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On integral representations for powers of the Riemann zeta-function

By ALEKSANDAR IVIĆ (Beograd)

Dedicated to Professor Imre Kátai on the occasion of his 60th birthday

Abstract. A new integral representation for $\zeta^r(s)$ is obtained, where $r \geq 3$ is a fixed natural number. The approach is due to A. Guthmann, who obtained the analogue of the classical Riemann-Siegel formula (for $\zeta(s)$) for several Dirichlet series, including $\zeta^2(s)$. The fundamental role is played by the Mellin inverse of $\pi^{-rs/2}\Gamma^r(s/2)\zeta^r(s)$. The properties of this function are studied in detail and in particular its asymptotic expansion is given.

1. Introduction

Integral representations of Dirichlet series are a major tool in Analytic Number Theory. Of special prominence is the classical Riemann-Siegel formula (see C.L. Siegel [12])

$$\begin{aligned}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\int_{0\swarrow 1}\frac{e^{i\pi x^2}x^{-s}}{e^{i\pi x}-e^{-i\pi x}}\mathrm{d}x \\ &\quad + \pi^{(s-1)/2}\Gamma\left(\frac{1-s}{2}\right)\int_{0\searrow 1}\frac{e^{-i\pi x^2}x^{s-1}}{e^{i\pi x}-e^{-i\pi x}}\mathrm{d}x.\end{aligned} \tag{1.1}$$

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This is valid for s not equal to the poles of $\Gamma(s)$, and $0 \swarrow 1$ (resp. $0 \searrow 1$) denotes a straight line which starts from infinity in the upper complex half-plane, has slope equal to 1 (resp. to -1), and cuts the real axis between 0 and 1. The integrals in (1.1) are of a fairly simple nature, and they can be evaluated asymptotically to provide precise formulas for $\zeta(s)$ (see [6], [12], [13] and (7.9)). Although (1.1) has been known for a long time, its direct generalization to other Dirichlet series, which possess functional equations with gamma-factors similar to the functional equation of $\zeta(s)$, remained an open problem. It is only in the early 1980's that Y. Motohashi [8]–[10] obtained the asymptotic expansion of $\zeta^2(s)$. His method, however, uses some intrinsic properties of the function $d(n)$ (the number of divisors of n), and cannot be readily generalized. Also due to some unfortunate circumstances (see the postscript in [10]) a detailed proof of his results was not appropriately published in due time.

It is only recently that A. Guthmann devised a general approach for obtaining integral representations for Dirichlet series, which may be regarded as a generalization of the Riemann-Siegel integral formula (1.1). In his Habilitation Thesis [2] and in [4] he obtained an analogue of (1.1) for zeta-functions of holomorphic cusp forms, and in [3] for $\zeta(s)\zeta(s+1)$. In [5] he further developed his ideas to tackle $\zeta^2(s)$. It is the purpose of this paper to obtain an analogous integral representation for $\zeta^r(s)$, where $r \ge 3$ is an arbitrary, but fixed natural number. This in turn depends on properties of the inverse Mellin transform of $\pi^{-rs/2}\Gamma^r(s/2)\zeta^r(s)$. This function, which we shall denote by $\psi_r(x)$, appears to be of intrinsic interest and it will be extensively studied in the sequel. Generalizations of our integral representations for $\zeta^r(s)$ to other Dirichlet series possessing functional equations with multiple gamma-factors are possible.

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2. The outline of the method

For $r \ge 1$ a fixed integer, $c > 0$ and $\Re\mathrm{e}\, x > 0$ let

$$f_r(x) = \frac{1}{2\pi i}\int_{(c)} \Gamma^r\left(\frac{s}{2}\right)\left(\frac{x}{2}\right)^{-s} \mathrm{d}s, \tag{2.1}$$

where the integral is absolutely convergent and, as usual,

$$\int_{(c)} F(s)\mathrm{d}s = \lim_{T\to\infty}\int_{c-iT}^{c+iT} F(s)\mathrm{d}s.$$

Then it follows by the Mellin inversion formula (see the Appendix of [6]) that

$$\Gamma^r\left(\frac{s}{2}\right)2^s = \int_0^\infty f_r(x)x^{s-1}\mathrm{d}x \quad (\Re\mathrm{e}\, s > 0). \tag{2.2}$$

If $d_r(n)$ is the number of ways in which $n\,(\in \mathbb{N})$ may be written as a product of r fixed factors ($d_1(n) = 1$, $d_2(n) = d(n)$), then for $\Re\mathrm{e}\, s > 1$

$$\zeta^r(s) = \sum_{n=1}^{\infty} d_r(n)n^{-s}.$$

Consequently by absolute convergence we have, for $\Re\mathrm{e}\, x > 0$, $c > 1$,

$$\psi_r(x) := \sum_{n=1}^{\infty} d_r(n) f_r\left(2\pi^{\frac{r}{2}}xn\right) = \frac{1}{2\pi i}\int_{(c)} \pi^{-\frac{rs}{2}}\zeta^r(s)\Gamma^r\left(\frac{s}{2}\right)x^{-s}\mathrm{d}s, \tag{2.3}$$

hence by the Mellin inversion formula we have

$$\pi^{-\frac{rs}{2}}\zeta^r(s)\Gamma^r\left(\frac{s}{2}\right) = \int_0^\infty \psi_r(x)x^{s-1}\mathrm{d}x \quad (\Re\mathrm{e}\, s > 1). \tag{2.4}$$

The shape of the left-hand side of (2.4) is such that it remains unchanged if s is replaced by $1-s$. This follows from the symmetric form of the functional equation for $\zeta(s)$ (see [6] or [13]), namely

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{(1-s)}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{2.5}$$

In fact it is precisely the symmetry furnished by (2.5) which is crucial in deriving integral representations for $\zeta^r(s)$. The function $\psi_r(x)$ is holomorphic for $\Re\mathrm{e}\, x > 0$, and it is of exponential decay as $x \to \infty$. To see this let $X = \pi^r x^2$, $c > 1$. Then from (2.3) we obtain

$$\psi_r(x) = \frac{1}{\pi i}\int_{(c)} \zeta^r(2w)\Gamma^r(w)X^{-w}\mathrm{d}w. \tag{2.6}$$

Since e^{-x}, $\Gamma(s)$ is a pair of Mellin transforms, the Parseval identity for Mellin transforms (see (A.5) of [6]) gives

$$\int_0^\infty |\Gamma(\sigma+it)|^2 \mathrm{d}t = \pi 2^{-2\sigma}\Gamma(2\sigma) \quad (\sigma > 0). \tag{2.7}$$

But for $\Re\mathrm{e}\, w \ge 2$ we have $|\zeta(w)| \le \frac{\pi^2}{6}$, and for $x > 0$ (see e.g. N.N. Lebedev [7])

$$|\Gamma(x+iy)| \le \Gamma(x) = \sqrt{2\pi}x^{x-\frac12}e^{-x}(1+r(x)), \quad |r(x)| \le e^{\frac{1}{12x}} - 1. \tag{2.8}$$

Assume now that $r \ge 2$. Since $e^x \le 1 + 2x$ for $0 \le x \le 1$ we have $|1 + r(x)| \le 7/6$ for $x \ge 1$, hence from (2.6)–(2.8) it follows that, for $x, c \ge 1$ and $r \ge 2$,

$$\begin{aligned}
|\psi_r(x)| &\le \frac{2}{\pi}\left(\frac{\pi^2}{6}\right)^r X^{-c}\int_0^\infty |\Gamma(c+it)|^2\Gamma^{r-2}(c)\mathrm{d}t \\
&= 2\left(\frac{\pi^2}{6}\right)^r X^{-c}2^{-2c}\Gamma(2c)\Gamma^{r-2}(c) \\
&\le \sqrt{2}\left(\frac{\pi^2}{6}\right)^r\left(\frac{7}{6}\sqrt{2\pi}\right)^{r-1}\exp\left(-c\log X + rc\log c - rc + \frac{1-r}{2}\log c\right) \\
&= \frac{6}{7\sqrt{\pi}}\left(\frac{7\pi^{5/2}2^{1/2}}{36}\right)^r X^{\frac{1-r}{2r}}\exp\left(-rX^{\frac1r}\right)
\end{aligned}$$

with the choice $c = X^{1/r}$ (≥ 1). Therefore we obtain

$$|\psi_r(x)| \le \frac{6}{7}\left(\frac{7\pi^2 2^{1/2}}{36}\right)^r x^{\frac{1-r}{r}}\exp\left(-r\pi x^{\frac2r}\right) \quad (x \ge 1,\ r \ge 2), \tag{2.9}$$

and from (3.2) it is seen that (2.9) also holds when $r = 1$. For an asymptotic expansion of $\psi_r(x)$ when r is fixed, see (5.11). Actually no absolute value signs are needed in (2.9), since $\psi_r(x) > 0$ (and we have $\psi_r'(x) < 0$) for $x > 0$. This follows from the series representation (2.3) and the properties of $f_r(x)$ (see (3.11) for the proof that $f_r(x) > 0$, $f_r'(x) < 0$ when $x > 0$).

