log z}{\log u}\right)^{122}\sqrt{R(Q,v,u,z)}.$$

Now we only have to prove that

$$R_1(Q,v,u,z) \ll \frac{x}{\sqrt[4]{u}}M_1(x)\left(\frac{\log z}{\log v}\right)^7 + x\frac{\log z}{\log v}(\log Q)(\log u)^2\left(\log\frac{2x}{z_1z_2}\right)^2 M_2(x)$$
$$\left(\frac{uQ\exp(\log^2 u)}{\sqrt{x}} + \frac{u^{\frac{3}{4}}}{\sqrt{z_1}} + \frac{u^{\frac{3}{4}}}{\sqrt{z_2}} + \frac{1}{\sqrt{u}}\right). \tag{12}$$

First, put

$$\Delta_1(x,d) = \sum_{\substack{nm\in r(x,v,u),\\ nm\equiv a \pmod d}} a_nb_m - \frac{1}{\varphi(d)}\sum_{\substack{nm\in r(x,v,u),\\ (nm,d)=1}} a_nb_m.$$

Then, since $(d, P(u)) = 1$ we have

$$1 - \frac{\varphi(d)}{d} = 1 - \prod_{p|d}\left(1 - \frac{1}{p}\right) \le \sum_{\substack{\delta>u,\\ \delta|d}} \frac{1}{\delta},$$

and therefore we obtain

$$\sum_{\substack{d\le Q,\\ d|P(u,z)}} |\Delta(x,d) - \Delta_1(x,d)|$$

$$\le \sum_{\substack{d\le Q,\\ d|P(u,z)}} \frac{1}{\varphi(d)}\left(\sum_{\substack{\delta>u,\\ \delta|d}} \frac{1}{\delta} \sum_{nm\le x} |a_n b_m| + \frac{\varphi(d)}{d} \sum_{\substack{nm\le x,\\ (nm,d)>1}} |a_n b_m|\right).$$

Using Cauchy's inequality again, we conclude that the righthand side is at most

$$\sqrt{x} M_1(x)\left(\sqrt{x}\sum_{\delta>u} \frac{1}{\delta\varphi(\delta)} \sum_{d|P(u,z)} \frac{1}{\varphi(d)} + \sum_{d|P(u,z)} \frac{1}{d}\left(\sum_{\substack{n\le x,\\ (n,d)>1}} 1\right)^{\frac{1}{2}}\right)$$

$$\ll x M_1(x)\left(\frac{1}{u}\frac{\log z}{\log u} + \sum_{\substack{\delta d|P(u,z),\\ \delta>u}} \frac{\tau(\delta d)}{\sqrt{\delta}\delta d}\right)$$

$$\ll \frac{x}{\sqrt{u}} M_1(x)\left(\frac{\log z}{\log u}\right)^4.$$

Thus

$$R_1(Q,v,u,z) \ll R_2(Q,v,u,z) + \frac{x}{\sqrt{u}} M_1(x)\left(\frac{\log z}{\log u}\right)^4,$$

where

$$R_2(Q,v,u,z) = \sum_{\substack{d|P(u,z),\\ d\le Q}} |\Delta_1(x,d)|.$$

Using Eratosthenes's sieve yields

$$R_2(Q,v,u,z)$$
$$\le \sum_{\substack{\delta\le y,\\ \delta|P(v,u)}}\ \sum_{\substack{d\le Q,\\ d|P(u,z)}} \left| \sum_{\substack{nm\le x,\\ nm\equiv a \pmod{d\delta)}}} a_n b_m - \frac{1}{\varphi(d)} \sum_{\substack{nm\le x,\\ nm\equiv a \pmod{\delta},\\ p(nm,d)=1}} a_n b_m \right|$$
$$+ R_4(Q,v,u,z) := R_3(Q,v,u,z) + R_4(Q,v,u,z),$$

where

$$R_4(Q,v,u,z) \ll \sum_{nm\le x} |a_n b_m| \Bigg(\sum_{\substack{\delta>y,\\ \delta|P(v,u)}}\ \sum_{\substack{d\le Q,\\ d|P(u,z),\\ \delta d|n-a}} 1 \Bigg) \frac{\log z}{\log u}.$$