It will turn out that $\psi_r(x)$ also satisfies a simple functional equation which relates its values at the points x and $1/x$. This result will be given as Theorem 4 in Section 6.

Let now $\xi = Re^{i\delta}$, where $R > 0$, $0 \le \delta < \frac{\pi}{2}$, and eventually we shall let $\delta \to \frac{\pi}{2}$. Then in (2.4) we may turn the line of integration by the angle δ around the origin to obtain

$$\begin{aligned}\int_0^\infty \psi_r(x)x^{s-1}\mathrm{d}x &= \int_0^{e^{i\delta}\infty} = \int_0^\xi + \int_\xi^{\xi\infty} \\ &= \int_{\xi^{-1}}^{\xi^{-1}\infty} \psi_r\left(\frac{1}{x}\right)x^{-s-1}\mathrm{d}x + \int_\xi^{\xi\infty} \psi_r(x)x^{s-1}\mathrm{d}x.\end{aligned} \tag{2.10}$$

Suppose temporarily that $\Re\mathrm{e}\, s > 1$. In the integral with $\psi_r(1/x)$ we use Theorem 4 (the functional equation for ψ_r) to obtain

$$\int_{\xi^{-1}}^{\xi^{-1}\infty} \psi_r\left(\frac{1}{x}\right)x^{-s-1}\mathrm{d}x = \int_{\xi^{-1}}^{\xi^{-1}\infty} \psi_r(x)x^{-s}\mathrm{d}x + H_r(s,\xi), \tag{2.11}$$

where the function $H_r(s,\xi)$ is defined by (2.17). It can be easily evaluated in terms of elementary functions, since

$$\int x^w \log^k x\mathrm{d}x = \frac{d^k}{dw^k}\left(\int x^w\mathrm{d}x\right) = \frac{d^k}{dw^k}\left(\frac{x^{w+1}}{w+1}\right) \tag{2.12}$$

for $k \in \mathbb{N}$ and $w \neq -1$. Hence applying (2.12) we obtain an analytic continuation of $H_r(s,\xi)$ which is valid for all complex s except $s = 0, 1$. For $r \ge 3$ we have, turning the line of integration so that it is again parallel to the real axis,

$$\begin{aligned}\lim_{\delta\to\frac{\pi}{2}} \int_\xi^{\xi\infty} \psi_r(x)x^{s-1}\mathrm{d}x &= \lim_{\delta\to\frac{\pi}{2}} \int_\xi^{\xi+\infty} \psi_r(x)x^{s-1}\mathrm{d}x \\ &= \int_{iR}^{iR+\infty} \psi_r(x)x^{s-1}\mathrm{d}x.\end{aligned} \tag{2.13}$$

Therefore from (2.4), (2.10) and (2.11) we obtain

$$\begin{aligned}\pi^{-\frac{rs}{2}}\Gamma^r\left(\frac{s}{2}\right)\zeta^r(s) &= \int_{iR}^{iR+\infty} \psi_r(x)x^{s-1}\mathrm{d}x \\ &\quad + \int_{\frac{1}{iR}}^{\frac{1}{iR}+\infty} \psi_r(x)x^{-s}\mathrm{d}x + H_r(s,\xi).\end{aligned} \tag{2.14}$$

Hence by analytic continuation we obtain from (2.14) the desired integral representation for $\zeta^r(s)$. It generalizes the case $r = 2$ of [5], where $R = p/q$ was a rational number. The remaining details of the proof will be given in Section 7, and the result is

Theorem 1. *For $R > 0, s \neq 0, 1,$ and $r \geq 3$ a fixed integer we have*

$$\zeta^r(s) = T_r(s, R) + X_r(s)\overline{T_r(1-\overline{s}, R^{-1})} + \pi^{\frac{rs}{2}}\Gamma^{-r}\left(\frac{s}{2}\right)H_r(s, iR), \tag{2.15}$$

where

$$X_r(s) = \pi^{rs-\frac{r}{2}}\frac{\Gamma^r\left(\frac{1-s}{2}\right)}{\Gamma^r\left(\frac{s}{2}\right)}, \tag{2.16}$$

$$T_r(s, R) = \pi^{\frac{rs}{2}}\Gamma^{-r}\left(\frac{s}{2}\right)\int_{iR}^{iR+\infty}\psi_r(x)x^{s-1}\mathrm{d}x,$$

$$H_r(s, \xi) := \int_{\xi^{-1}}^{\xi^{-1}\infty}\left(xp_{r-1}(-\log x) - p_{r-1}(\log x)\right)x^{-s-1}\mathrm{d}x, \tag{2.17}$$

and the polynomial $p_{r-1}(u)$ of degree r - 1 in u is defined by

$$p_{r-1}(\log x) = \operatorname*{Res}_{s=1}\pi^{-\frac{rs}{2}}\zeta^r(s)\Gamma^r\left(\frac{s}{2}\right)x^{1-s}.$$

3. Some properties of $f_r(x)$

It is clear that the study of the function $f_r(x)$, defined by (2.1), is essential for the understanding of the properties of the crucial function $\psi_r(x)$, defined by (2.3). The function $f_r(x)$ is in fact equal to $E_{r,0}(x/2)$ in the notation of A. Guthmann [1], where for nonnegative integers λ and ν such that $\lambda + \nu \geq 1$, and $x, c > 0$ he defined and studied the function

$$E_{\lambda,\nu}(x) := \frac{1}{2\pi i}\int_{(c)}\Gamma^\lambda\left(\frac{s}{2}\right)\Gamma^\nu(s)x^{-s}\mathrm{d}s,$$

with the aim of deriving approximate functional equations for a class of Dirichlet series.

We begin the discussion concerning $f_r(x)$ with the simplest case $r = 1$, when we have

$$f_1(x) = 2 \cdot \frac{1}{2\pi i} \int_{(c)} \Gamma(w) \left(\frac{x^2}{4} \right)^{-w} \mathrm{d}w = 2e^{-\frac{x^2}{4}}. \tag{3.1}$$

Consequently (2.4) becomes

$$\begin{aligned} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) &= \int_0^\infty \psi_1(x) x^{s-1} \mathrm{d}x, \\ \psi_1(x) &= 2 \sum_{n=1}^\infty e^{-\pi x^2 n^2} \quad (\Re\mathrm{e}\, s > 1), \end{aligned} \tag{3.2}$$

and (3.2) is in fact due already to B. Riemann [11]. Since ψ_1 is essentially a theta-function, the functional equation for the theta-function, namely

$$\theta(z) := \sum_{n=-\infty}^{\infty} e^{-\pi n^2 z} = z^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} e^{-\pi n^2 z^{-1}} \quad (\Re\mathrm{e}\, z > 0), \tag{3.3}$$

may be applied to yield a classical proof of the functional equation (2.5).