By applying Cauchy's inequality we obtain

$$R_4(Q,v,u,z) \ll \sqrt{x} M_1(x) \Bigg(\sum_{n\le x} \Bigg(\sum_{\substack{\delta>y,\\ \delta|P(v,u)}}\ \sum_{\substack{\delta d|n-a,\\ d|P(u,z)}} 1 \Bigg)^2 \Bigg)^{\frac12} \frac{\log z}{\log u}$$

$$\ll x M_1(x) \Bigg(\sum_{\substack{\delta>y,\\ \delta|P(v,u)}} \frac{\tau_3(\delta)}{\delta} \sum_{d|P(u,z)} \frac{\tau_3(d)}{d} \Bigg)^{\frac12} \frac{\log z}{\log u}.$$

Here we used again the fact that the number of representations of d as $[d_1, d_2]$ is less than $\tau_3(d)$. Taking $y = \exp(\ln^2 u)$ and $\alpha = 1/\ln u$ we see that (see the proof of Lemma 1)

$$\sum_{\substack{\delta>y,\\ \delta|P(v,u)}} \frac{\tau_3(\delta)}{\delta} \ll \frac{1}{y^\alpha} \exp\Bigg(3 \sum_{v\le p\le u} \frac{p^\alpha}{p} \Bigg) \ll \left(\frac{\log u}{\log v} \right)^{12} \frac{1}{u}$$

and therefore it follows that

$$R_4(Q,v,u,z) \ll x M_1(x) \frac{1}{\sqrt{u}} \left(\frac{\log z}{\log v} \right)^7.$$

Thus we have established the inequality

$$R_1(Q,v,u,z) \ll R_3(Q,v,u,z) + xM_1(x)\frac{1}{\sqrt{u}}\left(\frac{\log z}{\log v}\right)^7.$$

Now we want to make the summation over n, m independent from each other. For this aim we divide the intervals $z_1 < n \le \frac{x}{z_2}$, $z_2 < m \le \frac{x}{z_1}$ into intervals of type $\left(N, \left(1+\frac{1}{\sqrt{u}}\right)N\right]$ and $\left(M, \left(1+\frac{1}{\sqrt{u}}\right)M\right]$. The number of such full intervals lying in the interval $mn \le x$, $N \ge z_1$, $M \ge z_2$ is bounded by $O\left(u\left(\log\frac{2x}{z_1z_2}\right)^2\right)$. The last intervals for which $n \in I = \left(x\left(1+\frac{1}{\sqrt{u}}\right)^{-2}, \ x\left(1+\frac{1}{\sqrt{u}}\right)^2\right]$ contribute to $R_3(Q,v,u,z)$ not more than

$$\sum_{mn\in I}|a_n||b_m|\left(\sum_{\substack{d|P(v,z),\\ d|n-a,\\ d\le Qy}}1\right)\left(1+O\left(\frac{\log z}{\log v}\right)\right)$$

$$\ll \frac{x}{\sqrt[4]{u}}M_1(x)\left(\sum_{\substack{d\le (Qy)^2,\\ d|P(v,z)}}\frac{\tau_3(d)}{d}\right)^{\frac{1}{2}}\frac{\log z}{\log v} \ll \frac{x}{\sqrt[4]{u}}M_1(x)\left(\frac{\log z}{\log v}\right)^4.$$

Here we used the condition $(Qy)^2u \le x$.

Now we have proved that

$$R_1(Q,v,u,z) \ll \left(\sqrt{u}\log\frac{2x}{z_1z_2}\right)^2\max_{M,N}R_5(Q,M,N) + \frac{x}{\sqrt[4]{u}}M_1(x)\left(\frac{\log z}{\log v}\right)^7$$

where

$$R_5(Q,M,N) := \sum_{\substack{\delta\le y,\\ \delta|P(v,u)}}\sum_{\substack{d\le Q,\\ d|P(u,z)}}\left|\sum_{n\in\left(N,\left(1+\frac{1}{\sqrt{u}}\right)N\right]}\sum_{\substack{m\in\left(M,\left(1+\frac{1}{\sqrt{u}}\right)M\right],\\ nm\equiv a \pmod{d\delta}}}a_nb_m\right.$$

$$\left. - \frac{1}{\varphi(d)}\sum_{n\in\left(N,\left(1+\frac{1}{\sqrt{u}}\right)N\right]}\sum_{\substack{m\in\left(M,\left(1+\frac{1}{\sqrt{u}}\right)M\right],\\ nm\equiv a \pmod{\delta},\\ (nm,d)=1}}a_nb_m\right|$$