For $r = 2$ we have

$$f_2(x) = \frac{1}{2\pi i} \int_{(c)} \Gamma^2\big(\frac{s}{2}\big) \Big(\frac{x}{2}\Big)^{-s} \mathrm{d}s = 4K_0(x) \quad (c > 0,\ \Re\mathrm{e}\, x > 0), \tag{3.4}$$

where in standard notation

$$K_s(z) = \frac{1}{2} \int_0^\infty t^{s-1} \exp\left(-\frac{z}{2} \left(t + \frac{1}{t} \right) \right) \mathrm{d}t \quad (\Re\mathrm{e}\, z > 0) \tag{3.5}$$

is the modified Bessel function of the third kind (also called Macdonald's function). One can establish the second equality in (3.4) by noting that, for $\Re\mathrm{e}\, s > 0$,

$$\begin{aligned} \int_0^\infty K_0(x) x^{s-1} \mathrm{d}x &= \frac{1}{2} \int_0^\infty x^{s-1} \int_0^\infty t^{-1} \exp\left(-\frac{x}{2} \left(t + \frac{1}{t} \right) \right) \mathrm{d}t \mathrm{d}x \\ &= \Gamma(s) 2^{s-1} \int_0^\infty t^{-1} \Big(t + \frac{1}{t} \Big)^{-s} \mathrm{d}t = \Gamma(s) 2^{s-1} \int_0^\infty \frac{t^{s-1}}{(t^2+1)^s} \mathrm{d}t, \end{aligned}$$

where the change of the order of integration is justified by absolute convergence. Change of variable $x = (t^2+1)^{-1}$ in the last integral gives

$$\begin{aligned}\int_0^\infty K_0(x)x^{s-1}\mathrm{d}x &= \Gamma(s)2^{s-2}\int_0^1 x^{\frac{s}{2}-1}(1-x)^{\frac{s}{2}-1}\mathrm{d}x \\ &= \Gamma(s)2^{s-2}B\left(\frac{s}{2},\frac{s}{2}\right) = 2^{s-2}\Gamma^2\left(\frac{s}{2}\right),\end{aligned}\tag{3.6}$$

since the beta-function $B(a,b)$ satisfies

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{d}x = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad (\Re\mathrm{e}\, a,\ \Re\mathrm{e}\, b > 0).$$

From (3.6) one obtains (3.4) by Mellin inversion.

Also from the representation $K_0(z) = \int_1^\infty e^{-zt}(t^2-1)^{-1/2}\mathrm{d}t$ ($\Re\mathrm{e}\, z{>}0$) (see [7], p. 119), we obtain by change of variable $t = 1 + x/z$ that

$$\begin{aligned}K_0(z) = (2z)^{-1/2}e^{-z}\Bigg\{&\int_0^\infty e^{-x}x^{-1/2}\mathrm{d}x \\ &+ \int_0^\infty e^{-x}x^{-1/2}\left(\left(1+\frac{x}{2z}\right)^{-1/2}-1\right)\mathrm{d}x\Bigg\}.\end{aligned}\tag{3.7}$$

By analytic continuation (3.7) extends to the entire complex plane cut from 0 to $-\infty$. After some elementary estimations it follows from (3.7) that, in the complex z-plane cut from 0 to $-\infty$, we have

$$K_0(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}(1+H(z)),\tag{3.8}$$

where $H(z)$ is holomorphic and the principal branch of the square root is to be taken. Moreover, for $\Re\mathrm{e}\, z > 0$ one has $|H(z)| \le z$, while if additionally $|z| \ge 1$ is assumed, then $|H(z)| \le 1/|z|$. These properties of $K_0(z)$ were used in an essential way by A. Guthmann [5] in his work on the integral representations for $\zeta^2(s)$.

When $r \ge 3$ the situation becomes much more complicated, since it does not seem possible to find a simple expression for $f_r(x)$ in closed form, from which one can readily deduce its analytic properties and asymptotic behaviour as $x \to \infty$. However we can obtain a series expansion for $f_r(x)$ if in the defining relation (2.1) we take $c = -N-1/2$, where N is a natural number tending to infinity. In doing this we pass the poles of the integrand at $s = -2n$ ($n = 0, 1, 2, \dots$), which are of order r. We have

$$\operatorname*{Res}_{s=-2n} \Gamma^r\left(\frac{s}{2}\right)\left(\frac{x}{2}\right)^{-s} = x^{2n}(a_{0,r}(n)\log^{r-1} x + a_{1,r}(n)\log^{r-2} x + \dots + a_{r-1,r}(n))$$

with suitable constants $a_{j,r}(n)$ ($j = 0, 1, 2, \dots, r-1$) which may be explicitly evaluated. We recall Stirling's formula (asymptotic expansion) for $\Gamma(z)$ in the form (see N.N. Lebedev [7])

$$\Gamma(z) \sim e^{(z-\frac{1}{2})\log z - z + \frac{1}{2}\log(2\pi)}\left(1 + \frac{1}{12z} + \frac{1}{288z^2} + \cdots\right) \qquad (z \to \infty, |\arg z| < \pi) \tag{3.9}$$

and use it to obtain that the integral along the line $\Re\mathrm{e}\, w = -N - \frac{1}{2}$ tends to zero as $N \to \infty$. The meaning of the symbol $\sim$ in the asymptotic expansion is, as usual, that if we stop at the n-th term in the series, then the error that is made is $O_n(|z|^{-n-1})$. Hence we obtain by the residue theorem, for $\Re\mathrm{e}\, x > 0$,

$$f_r(x) = \sum_{n=0}^{\infty} x^{2n}(a_{0,r}(n)\log^{r-1} x + a_{1,r}(n)\log^{r-2} x + \dots + a_{r-1,r}(n)), \tag{3.10}$$

which shows that $\lim_{x\to 0+} f_r(x) = +\infty$. More precisely, since $a_{0,r}(0) = (-1)^{r-1}2^r/(r-1)!$ we have

$$\lim_{T\to\infty} \frac{f_r\left(\frac{1}{T}\right)}{\log^{r-1} T} = \frac{2^r}{(r-1)!} \quad (r = 1, 2, \dots).$$

For $x > 0$ the function $f_r(x)$ is positive, monotonically decreasing and (3.10) gives $\lim_{x\to\infty} f_r(x) = 0$. Namely with the change of variable

$u = xt^{-1/2}$ we have, for $x > 0$,

$$
\begin{aligned}
2\int_0^\infty e^{-\frac{x^2}{u^2}} f_r(u)\frac{\mathrm{d}u}{u} &= 2\cdot\frac{1}{2\pi i}\int_{(c)}\Gamma^r\left(\frac{s}{2}\right)2^s\left(\int_0^\infty e^{-\frac{x^2}{u^2}}u^{-1-s}\mathrm{d}u\right)\mathrm{d}s\\
&= \frac{1}{2\pi i}\int_{(c)}\Gamma^r\left(\frac{s}{2}\right)2^s x^{-s}\int_0^\infty e^{-t}t^{\frac{s}{2}-1}\mathrm{d}t\,\mathrm{d}s\\
&= \frac{1}{2\pi i}\int_{(c)}\Gamma^{r+1}\left(\frac{s}{2}\right)\left(\frac{x}{2}\right)^{-s}\mathrm{d}s = f_{r+1}(x),
\end{aligned}
$$

where the interchange of integration is permitted by absolute convergence. Hence for $r \ge 1$ we have

$$
\begin{aligned}
f_{r+1}(x) &= 2\int_0^\infty e^{-\frac{x^2}{u^2}} f_r(u)\frac{\mathrm{d}u}{u},\\
f'_{r+1}(x) &= -4x\int_0^\infty e^{-\frac{x^2}{u^2}} f_r(u)\,\frac{\mathrm{d}u}{u^3}.
\end{aligned}
\tag{3.11}
$$

In view of (3.1) we easily conclude from (3.11) by induction that $f_r(x) > 0$, $f'_r(x) < 0$ for $x > 0$ and any $r \ge 1$.

Since the residue of $\Gamma(s)$ at $s = -n$ $(n = 0, 1, 2, \dots)$ is $(-1)^n/n!$, we easily see that for $r = 1$ formula (3.10) reduces to

$$
f_1(x) = 2\sum_{n=0}^\infty \frac{(-1)^n x^{2n}}{4^n n!} = 2e^{-\frac{x^2}{4}}.
$$

To see explicitly the shape of (3.10) for $r = 2$ write

$$
f_2(x) = \frac{2}{2\pi i}\int_{(c)}\Gamma^2(w)\Big(\frac{x}{2}\Big)^{-2w}\mathrm{d}w,
\tag{3.12}
$$

so that now the integrand has poles of second order at $n = 0, -1, -2, \dots$. Near $w = -n$ we have the expansions

$$
\begin{aligned}
z^{-2w} &= z^{2n}(1 - 2(w+n)\log z + (w+n)^2\log^2 z + \cdots),\\
\Gamma(w) &= \frac{(-1)^n}{n!(w+n)} + c(n) + c_1(n)(w+n) + \cdots\,,\\
\Gamma^2(w) &= \frac{1}{(n!)^2(w+n)^2} + \frac{2(-1)^n c(n)}{n!(w+n)} + \cdots\,.
\end{aligned}
$$

Note that the constant $c(n)$ satisfies, by l'Hospital's rule,

$$
\begin{aligned}
c(n) &= \lim_{w\to -n}\left(\Gamma(w) - \frac{(-1)^n}{n!(w+n)}\right) = \lim_{w\to -n}\frac{(w+n)\Gamma(w) - \frac{(-1)^n}{n!}}{w+n}\\
&= \lim_{w\to -n}(\Gamma(w)+\Gamma'(w)(w+n)) = \lim_{w\to -n}(w+n)\Gamma(w)\left(\frac{1}{w+n}+\frac{\Gamma'(w)}{\Gamma(w)}\right)\\
&= \frac{(-1)^n}{n!}\lim_{w\to -n}\left(\frac{1}{w+n}+\psi(w)\right),
\end{aligned}
$$

where as usual

$$
\psi(w) := (\log\Gamma(w))' = \frac{\Gamma'(w)}{\Gamma(w)}.
$$

But from the reflection property $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ it follows by logarithmic differentiation that

$$
\psi(z) = \psi(1-z) - \pi\cot(\pi z),
$$

which gives

$$
c(n) = \frac{(-1)^n}{n!}\,\psi(n+1),
$$

with (see N.N. Lebedev [7])

$$
\psi(n+1) = 1+\frac{1}{2}+\cdots+\frac{1}{n}-\gamma, \quad \gamma=-\Gamma'(1)=0.57721\ldots \text{ (Euler's constant).}
$$

Therefore we have

$$
\operatorname*{Res}_{w=-n}\Gamma^2(w)z^{-2w} = -\frac{2z^{2n}}{(n!)^2}(\log z - \psi(n+1)) \quad (n=0,1,2,\ldots),
$$

and from (3.12) we obtain by the residue theorem

$$
f_2(x) = -4\sum_{n=0}^{\infty}\frac{(\frac{x}{2})^{2n}}{(n!)^2}(\log\frac{x}{2} - \psi(n+1)). \tag{3.13}
$$

On comparing (3.4) and (3.13) we obtain

$$
K_0(x) = -\sum_{n=0}^{\infty}\frac{(\frac{x}{2})^{2n}}{(n!)^2}(\log\frac{x}{2} - \psi(n+1)),
$$

which is the well-known series expansion (see N.N. Lebedev [7]) of $K_0(x)$.

The analysis that led to (3.10) can be carried further, as was done for the functions $E_{\lambda,\nu}(x)$ by Guthmann [1]. In particular, his Lemma 3 gives that there exist entire functions $H_{r,j}(x)$ such that, for x in the complex plane cut form $-\infty$ to 0,

$$f_r(x) = \sum_{j=0}^{r-1} H_{r,j}(x) \log^j x,$$

and the coefficients in the power series expansion for $H_{r,j}(x)$ can be evaluated explicitly.

4. The differential equation satisfied by $f_r(x)$

The function $f_r(x)$, defined by (2.1), satisfies the differential equations

$$\begin{aligned}
&\frac{y'}{x} = -\frac{1}{2}y \quad (r=1), \qquad y'' + \frac{y'}{x} = y \quad (r=2),\\
&xy''' + 3y'' + \frac{y'}{x} = -2y \quad (r=3),\\
&x^2y^{(4)} + 6xy''' + 7y'' + \frac{y'}{x} = 4y \quad (r=4),\\
&x^3y^{(5)} + 10x^2y^{(4)} + 25xy''' + 15y'' + \frac{y'}{x} = -8y \quad (r=5),
\end{aligned}$$

etc. Note that the differential equation for $r = 1$ trivially follows from $f_1(x) = 2e^{-x^2/4}$, while the one for $r = 2$ is a consequence of the fact that the Bessel functions $I_\nu(z)$ and $K_\nu(z)$ are (linearly independent) solutions of the differential equation

$$y'' + \frac{y'}{x} - \left(1 + \frac{\nu^2}{x^2}\right) y = 0.$$

In general, $f_r(x)$ satisfies a relatively simple differential equation of order r, which is linear and homogeneous, and whose special cases are the examples given above. This fact may provide useful information on the behaviour of $f_r(x)$. Recall that the Stirling numbers $S(n,m)$ of the second kind denote the number of ways of partitioning a set of $n\,(\geq m) \geq 1$ elements into m non-empty subsets. They satisfy the relation

$$x^n = \sum_{m=1}^{n} S(n,m)x(x-1)\dots(x-m+1). \tag{4.1}$$

Then we have

Theorem 2. *For $r \geq 1$ the function $y = f_r(x)$ satisfies the differential equation*

$$\sum_{j=1}^{r} S(r,j)x^{j-2}y^{(j)} = (-1)^r 2^{r-2} y. \tag{4.2}$$

Proof. Setting $x = -s$ in (4.1) we obtain

$$(-1)^n s^n = \sum_{m=1}^{n} (-1)^m S(n,m)s(s+1)\dots(s+m-1). \tag{4.3}$$

From (2.1) we obtain

$$y^{(j)} = f_r^{(j)}(x) = \frac{(-1)^j}{2\pi i}\int_{(c)} \Gamma^r\left(\frac{s}{2}\right) 2^s s(s+1)\dots(s+j-1)x^{-s-j}\mathrm{d}s$$
$$(c > 0,\ j = 1,2,\dots).$$

In view of $z\Gamma(z) = \Gamma(z+1)$ we obtain from (4.3) (with $n = r$), for $c > 2$,

$$\begin{aligned}
\sum_{j=1}^{r} S(r,j)x^{j-2}y^{(j)} &= \frac{1}{2\pi i}\int_{(c)} \Gamma^r\left(\frac{s}{2}\right) 2^s x^{-s-2} \\
&\quad \times \sum_{j=1}^{r} (-1)^j S(r,j)s(s+1)\dots(s+j-1)\mathrm{d}s \\
&= \frac{(-1)^r}{2\pi i}\int_{(c)} \Gamma^r\left(\frac{s}{2}\right) 2^s s^r x^{-s-2}\mathrm{d}s \\
&= \frac{(-1)^r}{2\pi i} 2^{r-2}\cdot \int_{(c)} \Gamma^r\left(\frac{s+2}{2}\right) 2^{s+2} x^{-(s+2)}\mathrm{d}s \\
&= (-1)^r 2^{r-2}\cdot \frac{1}{2\pi i}\int_{(c-2)} \Gamma^r\left(\frac{w}{2}\right)\left(\frac{x}{2}\right)^{-w}\mathrm{d}w = (-1)^r 2^{r-2} y.
\end{aligned}$$

Since $S(r,1) = S(r,r) = 1, S(r,r-1) = \frac{r}{2}(r-1)$ we easily compute $S(r,j)$ for $1 \le r,\, j \le 5$ to obtain from (4.2) the examples stated at the beginning of this section. Further examples can be obtained by using a table of Stirling numbers of the second kind.

One can obtain another variant of the differential equation satisfied by $f_r(x)$. Namely, let

$$\varphi_r(x) := f_r(2\sqrt{x}), \tag{4.4}$$

so that

$$f_r(x) = \varphi_r\left(\frac{x^2}{4}\right). \tag{4.5}$$

Then we obtain

$$\varphi_r(x) = \frac{2}{2\pi i}\int_{(c)} \Gamma^r(w)x^{-w}\mathrm{d}w \quad (c > 0,\ \Re\mathrm{e}\,, x > 0). \tag{4.6}$$

If Δ is the differential operator defined by

$$\Delta^1 f(x) = \Delta f(x) := xf'(x), \quad \Delta^r = \Delta(\Delta^{r-1}) \quad (r \ge 2),$$

then from (4.6) and the functional equation $z\Gamma(z) = \Gamma(z+1)$ we obtain the differential equation satisfied by $\varphi_r(x)$ in the form

$$\Delta^r\,\varphi_r(x) = (-1)^r x\varphi_r(x). \tag{4.7}$$

Similarly to (4.2), we can ascertain that (4.7) is a also linear, homogeneous differential equation of order r. The equation (4.7) is in fact equivalent to equation (42) of A. Guthmann [1], but the advantage of (4.2) over (4.7) is that (4.2) gives quite explicitly the shape of the differential equation in question, whereas (4.7) does not.