and the maximum has been taken over $MN \le x$, $M \ge z_2$, $N \ge z_1$. Using Dirichlet-characters we obtain

$$R_5(Q,M,N) \le \sum_{\substack{\delta \le y,\\ \delta | P(v,u)}} \sum_{\substack{d \le Q,\\ d|P(u,z)}} \frac{1}{\varphi(d\delta)} \sum_{\chi_\delta} \sum_{\chi_d \ne \chi_0} \left| \sum_{n \sim N} a_n \chi_{d\delta}(n) \right| \left| \sum_{m \sim M} b_m \chi_{d\delta}(m) \right|,$$

where $n \sim N$ and $m \sim M$ mean that $n \in \left(N, \left(1+\frac{1}{\sqrt{u}}\right)N\right]$ and $m \in \left(M, \left(1+\frac{1}{\sqrt{u}}\right)M\right]$, respectively. Let χ_k^* be the primitive character which generates $\chi_{d\delta}$. We have $\chi_d \ne \chi_0$ and $d|P(u,z)$, $\delta|P(v,u)$ and therefore $k > u$ and $k|P(v,z)$. Thus

$$R_5(Q,M,N) \le \sum_{d|P(v,z)} \frac{1}{\varphi(d)} \sum_{\substack{u \le k \le Qy,\\ k|P(v,z)}} \frac{1}{\varphi(k)} \sum_{\chi_k^*} |S_N(\chi_k^*)|\, |F_M(\chi_k^*)|\,,$$

where

$$S_N(\chi_k^*) = \sum_{\substack{n \sim N,\\ (n,d)=1}} a_n \chi_k^*(n), \qquad F_M(\chi_k^*) = \sum_{\substack{m \sim M,\\ (m,d)=1}} b_m \chi_k^*(m).$$

We divide the interval $u < k \le Qy$ into intervals $[Q_1, 2Q_1]$. Using Cauchy's intquality yields

$$R_5(Q,M,N) \ll \log(Qy) \frac{\log z}{\log v} \max_{d \le x} \max_{Q_1} \frac{1}{Q_1} \sum_{k \le 2Q_1} \frac{k}{\varphi(k)} \sum_{\chi_k^*} |S_N(\chi_k^*) F_M(\chi_k^*)|$$

$$\ll \log(Qy) \frac{\log z}{\log v} \max_{d \le x} \max_{Q_1} \frac{1}{Q_1} \left(\sum_{k \le 2Q_1} \frac{k}{\varphi(k)} \sum_{\chi_k^*} |F_M(\chi_k^*)|^2 \right)^{\frac{1}{2}}$$

$$\times \left(\sum_{k \le 2Q_1} \frac{k}{\varphi(k)} \sum_{\chi_k^*} |S_N(\chi_k^*)|^2 \right)^{\frac{1}{2}}.$$

Hence by Gallagher's inequality (see [8])

$$\sum_{k \le Q} \frac{k}{\varphi(k)} \sum_{\chi_k^*} \left| \sum_{n=M+1}^{M+N} \chi_k^*(n) a_n \right|^2 \le (Q^2 + \pi N) \sum_{n=M+1}^{M+N} |a_n|^2$$

we have

$$R_5(Q,M,N) \ll \log(Qy)\frac{\log z}{\log v}\max_{Q_1\in[u,Qy]}\sqrt{x}M_2(x)Q_1^{-1}\left(\left(Q_1^2+\frac{\pi N}{\sqrt{u}}\right)\left(Q_1^2+\frac{\pi M}{\sqrt{u}}\right)\right)^{\frac{1}{2}}$$

$$\ll x\log(Qy)\frac{\log z}{\log v}\left(\frac{Qy}{\sqrt{x}}+\frac{1}{\sqrt[4]{u}\sqrt{z_1}}+\frac{1}{\sqrt[4]{u}\sqrt{z_2}}+\frac{1}{u\sqrt{u}}\right)\cdot M_2(x)$$

and therefore

$$R_1(Q,v,u,z) \ll$$

$$\ll x(\log Q)(\log u)^2\frac{\log z}{\log v}\left(\log\frac{2x}{z_1z_2}\right)^2\left(\frac{Qu\exp(\log^2 u)}{\sqrt{x}}+\frac{u^{\frac{3}{4}}}{\sqrt{z_1}}+\frac{u^{\frac{3}{4}}}{\sqrt{z_2}}+\frac{1}{\sqrt{u}}\right)M_2(x)$$

$$+\frac{x}{\sqrt[4]{u}}M_1(x)\left(\frac{\log z}{\log v}\right)^7.$$

Thus we have proved (16), and this gives Lemma 4.