5. The asymptotic expansion of $f_r(x)$

It seems of interest to find the asymptotic expansion of $f_r(x)$ and $\psi_r(x)$ as $x \to \infty$ in terms of elementary functions. Consider first $f_r(x)$.

For $r = 1$ there is nothing to be done since $f_1(x) = 2e^{-x^2/4}$, and for $r = 2$ we have

$$f_2(x) \sim \left(\frac{8\pi}{x}\right)^{1/2} e^{-x}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n[(2n-1)!!]^2}{2^{3n}n!}\, x^{-n}\right) \qquad (\Re\mathrm{e}\, x > 0). \tag{5.1}$$

The asymptotic expansion (5.1) follows from (3.4) and the corresponding asymptotic expansion (see (5.11.9) of N.N. Lebedev [7])

$$K_\nu(z) \sim \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}\sum_{n=0}^{\infty}(\nu, n)(2z)^{-n} \quad (|\arg z| \le \pi - \delta),$$

where $0 < \delta < \frac{\pi}{2}$, $(\nu, 0) = 1$ and for $k \ge 1$

$$(\nu, k) = \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)\ldots(4\nu^2 - (2k-1)^2)}{2^{2k}k!}.$$

For the general case of $f_r(x)$ we could appeal to Lemma 6 of A. Guthmann [1], who gave the asymptotic expansion of the function

$$E_{\lambda,\nu}(x) = \frac{1}{2\pi i}\int_{(c)} \Gamma^\lambda\left(\frac{s}{2}\right)\Gamma^\nu(s)x^{-s}\mathrm{d}s \quad (c > 0),$$

and use the fact that $f_r(x) = E_{r,0}(x/2)$. Guthmann's proof of the asymptotic expansion of $E_{\lambda,\nu}(x)$ is long and complicated. It uses, among other things, the convolution property that

$$\int_0^\infty f(u)g\left(\frac{x}{u}\right)\frac{\mathrm{d}u}{u} = \frac{1}{2\pi i}\int_{(c)} F(s)G(s)x^{-s}\mathrm{d}s,$$

if f, F and g, G are two pairs of Mellin transforms, and certain conditions are satisfied. We shall obtain here the asymptotic expansion of $f_r(x)$ by another method, which is simpler and can be readily generalized. Namely if $a_1, a_2, \ldots, a_r > 0$, let us define

$$\begin{aligned} H(x) &= H(x; a_1, \ldots, a_r) \\ &= \frac{1}{2\pi i}\int_{(c)} \Gamma(a_1 w)\Gamma(a_2 w)\cdots\Gamma(a_r w)x^{-w}\mathrm{d}w \quad (c > 0). \end{aligned} \tag{5.2}$$

Then for $a_1 = \ldots = a_r = 1/2$ we obtain $H(x) = f_r(2x)$, and for $a_1 = \ldots = a_\lambda = 1/2, a_{\lambda+1} = \ldots = a_r = 1$ $(r = \lambda + \nu)$ we obtain $H(x) = E_{\lambda,\nu}(x)$. We have

Theorem 3. *Let $x \to \infty$ and*

$$B = \frac{(2\pi)^{\frac{r-1}{2}}}{\sqrt{a_1 \cdots a_r(a_1 + \cdots + a_r)}}\left(a_1^{a_1} \cdots a_r^{a_r}\right)^{\frac{r-1}{2(a_1+\cdots+a_r)}},$$

$$D = \frac{a_1 + \cdots + a_r}{\left(a_1^{a_1} \cdots a_r^{a_r}\right)^{\frac{1}{a_1+\cdots+a_r}}}, \quad X = Dx^{\frac{1}{a_1+\cdots+a_r}}. \tag{5.3}$$

Then there exist constants $e_1, e_2, \ldots$, which can be explicitly evaluated and which depend on $a_1, \ldots , a_r$, such that

$$H(x) \sim Bx^{\frac{1-r}{2(a_1+\cdots+a_r)}} e^{-X} \left(1 + \frac{e_1}{X} + \frac{e_2}{X^2} + \cdots\right). \tag{5.4}$$

Proof. The basic tool in the proof is Stirling's formula (3.9), which for $a > 0$ gives, with $|\arg z| < \pi$ and some constants $c_1, c_2, \ldots$ depending on a and b,

$$\Gamma(az + b) \sim \sqrt{2\pi}e^{-az}(az)^{az+b-1/2}\left(1 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots\right). \tag{5.5}$$

Using (5.5) we obtain, for some constants $d_1, d_2, \ldots$ which depend on $a_1, \ldots , a_r$,

$$\begin{aligned}
&\frac{\Gamma(a_1w) \cdots \Gamma(a_rw)}{\Gamma\Big((a_1 + \cdots + a_r)w + \frac{1-r}{2}\Big)} \sim \left(1 + \frac{d_1}{w} + \frac{d_2}{w^2} + \cdots\right)\\
&\times (2\pi)^{\frac{r-1}{2}} \exp\Big\{w\Big(a_1 \log a_1 + \ldots + a_r \log a_r\\
&- (a_1 + \ldots + a_r) \log(a_1 + \ldots + a_r)\Big)\\
&- \frac{1}{2}\log a_1 - \ldots - \frac{1}{2}\log a_r + \frac{r}{2}\log(a_1 + \ldots + a_r)\Big\}.
\end{aligned} \tag{5.6}$$

The transformation formula (5.6) is the crucial step in deriving (5.4). Now we take $c(a_1 + \cdots a_r) > N + 1$ in (5.2), where $N \ge 1$ is any fixed integer, insert (5.6) and make the change of variable $(a_1 + \cdots a_r)w + \frac{1-r}{2} = s$. If B and D are given by (5.3), we obtain, for a suitable constant $C > N + 1$, suitable constants e_j (which depend on $a_1, a_2, \dots$ and may be evaluated explicitly) and a function $h_N(s)$ which is regular and $\ll 1$ for $\Re\text{e}\, s \ge N+1$,

$$
\begin{aligned}
H(x) = \frac{B}{2\pi i}\int_{(c)} \Gamma(s) D^{-s} x^{\frac{1-r-2s}{2(a_1+\cdots+a_r)}} \\
\times\Bigl(1+\frac{e_1}{s-1}+\ldots+\frac{e_N}{(s-1)(s-2)\cdots(s-N)} \\
+\frac{h_N(s)}{(s-1)(s-2)\cdots(s-N-1)}\Bigr)\mathrm{d}s.
\end{aligned}
\tag{5.7}
$$

If we use

$$
z\Gamma(z) = \Gamma(z+1), \quad e^{-x} = \frac{1}{2\pi i}\int_{(c)} \Gamma(s) x^{-s}\mathrm{d}s \quad (c>0, \Re\text{e}\ x>0),
$$

then we obtain from (5.7)

$$
\begin{aligned}
H(x) = Bx^{\frac{1-r}{2(a_1+\cdots+a_r)}}\Bigl\{e^{-X}\Bigl(1+\frac{e_1}{X}+\cdots+\frac{e_N}{X^N}\Bigr) \\
+ O_N\Bigl(\Bigl|\int_{(C)} \Gamma(s-N-1)h_N(s)X^{-s}\mathrm{d}s\Bigr|\Bigr)\Bigr\} \quad (C>N+1),
\end{aligned}
\tag{5.8}
$$

where $X = Dx^{1/(a_1+\cdots+a_r)}$. Hence (5.4) will follow if we can show that, for $C > N+1$ and $Y \to \infty$,

$$
I_C(Y) := \frac{1}{2\pi i}\int_{(C)} \Gamma(s-N-1)h_N(s)Y^{-s}\mathrm{d}s \ll Y^{-N-1}e^{-Y}.
\tag{5.9}
$$