Proof of Theorem 1. Let $n=n_1n_2$, where $(n,a)=1$, $p(n_1)\le z_1$, $(n_1,a)=1$ and $(n_2,P(z_1))=1$. For $\sqrt[10]{x}<t\le\sqrt{x}$ we have

$$|\{p: p\le x, p+a\in W\}|$$

$$\le |\{n_1n_2\le x, n_1\ge z_2, n_2\ge z_1, n_1n_2\in W, (n_1n_2-a,P(t))=1\}|$$

$$+|\{n_1n_2\le x, n_1<z_2, (n_1n_2-a,P(t))=1\}|$$

$$+|\{n_1\le x\}|+O(t).$$

Using Selberg's sieve (see [9]) we obtain that the second term of the right-hand side is

$$\le |\{n_1n_2\le x, n_1\le z_2, (n_2(n_1n_2-a),P(z_1))=1\}|$$

$$\ll \sum_{n_1\le z_2}\frac{1}{\varphi(n_1)}\frac{x}{\log z_1\log z_1}\ll\frac{x}{\log z_1}\frac{\log z_2}{\log z_1}.$$

By Lemma 1 we get

$$|\{n_1: n_1\le x\}|\le\sqrt{x}+x\sum_{\sqrt{x}<n_1}\frac{1}{n_1}\ll x\exp\left(-\frac{1}{2}\frac{\log x}{\log z_1}\right)(\log z_1)^4.$$

Let $z_1 = \exp\left(\dfrac{\log x}{12\log\log 10x}\right)$, $z_2 = (\log x)^A$, $A > 0$. Then

$$(13) \qquad |\{p : p \le x, p + a \in W\}|$$
$$\le |\{n_1n_2 : n_1n_2 \le x, n_1 \ge z_2, n_2 \ge z_1, n_1n_2 \in W, (n_1n_2 - a, P(t)) = 1\}|$$
$$+ O\left(\frac{x}{\log^2 x}\cdot(\log\log 10x)^3\right).$$

By the definition of W we have

$$|\{n_1n_2 : n_1n_2 \le x, n_1 \ge z_2, n_2 \ge z_1, n_1n_2 \in W, (n_1n_2 - a, P(t)) = 1\}|$$
$$\le \sum_i |A_i(t)|$$

where

$$A_i(t) =$$
$$|\{n_1n_2 : n_1n_2 \le x, n_1 \in U_i, n_2 \in V_i, n_1 \ge z_2, n_2 \ge z_1, (n_1, a) = 1, (n_1n_2 - a, P(t)) = 1\}|\,.$$

To estimate $|A_i(t)|$ we apply Selberg's linear sieve (see Theorem 8.3 [9]). Let $\Pi = \{p : u < p \le t\}$, let η be a multiplicative function such that $\eta(p) = 1$ for $p > u$ and $\eta(p) = 0$ for $p \le u$ and

$$X_i =$$
$$|\{n_1n_2 : n_1n_2 \le x, n_1 \in U_i, n_2 \in V_i, n_1 \ge z_2, n_2 \ge z_1, (n_1, a) = 1, (n_1n_2 - a, P(u)) = 1\}|\,.$$

For $2 \le u \le z$ we have

$$-\log u - C_1 \le \sum_{u \le p < z} \eta(p)\frac{\log p}{p} - \log\frac{z}{u} \le C_2.$$

Thus the conditions of Theorem 8.3 [9] is satisfied. Therefore

$$|A_i(t)| \le \prod_{u < p \le t}\left(1 - \frac{1}{p}\right)|A_i(u)|\left\{F\left(\frac{\log \xi^2}{\log t}\right) + O\left(\frac{\log u}{(\log \xi)^{1/14}}\right)\right\}$$
$$(14) \qquad + \sum_{\substack{d \le \xi^2 \\ d | P(u,t)}} \mu^2(d)3^{\omega(d)}|\eta_i(x,d)|,$$

where

$$\eta_i(x,d) = |\{n_1n_2 : n_1n_2 \in A_i(u), n_1n_2 \equiv a \pmod d\}| - \frac{1}{d}|A_i(u)|.$$

Now we apply Lemma 4. Let $a(n,i) = 1$ if $n = n_1 \in U_i$, $(n_1, a) = 1$, $n_1 \ge z_2$ and $b(n,i) = 1$ if $n = n_2 \in V_i$, $n_2 \ge z_1$ and $a(n,i) = b(n,i) = 0$ otherwise. In the notation used in Lemma 4 we have

$$\sum_{\substack{d \le \xi^2 \\ d | P(u,t)}} |\eta_i(x,d)| = R(\xi^2, 1, u, t).$$

Let $\xi^2 = Q = \dfrac{\sqrt{x}}{u^2 \exp(\log^2 u)}$, $u = \log^{A/4} x$, $t = \xi$. By definition $z_1 = \exp\left(\dfrac{\log x}{12 \log\log x}\right)$ and $z_2 = \log^A x$. Hence

$$R(\xi^2, 1, u, t) \ll x(\log x)^{126 - \frac{A}{32}} \cdot \left(M_{1,i}(x) + \sqrt{M_{1,i}(x)}\sqrt{M_{2,i}(x)} \right).$$

Applying Cauchy's inequality leads to

$$\sum_i M_{1,i}(x) = \sum_i \max_{y \le x} \left(\frac{1}{y} \sum_{\substack{n_1n_2 \le y \\ n_1 \in U_i, n_2 \in V_i}} 1 \right)^{\frac{1}{2}}$$

$$\le \left(\sum_i \sum_{\substack{n_1 \le x \\ n_1 \in U_i}} \frac{1}{n_1} \right)^{\frac{1}{2}} \left(\sum_i \sum_{\substack{n_2 \le x \\ n_2 \in V_i}} \frac{1}{n_2} \right)^{\frac{1}{2}}.$$

It follows from the definition of W that $U_i \cap U_j = \emptyset$ for $i \ne j$ and $V_{i_1} \cap \ldots \cap V_{i_s} = \emptyset$ if $i_1 < \ldots < i_s$ and $s \ge s(z_1)$. Thus

$$\sum_i M_{1,i}(x) \ll \sqrt{s(z_1)} \cdot \log x.$$

In the same way we obtain

$$\sum_i \sqrt{M_{1,i}(x)} \cdot \sqrt{M_{2,i}(x)} \ll \sqrt{\sqrt{s(z_1)} \cdot \log x.} \cdot \left(\sum_i \sum_{\substack{n_1 \in U_i \\ n_1 \le 2x}} \frac{1}{n_1} \sum_i \sum_{\substack{n_2 \in V_i \\ n_2 \le 2x}} \frac{1}{n_2} \right)^{\frac{1}{2}}$$

$$\ll \sqrt{s(z_1)} \log x.$$

By (18) it follows that

$$\sum_i |A_i(t)| \le \prod_{u<p\le t}\left(1-\frac{1}{p}\right)$$
$$\times\left\{F(2)+O(\log^{-1/15}x)\right\}\sum_i |A_i(u)| + O\left(\sqrt{s(z_1)}\cdot(\log x)^{127-\frac{A}{32}}\right).$$

We have (see [9], p. 226) $F(2)=e^{\gamma}$ and

$$\prod_{p\le x}\left(1-\frac{1}{p}\right)=\frac{e^{-\gamma}}{\log x}\left(1+O\left(\frac{1}{\log x}\right)\right).$$

Hence, by (17), we obtain

$$|\{p : p\le x, p+a\in W\}|$$
$$\le 4\prod_{p\le u}\left(1-\frac{1}{p}\right)^{-1}\left(1+O(\log^{-1/15}x)\right)\cdot\frac{1}{\log x}\cdot\sum_i |A_i(u)|$$
$$+O\left(\sqrt{s(z_1)}x(\log x)^{127-\frac{A}{32}}\right)+O\left(\frac{(\log\log 10x)^3}{\log^2 x}\right).$$