To obtain (5.9) let $s = N+1+w$ and use the duplication formula for $\Gamma(s)$ in the form

$$
\Gamma(s) = \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)2^{s-1}\pi^{-1/2}.
$$

Then by the Cauchy–Schwarz inequality for integrals and (2.7)–(2.8) we obtain (changing $C-N-1$ to C)

$$
\begin{aligned}
I_C(Y) &= \frac{Y^{-N-1}}{2\pi i}\int_{(C)}\Gamma(w)h_N(N+1+w)Y^{-w}\mathrm{d}w \\
&\ll Y^{-N-1-C}\int_0^\infty |\Gamma(C+iv)|\mathrm{d}v \\
&\ll 2^C Y^{-N-1-C}\int_0^\infty \left|\Gamma\left(\frac{C}{2}+\frac{iv}{2}\right)\Gamma\left(\frac{C+1}{2}+\frac{iv}{2}\right)\right|\mathrm{d}v \\
&\ll 2^C Y^{-N-1-C}\Big(\int_0^\infty \left|\Gamma\left(\frac{C}{2}+\frac{iv}{2}\right)\right|^2\mathrm{d}v \\
&\quad\times\int_0^\infty \left|\Gamma\left(\frac{C+1}{2}+\frac{iy}{2}\right)\right|^2\mathrm{d}y\Big)^{1/2} \\
&\ll \Big(\Gamma(C)\Gamma(C+1)\Big)^{1/2}Y^{-N-1-C} = C^{1/2}\Gamma(C)Y^{-N-1-C} \\
&\ll e^{C\log C-C-C\log Y}Y^{-N-1} \ll e^{-Y}Y^{-N-1}
\end{aligned}
$$

with the choice $C=Y$. Thus (5.4) follows from (5.8) and (5.9), since N may be arbitrary.

From (5.3) we obtain the asymptotic expansion of $f_r(x)=H(x/2)$ for $a_1=\cdots=a_r=1$. In this case we can calculate explicitly without much trouble the first few coefficients e_j. The result is

Corollary 1. *If $r\ge 1$, $\Re\mathrm{e}\, x>0$ and $x\to\infty$, then we have*

$$
\begin{aligned}
f_r(x) &\sim 2r^{-\frac12}(2\pi)^{\frac{r-1}{2}}2^{\frac{r-1}{r}}x^{\frac{1-r}{r}}e^{-r(\frac{x}{2})^{\frac{2}{r}}} \\
&\quad\times\Big(c_{0,r}+c_{1,r}x^{-\frac{2}{r}}+c_{2,r}x^{-\frac{4}{r}}+\dots\Big),
\end{aligned}
\tag{5.10}
$$

where $c_{0,r}=1$, $c_{1,r}=2^{2/r}(1-r^2)/(24r)$, and the other $c_{j,r}$'s are also effectively computable. Note that the range of validity of the asymptotic expansion in Theorem 3 can be extended to $|\arg x|<\frac{\pi}{2}(a_1+\ldots+a_r)$ and in Corollary 1 to $|\arg x|<\frac{\pi r}{4}$.

Setting $r=2$ in (5.10) we obtain

$$
f_2(x) = \left(\frac{8\pi}{x}\right)^{1/2}e^{-x}\left(1-\frac{1}{8x}+O\left(\frac{1}{|x|^2}\right)\right),
$$

which is in accordance with (5.1). Also we may note that by the method of proof of (5.10) we may similarly obtain an asymptotic expansion of $\psi_r(x)$, starting from (2.3) and noting that

$$\zeta^r(s) = 1 + O_r\left(\frac{1}{2^\sigma}\right) \quad (\sigma = \Re\mathrm{e}\, s \to \infty).$$

We shall obtain

Corollary 2. *For* $r \ge 1$, $\Re\mathrm{e}\, x > 0$ *and* $x \to \infty$

$$\psi_r(x) \sim x^{\frac{1-r}{r}} e^{-\pi r x^{\frac{2}{r}}} \left(b_{0,r} + b_{1,r}x^{-\frac{2}{r}} + b_{2,r}x^{-\frac{4}{r}} + \dots\right) \tag{5.11}$$

with suitable coefficients $b_{j,r}$ $(j = 0, 1, 2, \dots)$, *which may be explicitly evaluated.*

If in (2.1) we take $0 < c < 1$, replace s by w then by absolute convergence for $\Re\mathrm{e}\, s > 0$ we may change the order of integration to obtain

$$\begin{aligned} \int_0^\infty f_r(x)e^{-sx}\mathrm{d}x &= \frac{1}{2\pi i}\int_{(c)} \Gamma^r\left(\frac{w}{2}\right) 2^w \int_0^\infty x^{-w}e^{-sx}\mathrm{d}x\mathrm{d}w \\ &= \frac{1}{2\pi i}\int_{(c)} \Gamma^r\left(\frac{w}{2}\right) 2^w\Gamma(1-w)s^{w-1}\mathrm{d}w. \end{aligned} \tag{5.12}$$

Now we take $s = 1/T$, $T \to \infty$, $c = 1/2$ and make the substitution $1 - w = z$. We obtain from (5.12)

$$\int_0^\infty f_r(x)e^{-x/T}\mathrm{d}x = \frac{1}{2\pi i}\int_{(c)} \Gamma^r\left(\frac{1-z}{2}\right) 2^{1-z}\Gamma(z)T^z\mathrm{d}z.$$

Shifting the line of integration in the last integral to $-\infty$ and applying the residue theorem we obtain

$$\int_0^\infty f_r(x)e^{-x/T}\mathrm{d}x \sim \sum_{n=0}^\infty \Gamma^r\big(\frac{n+1}{2}\big)\frac{2^{n+1}}{n!}\cdot\left(\frac{-1}{T}\right)^n \quad (T \to \infty),$$

which is the asymptotic expansion of the Laplace transform of $f_r(x)$ as $s = 1/T \to 0+$.

Note also that the Parseval formula for Mellin transforms gives the identity

$$\int_0^\infty f_r^2(x)x^{2\sigma-1}\mathrm{d}x = \frac{2^{1+2\sigma}}{\pi}\int_0^\infty \left|\Gamma\left(\frac{\sigma}{2} + ix\right)\right|^{2r}\mathrm{d}x \quad (\sigma > 0),$$

which for $r = 1$ reduces to (2.7).

6. The functional equation for $\psi_r(x)$

As mentioned in Section 2, the function $\psi_r(x)$ satisfies a simple functional equation relating its values at the points x and $1/x$. This result, which is an essential ingredient in the proof of the integral representation for $\zeta^r(s)$ furnished by Theorem 1, may be obtained as follows. From (2.3) we have by the residue theorem

$$\begin{aligned}\psi_r(x) &= \operatorname*{Res}_{s=1} \pi^{-\frac{rs}{2}} \zeta^r(s)\Gamma^r\left(\frac{s}{2}\right) x^{-s} \\ &\quad + \frac{1}{2\pi i}\int_{(\frac{1}{2})} \pi^{-\frac{rs}{2}} \zeta^r(s)\Gamma^r\left(\frac{s}{2}\right) x^{-s}\mathrm{d}s. \end{aligned}\tag{6.1}$$

Setting

$$g_r(x) := \frac{1}{2\pi i}\int_{(\frac{1}{2})} \pi^{-\frac{rs}{2}} \zeta^r(s)\Gamma^r\left(\frac{s}{2}\right) x^{-s}\mathrm{d}s \quad (\Re\mathrm{e}\, x > 0)$$

we have, by the functional equation (2.5) (raised to the r-th power) and the change of variable $1 - s = w$,

$$\begin{aligned} g_r(\frac{1}{x}) &= \frac{1}{2\pi i}\int_{(\frac{1}{2})} \pi^{-\frac{rs}{2}} \zeta^r(s)\Gamma^r\left(\frac{s}{2}\right) x^{s}\mathrm{d}s \\ &= \frac{1}{2\pi i}\int_{(\frac{1}{2})} \pi^{-\frac{r(1-s)}{2}} \zeta^r(1-s)\Gamma^r\left(\frac{1-s}{2}\right) x^{s}\mathrm{d}s \\ &= \frac{1}{2\pi i}\int_{(\frac{1}{2})} \pi^{-\frac{rw}{2}} \zeta^r(w)\Gamma^r\left(\frac{w}{2}\right) x^{1-w}\mathrm{d}w = x g_r(x), \end{aligned}$$

hence

$$g_r(x) = \frac{1}{x} g_r\left(\frac{1}{x}\right) \quad (\Re\mathrm{e}\, x > 0).\tag{6.3}$$