By the definition of W we know that if $n_1\in U_i$ and $n_2\in V_i$, then $n=n_1n_2\in W(z_1)$ and further that $U_i\cap U_j=\emptyset$ means, for $i\ne j$, that $n_1n_2\ne n_1'n_2'$ if $n_1\in U_i$, $n_2\in V_i$, $n_1'\in U_j$, $n_2'\in V_j$. Therefore if $A\ge 32\cdot 127+16$ we see that

$$(15)\qquad |\{p : p\le x, p+a\in W\}|$$
$$\le 4\prod_{p\le u}\left(1-\frac{1}{p}\right)^{-1}\left(1+O(\log^{-1/15}x)\right)\cdot\frac{1}{\log x}$$
$$|\{n : n\le x, n\in W(z_1), (n-a, P(u)=1)\}|+O\left(\frac{x}{\log^2 x}(\log\ln 10x)^3\right).$$

Now we show that for any $|a| < \sqrt{u} < u < \log^A x,\ A \geq 32 \cdot 127 + 16$, the estimate

$$|\{n \leq x : n \in W, (n-a, P(u)) = 1\}|$$

$$\leq \prod_{\sqrt{u} \leq p < u} \left(1 - \frac{1}{p}\right) \cdot \left|\{n : n \leq x, n \in W(z_3), \left(n - a, P\left(\sqrt{u}\right)\right) = 1\}\right|$$

$$+O\left(\frac{x}{u^\lambda \log u}\right) + O\left(x \cdot \frac{(\log\log 10x)^2}{\log x}\right)$$

holds, where $\log z_3 = \min\left(u^\lambda \log^2 u^4, \dfrac{\log x}{4A \log\log x}\right)$ with $\lambda = \dfrac{1}{48}$.

Let $p(n_1) \leq z_3$ and $(n_2, P(z_3)) = 1$. Following the same argument as before, we deduce

$$|\{n_1 n_2 \leq x, n_1 n_2 \in W, (n_1 n_2 - a, P(u)) = 1\} t|$$

$$\leq \left|\left\{n_1 n_2 \leq x, n_1 n_2 \in W, (n_1 n_2 - a, P(u)) = 1, z_4 < n_1 \leq z_5, \frac{x}{z_5^2} < n_2 \leq \frac{x}{z_4}\right\}\right|$$

$$+ |\{n_1 n_2 : n_1 \leq z_4, n_1 n_2 \leq x\}| + |\{n_1 n_2 : n_1 > z_5, n_1 n_2 \leq x\}|$$

$$+ \left|\{n_1 n_2 : n_1 \leq z_5, n_2 \leq \frac{x}{z_5^2}\}\right|.$$

The second term is (see Selberg's sieve [9])

$$\ll \sum_{n_1 \leq z_4} \frac{1}{\varphi(n_1)} \frac{x}{\log z_3} \ll x \frac{\log z_4}{\log z_3}.$$

The third term can be estimated by (see Lemma 1)

$$\ll x \sum_{z_5 < n_1 \leq x} \frac{1}{n_1} \ll x \exp\left(-\frac{\log z_5}{\log z_3}\right) \cdot \log^4 z_3,$$

and the fourth term does not exceed x/z_5.

Let $z_4 = u^4, \log z_3 = \min\left(u^\lambda \log^2 z_4, \dfrac{\log x}{4A \log\log 10x}\right)$ and $\log z_5 = (\log z_3) \cdot \log(u^\lambda \ln^5 z_3)$. Obviously

$$\log z_5 \leq \frac{\log x}{4A \log\log 10x} \cdot (\lambda A + 5) \log\log 10x \leq \frac{1}{10} \log x.$$

Hence $z_5 \le x^{1/10}$ and

$$|\{n_1n_2 : n_1n_2 \le x, (n_1n_2 - a, P(u)) = 1, n_1n_2 \in W\}| \le \sum_i \Lambda_i(u, z_3, z_4, z_5, 1) + O\left(\frac{x}{u^\lambda \log u}\right) + O\left(x \cdot \frac{(\log\log 10x)^2}{\log x}\right) \tag{16}$$

where

$$\Lambda_i(u, z_3, z_4, z_5, d)$$
$$= \Big|\{n_1n_2 : n_1n_2 \le x, n_1 \in U_i, n_2 \in V_i, (n_1, a) = 1, (n_1n_2 - a, P(u)),$$
$$z_4 < n_1 \le z_5, \frac{x}{z_5^2} < n_2 \le \frac{x}{z_4}, n_1n_2 \equiv a \pmod d\}\Big|$$