Since $\zeta^r(s)$ has at $s = 1$ a pole of order r, it follows that

$$\operatorname*{Res}_{s=1} \pi^{-\frac{rs}{2}} \zeta^r(s)\Gamma^r\left(\frac{s}{2}\right) x^{-s} = \frac{1}{x} p_{r-1}(\log x),\tag{6.4}$$

where $p_{r-1}(u)$ is a polynomial of degree $r-1$ in u whose coefficients, which depend on r, may be effectively evaluated, since the Laurent expansions at $s=1$ of $\zeta(s)$ and $\Gamma(s)$ are well-known. From (6.1), (6.2) and (6.4) it follows that

$$g_r(x) = \psi_r(x) - \frac{1}{x}p_{r-1}(\log x),$$

and (6.3) yields then

$$\frac{1}{x}\psi_r\left(\frac{1}{x}\right) - p_{r-1}(-\log x) = \frac{1}{x}g_r\left(\frac{1}{x}\right) = g_r(x) = \psi_r(x) - \frac{1}{x}p_{r-1}(\log x).$$

This means that we have proved

Theorem 4. *If $\psi_r(x)$ is defined by* (2.3) *and $p_{r-1}(\log x)$ by* (6.4), *then for $r \ge 1$ and $\Re\mathrm{e}\, x > 0$ we have*

$$\psi_r\left(\frac{1}{x}\right) = x\psi_r(x) - p_{r-1}(\log x) + xp_{r-1}(-\log x). \tag{6.5}$$

The functional equation (6.5) shows that $\psi_r(x) \asymp x^{-1}\log^{r-1} x$ as $x \to 0+$. In particular we have

$$p_0(\log x) = \operatorname*{Res}_{s=1} \pi^{-\frac{s}{2}}\zeta(s)\Gamma\left(\frac{s}{2}\right) = 1,$$

since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\lim_{s\to 1} \zeta(s)(s-1) = 1$. Hence

$$\psi_1\left(\frac{1}{x}\right) = x\psi_1(x) + x - 1,$$

which also follows from (3.3), since $\psi_1(x) = \theta(x^2) - 1$. We also have

$$p_1(u) = \gamma - 2\log 2 - \log\pi - u, \tag{6.6}$$

where γ is Euler's constant as before. Namely near $s=1$ we have

$$\zeta^2(s) = \frac{1}{(s-1)^2} + \frac{2\gamma}{s-1} + a + \dots\,,$$

$$(\pi x)^{-(s-1)} = 1 - (s-1)\log(\pi x) + \dots\,,$$

$$\Gamma(\frac{s}{2}) = \Gamma\left(\frac{1}{2}\right) + \frac{1}{2}\Gamma'\left(\frac{1}{2}\right)(s-1) + \dots\,.$$

But in view of $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and (see N.N. Lebedev [7]) $\psi(\frac{1}{2}) = \frac{\Gamma'}{\Gamma}(\frac{1}{2}) = -\gamma - 2\log 2$, we have

$$\Gamma'\left(\frac{1}{2}\right) = -\sqrt{\pi}(\gamma + 2\log 2),$$
$$\Gamma^2\left(\frac{s}{2}\right) = \pi - \pi(\gamma + 2\log 2)(s-1) + \cdots .$$

We obtain

$$p_1(\log x) = \frac{1}{\pi}\operatorname*{Res}_{s=1}(\pi x)^{-(s-1)}\zeta^2(s)\Gamma^2\left(\frac{s}{2}\right) = \gamma - 2\log 2 - \log\pi - \log x,$$

so that (6.6) follows. By more cumbersome calculations we may evaluate p_{r-1} for $r \geq 3$.

7. Integral representations of $\zeta^r(s)$

In this section we shall complete the proof of Theorem 1, outlined in Section 2, and then furnish yet another integral representation of $\zeta^r(s)$. To complete the proof of Theorem 1 we need to show that (2.9) holds, and that (2.11)–(2.13) follows from (2.10). To prove (2.9) it is enough to show that

$$\lim_{\delta\to\frac{\pi}{2}}\int_{\xi}^{\xi+1}\psi_r(x)x^{s-1}\mathrm{d}x = \int_{iR}^{iR+1}\psi_r(x)x^{s-1}\mathrm{d}x \tag{7.1}$$
$$\left(\xi = Re^{i\delta},\ R>0,\ r\geq 3,\ 0\leq\delta<\frac{\pi}{2}\right),$$

since $\psi_r(x)$ decays exponentially at ∞ by Theorem 3. If $r \geq 3$, $\Re\mathrm{e}\, x > 0$, $c > 1$ is fixed, then from (2.3) and (3.9) we obtain (since $|\arg\frac{x}{2}| \leq \frac{\pi}{2}$)

$$\psi_r(x) \ll \left|\frac{x}{2}\right|^{-c}\left(1 + \int_{t_0}^{\infty} e^{t|\arg\frac{x}{2}|}t^{\frac{r(c-1)}{2}}e^{-\frac{\pi r t}{4}}\mathrm{d}t\right) \ll |x|^{-c}, \tag{7.2}$$

and a bound analogous to (7.2) will also hold for $\psi_r'(x)$. Hence from (7.2) we obtain ($\xi = Re^{i\delta}$)

$$\begin{aligned}
&\int_{\xi}^{\xi+1}\psi_r(x)x^{s-1}\mathrm{d}x - \int_{iR}^{iR+1}\psi_r(x)x^{s-1}\mathrm{d}x\\
&= \int_0^1\Big(\psi_r(Re^{i\delta}+u)(Re^{i\delta}+u)^{s-1} - \psi_r(iR+u)(iR+u)^{s-1}\Big)\mathrm{d}u\\
&= \int_0^1\int_{Re^{\frac{i\pi}{2}}}^{Re^{i\delta}}\Big(\psi_r'(v+u)(v+u)^{s-1} + \psi_r(v+u)(s-1)(v+u)^{s-2}\Big)\mathrm{d}v\mathrm{d}u\\
&\ll_{R,s} \max_{0\le u\le 1}|e^{i\delta} - e^{\frac{i\pi}{2}}| \ll_{R,s} \Big|\delta - \frac{\pi}{2}\Big|
\end{aligned}$$

since $R \le |v+u| \le R+1$. Letting $\delta \to \frac{\pi}{2}$ we obtain (7.1).

To complete the proof of Theorem 1, note that (2.10) gives

$$\zeta^r(s) = T_r(s,R) \tag{7.3}$$
$$+\,\pi^{\frac{rs}{2}}\Gamma^{-r}\left(\frac{s}{2}\right)\int_{\frac{1}{iR}}^{\frac{1}{iR}+\infty}\psi_r(x)x^{-s}\mathrm{d}x + \pi^{\frac{rs}{2}}\Gamma^{-r}\left(\frac{s}{2}\right)H_r(s,iR),$$

where $T_r(s,R)$ is given by (2.16) and $H_r(s,iR)$ by (2.17). Since $\overline{\psi_r(x)} = \psi_r(\overline{x})$ and $\overline{a^b} = \overline{a}^{\overline{b}}$, we obtain

$$\begin{aligned}
\overline{T_r(1-\overline{s},R^{-1})} &= \pi^{\frac{r(1-s)}{2}}\Gamma^{-r}\left(\frac{1-s}{2}\right)\overline{\int_{\frac{1}{iR}}^{\frac{1}{iR}+\infty}\psi_r(x)x^{-\overline{s}}\mathrm{d}x}\\
&= \pi^{\frac{r(1-s)}{2}}\Gamma^{-r}\left(\frac{1-s}{2}\right)\overline{\int_0^{\infty}\psi_r\left(\frac{i}{R}+u\right)\left(\frac{i}{R}+u\right)^{-\overline{s}}\mathrm{d}u}\\
&= \pi^{\frac{r(1-s)}{2}}\Gamma^{-r}\left(\frac{1-s}{2}\right)\int_0^{\infty}\psi_r\left(-\frac{i}{R}+u\right)\left(-\frac{i}{R}+u\right)^{-s}\mathrm{d}u\\
&= \pi^{\frac{r(1-s)}{2}}\Gamma^{-r}\left(\frac{1-s}{2}\right)\int_0^{\infty}\psi_r\left(\frac{1}{iR}+u\right)\left(\frac{1}{iR}+u\right)^{-s}\mathrm{d}u\\
&= \pi^{\frac{r(1-s)}{2}}\Gamma^{-r}\left(\frac{1-s}{2}\right)\int_{\frac{1}{iR}}^{\frac{1}{iR}+\infty}\psi_r(x)x^{-s}\mathrm{d}x.
\end{aligned}$$

Therefore

$$\int_{\frac{1}{iR}}^{\frac{1}{iR}+\infty} \psi_r(x)x^{-s}\mathrm{d}x = \pi^{\frac{r(s-1)}{2}}\Gamma^r\left(\frac{1-s}{2}\right)\overline{T_r(1-\overline{s},R^{-1})},$$

and if we insert this expression in (7.3) we obtain (2.15). Although the value of R is not specified, from the definition of T_r it follows that optimal symmetry in (2.15) will be attained in the case when $R=1$.

Another integral representation for $T_r(s,R)$, which generalizes equation (3.1) of A. Guthmann [5], is given by

Theorem 5. *If $T_r(s,R)$ is given by* (2.16) *and $0 \le \varphi \le \frac{\pi}{2}$, then for $r \ge 1$ fixed we have*

$$T_r(s,R) = \sin^r\left(\frac{\pi s}{2}\right)\int_0^{\infty e^{-i\varphi}} u^{1-s}\left(\int_{iR}^{iR+\infty} x\psi_r(x)f_r(2\pi^{\frac{r}{2}}xu)\mathrm{d}x\right)\mathrm{d}u. \tag{7.4}$$

Proof. The result is formulated for $r \ge 1$, since it does not depend on (7.2), like the proof of (2.13) does. For $\Re\mathrm{e}\, s < 2$ we have, from (2.2),

$$\Gamma^r\left(\frac{2-s}{2}\right)2^{2-s} = \int_0^\infty f_r(x)x^{1-s}\mathrm{d}x. \tag{7.5}$$

From $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ with $z=(2-s)/2$ we have

$$\Gamma^r\left(\frac{2-s}{2}\right) = \frac{\pi^r}{\sin^r\left(\frac{\pi s}{2}\right)\Gamma^r\left(\frac{s}{2}\right)}.$$

Hence (7.5) yields

$$\frac{1}{\Gamma^r\left(\frac{s}{2}\right)} = 2^{s-2}\pi^{-r}\sin^r\left(\frac{\pi s}{2}\right)\int_0^\infty f_r(z)z^{1-s}\mathrm{d}z. \tag{7.6}$$

In (7.6) we turn the line of integration about the origin by the angle φ and make the change of variable $z=\alpha u$ to obtain

$$\frac{1}{\Gamma^r\left(\frac{s}{2}\right)} = 2^{s-2}\pi^{-r}\sin^r\left(\frac{\pi s}{2}\right)\alpha^{2-s}\int_0^{\infty e^{-i\varphi}} f_r(\alpha u)u^{1-s}\mathrm{d}u. \tag{7.7}$$

Taking $\alpha = 2\pi^{\frac{r}{2}}x$ we obtain then from (7.7)

$$\frac{\pi^{\frac{rs}{2}}}{\Gamma^r\left(\frac{s}{2}\right)} = \sin^r\left(\frac{\pi s}{2}\right)x^{2-s}\int_0^{\infty e^{-i\varphi}} f_r(2\pi^{\frac{r}{2}}xu)u^{1-s}\mathrm{d}u.$$

From this expression and the definition (2.16) of $T_r(s,R)$ we have

$$T_r(s,R) = \sin^r\left(\frac{\pi s}{2}\right)\int_{iR}^{iR+\infty} x\psi_r(x)\int_0^{\infty e^{-i\varphi}} u^{1-s}f_r(2\pi^{\frac{r}{2}}xu)\mathrm{d}u\mathrm{d}x. \tag{7.8}$$

By absolute convergence of the double integral we may change the order of integration in (7.8), and (7.4) follows.

The integral representations for $\zeta^r(s)$ furnished by Theorem 1 and Theorem 5 may be regarded as initial steps towards an asymptotic expansion of $\zeta^r(s)$ (in terms of elementary functions). Namely from (1.1) it is possible to derive an asymptotic expansion of $\zeta(s)$, as was shown by C.L. Siegel [12]. A variant of this important formula states that

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^{N} n^{-s} + \chi(s)\sum_{n=1}^{N} n^{s-1} \\ &\quad + (-1)^{N-1}(2\pi)^{\frac{s+1}{2}}\Gamma^{-1}(s)t^{\frac{s-1}{2}}e^{\pi i s-\frac{it}{2}-\frac{\pi i}{8}}S,\end{aligned} \tag{7.9}$$

where $0 \le \Re\mathrm{e}\, s \le 1$, $t = \Im\mathrm{m} s \to \infty$, $N = \left[\sqrt{\frac{t}{2\pi}}\right]$, $\chi(s) = \zeta(s)/\zeta(1-s)$,

$$S = \sum_{k=0}^{\nu-1} a_k \sum_{0\le 2r\le k} b_{kr}F^{(k-2r)}(\delta) + O\big((n/t)^{\nu/6}\big), \quad \delta = \sqrt{t} - \left(N+\frac{1}{2}\right)\sqrt{2\pi},$$

a_k, b_{kr} are certain complex constants with $a_k \ll t^{-k/6}$, $\nu(\le 2\cdot 10^{-8}t)$ is a natural number, and

$$F(z) := \frac{\cos(z^2+\frac{3\pi}{8})}{\cos(\sqrt{2\pi}z)}.$$

The integral representations for $\zeta^2(s)$ of A. Guthmann [5] have not yielded yet an asymptotic expansion for $\zeta^2(s)$ of the desired form

$$\zeta^2(s) = \sum_{n\le\frac{t}{2\pi}} d(n)n^{-s} + \chi^2(s)\sum_{n\le\frac{t}{2\pi}} d(n)n^{s-1} + R\left(s, \frac{t}{2\pi}\right) \tag{7.10}$$

with an explicit expression for $R(s, \frac{t}{2\pi})$ in terms of elementary functions, of the type obtained by Y. Motohashi [8]–[10]. The latter involve the function $\Delta(x)$, which represents the error term in the Dirichlet divisor problem and a related function. In the general case one would like to obtain

$$\zeta^r(s) = \sum_{n\le(\frac{t}{2\pi})^{r/2}} d_r(n)n^{-s} + \chi^r(s) \sum_{n\le(\frac{t}{2\pi})^{r/2}} d_r(n)n^{s-1} + R_r(s, \frac{t}{2\pi}), \tag{7.11}$$

where $R_r(s, \frac{t}{2\pi})$ is to be considered as the error term in the approximate functional equation (7.11). In the most important case $s = \frac{1}{2} + it$ some results on $R_r(s, \frac{t}{2\pi})$ are to be found in Ch. 4 of the author's monograph [6]. It would be certainly interesting if one could use the integral representations furnished by Theorem 1 or Theorem 5 to improve the bounds for $R_r(s, \frac{t}{2\pi})$ given in [6].

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ALEKSANDAR IVIĆ
KATEDRA MATEMATIKE RGF–A
UNIVERSITETA U BEOGRADU
ĐUŠINA 7, 11000 BEOGRAD
SERBIA (YUGOSLAVIA)

E-mail: aleks@ivic.matf.bg.ac.yu, eivica@ubbg.etf.bg.ac.yu, aivic@rgf.rgf.bg.ac.yu

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