The condition $(n_1n_2 - a, P(u)) = 1$ can be expressed by a sum over the Möbius function. This leads to

$$\sum_i \Lambda_i(u, z_3, z_4, z_5, 1) \le \prod_{\sqrt{u} \le p < u} \left(1 - \frac{1}{p}\right) \sum_i \Lambda_i(\sqrt{u}, z_3, z_4, z_5, 1)$$
$$+ \sum_i \sum_{\substack{d \le Q \\ d | P(\sqrt{u}, u)}} \left|\Lambda_i(u, z_3, z_4, z_5, d) - \frac{1}{d}\Lambda_i(u, z_3, z_4, z_5, 1)\right|$$
$$+ s(z_3) \sum_{\substack{d > Q \\ d | P(u)}} \frac{x}{d}. \tag{17}$$

Using Lemma 1 we see that the third sum has an upper bound

$$\ll s(z_3) \cdot x \exp\left(-\frac{\log Q}{\log u}\right) \cdot \log^4 u \ll \frac{x}{u^\lambda \log u}$$

if $\log Q = (\log u)\log(u^{2\lambda}(\log u)^7)$. The second sum will be estimated by Lemma 4. Put $z_1 := z_4$, $u := \sqrt{u}$, $z = u$, $z_2 := \frac{x}{z_5^2} \ge x^{4/5}$ so that $Q = \exp\big((\log u)\log(u^{2\lambda} \log^7 u)\big) \le x^\varepsilon$. Let $a(n, i) = 1$ if $n = n_1 \in U_i$, $z_4 < n_1 \le z_5$, $(n_1, a) = 1$ and $a(n, i) = 0$ otherwise, and further $b(n, i) = 1$

if $n = n_2 \in V_i$, $\frac{x}{z_5^2} < n_2 \le \frac{x}{z_4}$ and $b(n,i) = 0$ otherwise. Using Lemma 4 we conclude that the second sum is bounded by

$$\ll \frac{x}{\sqrt[16]{u}} (\log u)^{126} \sum_i M_{1,i}(x) + x(\log u)^{129} \sqrt{\log^2 u \frac{\log z_5^2}{\sqrt[4]{u}}} \sum_i \sqrt{M_{2,i}(x)}.$$

We have $\log z_5 = \log z_3 \cdot \log(u^\lambda \log^5 z_3) \ll u^\lambda \cdot \log^3 u$ with $\lambda \le \frac{1}{48}$. From this we conclude that the second sum on the right-hand side of (17) is

$$\ll \frac{x}{\sqrt[17]{u}} \left(\sum_i M_{1,i}(x) + \sum_i \sqrt{M_{1,i}(x)} \sqrt{M_{2,i}(x)} \right)$$

$$\ll \frac{x}{\sqrt[17]{u}} \left(\sum_i \left(\sum \frac{a(n,i)}{n} \right) \right)^{\frac{1}{2}} \cdot \left(\sum_i \left(\sum \frac{b(n,i)}{n} \right) \right)^{\frac{1}{2}}$$

$$\ll \frac{x}{\sqrt[17]{u}} \cdot \sqrt{s(z_3)} \cdot \ln z_5 \ll \frac{x}{\sqrt[17]{u}} \cdot u^{\frac{3}{2}\lambda} \ln^3 u \ll \frac{x}{u^{1/48} \log u}$$

if $\lambda \le \frac{1}{48}$. Here we used $s(z_3) \ll \ln z_3$. By (17) and (16) we have

$$\text{(18)} \qquad |\{n : n \le x, (n-a, P(u)) = 1, n \in W\}|$$

$$\le \prod_{\sqrt{u} \le p < u} \left(1 - \frac{1}{p}\right) \left|\{n : n \le x, \left(n - a, P(\sqrt{u})\right) = 1, n \in W(z_3)\}\right|$$

$$+ O\left(\frac{x}{u^\lambda \log u} + x \cdot \frac{(\log\log 10x)^2}{\log x} \right).$$

Put $u_1 = y^{2^k}$, $u_2 = y^{2^{k-1}}$, $\dots$, $u_k = y$, $L = \log x$, $l = \log\log 10x$, $M(x) = \left(1 + O(\log^{-1/15} x)\right)$, $k = \left[\log_2 \frac{\log(\log x)^A}{\log y}\right] + 1$, and let $|a| < y \le L^A$, $A = 32 \cdot 127 + 16$, $\lambda = \frac{1}{48}$. Using (15) and (18) we obtain

$$|\{p : p \le x, p + a \in W\}|$$

$$\le 4 \prod_{p \le u_1} \left(1 - \frac{1}{p}\right)^{-1} \frac{M(x)}{L} |\{n : n \le x, n \in W(t_0), (n - a, P(u_1)) = 1\}| + O\left(x \cdot \frac{l^3}{L^2}\right)$$

$$\leq 4 \prod_{p \leq u_2} \left(1 - \frac{1}{p}\right)^{-1} \frac{M(x)}{L} |\{n : n \leq x, n \in W(t_0, t_1), (n - a, P(u_2)) = 1\}|$$

$$+ O\left(\frac{x}{u_1^{\lambda} L} + x \cdot \frac{l^2}{L^2} \log u_1 + x \cdot \frac{l^3}{L^2}\right)$$

$$\leq 4 \prod_{p \leq y} \left(1 - \frac{1}{p}\right)^{-1} |\{n : n \leq x, n \in W(t_0, \dots, t_k), (n - a, P(y)) = 1\}| \frac{M(x)}{L}$$

$$+ O\left(\frac{x}{L} \sum_{i \geq 0} y^{-\lambda 2^i} + x \cdot \frac{l^2}{L^2} \sum_{i \leq k} 2^i \log y + x \cdot \frac{l^3}{L^2}\right),$$

Where $t_0 = \exp\left(\frac{\log x}{12 \log\log 10x}\right)$, $\log t_i = \min\left(u_i^{\lambda} \log^2 u_i^4, \frac{\log x}{4A \log\log x}\right)$, $i = 1, \dots, k$, $\lambda = \frac{1}{48}$. This ends the proof of Theorem 1.

References

[1] G. Halász, On the distribution of additive arithmetical functions, *Acta. Arithm.* **27** (1975), 143–152.

[2] I. Ruzsa, On the concentration of additive functions, *Acta Math. Acad. Sci. Hungar.* **36** (1980), 215–232.

[3] N. M. Timofeev, The Erdös–Kubilius conjecture on the distribution of the values of additive functions on the sequence of shifted primes, *Acta Arithm.* **52** (1991), 113–131.

[4] P. D. T. A. Elliott, The concentration function of additive function on shifted primes, *Acta Math. Mittag Lefler* **173** (1994), 1–35.

[5] G. Halász, Über die Mittelwerte multiplikativer zahlentheoretischer Funktionen, *Acta Math. Acad. Sci. Hugar.* **19** (1968), 365–403.

[6] N. M. Timofeev, Multiplicative functions on the set of shifted prime numbers, *Math. USSR Izvestiya* **39** (1992), 1189–1207.

[7] P. D. T. A. Elliott, The multiplicative group of rationals generated by the shifted primes, I, *Journal für die reine und angewandte Mathematik* **463** (1995), 169–216.

[8] P. X. Gallagher, Bombieri's mean value theorem, *Mathematika* **15** (1968), 1–6.

[9] H. Halberstam and H.E. Richert, Sieve methods, *Academic Press*, *New York*, 1974.

[10] G. Halász, Remark to my paper "On the distribution of additive and the mean values of multiplicative arithmetic functions", *Acta Math. Acad. Sci. Hung.* **8** (1977), 135–150.

[11] K. K. Norton, On the number of restricted prime factors of an integer I, *Ill. J. Math.* **20** (1976), 681–705.

[12] N. M. Timofeev, The Hardy-Ramanujan and Halász inequalities for shifted primes, *Mathematical Notes* **57** (1995), 522–535.

[13] I. Kátai, Research problems in number theory, *Publ. Math. Debrecen* **24** (1977), 263–276.

[14] I. Kátai, Research problems in number theory II, *Annales Univ. Sci. Budapest, Sect. Comp.* **16** (1996), 223–251.

KARL–HEINZ INDLEKOFER
FACULTY OF MATHEMATICS AND INFORMATICS
UNIVERSITY OF PADERBORN
WARBURGER STRASSE 100
33098 PADERBORN
GERMANY

E-mail: k-heinz@uni-paderborn.de

NIKOLAI M. TIMOFEEV
STROITELEI 11
VLADIMIR STATE PED. UNIVERSITY
600024 VLADIMIR
RUSSIA

E-mail: timofeev@vgpu.elcom.ru

(Received November 18, 1997)