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A generalization of a theorem of Piccard

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Dedicated to Professors Zoltán Daróczy and Imre Kátai on the occasion of their sixtieth birthday

Abstract. In this paper the theorem of Piccard stating that the sum of two Baire sets having second Baire category contains a nonvoid open set is generalized: addition is replaced by a smooth function with several variables.

A famous theorem of Steinhaus asserts that the sum of two measurable subsets of the real line with positive measure contains an interval. This theorem has numerous generalizations: $\mathbb{R}$ can be replaced by other topological measure spaces, and addition can be replaced by a function of two or more variables. The results have applications in the investigation of regular solutions of functional equations: see the papers SANDER [1976] and JÁRAI [1993], in which further references can be found. We remark that as Sander pointed out, the measurability of one of the sets can be omitted.

The analogous result of Piccard states that the sum of two Baire sets having second Baire category has an inner point. Very strong generalizations exists; in this case also addition can be replaced by a two variable function with weak solvability conditions. These results are useful in the proof of "Baire property implies continuity" and "Baire property implies boundedness" type regularity theorems for functional equations. We refer the reader to the papers SANDER [1978], [1979], [1981],

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Kominek [1973], Járai [1986] and Grosse-Erdmann [1989] and the references cited therein.

The purpose of this paper is to give a generalization of the theorem of Piccard in which addition is replaced by a smooth function F with several variables. This will include the well-known special case stating that if $F : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ then $F(A_1 \times A_2)$ has an interior point. Let us observe that considering a function $F : \mathbb{R}^{kn} \to \mathbb{R}^k$, $n > 2$ is a trivial generalization. To obtain a proper generalization we have to consider a function $F : \mathbb{R}^{kn} \to \mathbb{R}^{k(n-1)}$ or more generally a function $F : \mathbb{R}^r \to \mathbb{R}^m$ where m is not much less than r.

Notations. All *normed spaces* are assumed to be *real*; the norm will be denoted by $|\ |$. The notation $\|\ \|$ will be used only for the operator norm of linear operators. If $f : D \to Y$ is a function mapping an open subset of a normed space into a normed space, then f' will denote the *derivative* of f. If $D \subset X_1 \times X_2 \times \ldots \times X_n$ we will use the notations

$$D_{x_i} = \big\{(x_1, \ldots , x_{i-1}, x_{i+1}, \ldots , x_n) : (x_1, \ldots , x_n) \in D\big\}$$

for the *partial sets*. The *partial functions* $f_{x_i} : D_{x_i} \to Y$ are defined by

$$f_{x_i}(x_1, \ldots , x_{i-1}, x_{i+1}, \ldots , x_n) = f(x_1, \ldots , x_n)$$

whenever $(x_1, \ldots , x_n) \in D$. The sets $D_{x_{i_1},\ldots,x_{i_r}}$ and functions $f_{x_{i_1},\ldots,x_{i_r}}$ are defined similarly. If X_i and Y are normed spaces and the domain of the partial function $f_{x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n}$ is an open subset of X_i, then the derivative of this function defines the *partial derivative* denoted by

$$\partial_i f, \quad \partial_{x_i} f \quad \text{or} \quad \frac{\partial f}{\partial x_i}$$

if it exists. Concerning topology we follow the terminology and notations of Bourbaki [1966]. Hence every regular, completely regular, normal, compact and locally compact space is *supposed to be Hausdorff*. We will say that a subset A of a topological space X *is of first category*, if A can be represented as a countable union of nowhere dense sets, otherwise A is of *second category*. X is called a *Baire space* if every nonvoid open subset of X is of second category. A will be called a *Baire set* if there exists an open set V such that the symmetric difference $A \triangle V$ has first category. The function f has the *Baire property on A* if the domain of f contains

A except a set of first category, the range of f is in a topological space Y and $A \cap f^{-1}(W)$ is a Baire set in X for every open subset W of Y. We note that several authors, including BOURBAKI [1966], use another terminology. The most important facts concerning these notions can be found in BOURBAKI [1966]; see Chapter IX, §5, and the corresponding exercises. Combining these facts with the proof in OXTOBY [1971], Chapter 15, we get the following form of a well-known theorem of Kuratowsky and Ulam:

Theorem [Kuratowsky, Ulam]. *Let X and Y be topological spaces, and suppose that Y has a countable base. Let E be a Baire set in $X \times Y$. Then except for a set of points x of X which is of first category the set E_x is a Baire set. Moreover E is of first category if and only if the set E_x is of first category in Y with the exception of a set of x's of first category.*

The following theorem is an abstract version of our generalization of the theorem of Piccard.

Theorem. *Let T, Y and X_i be topological spaces, $g_i : T \times Y \to X_i$ continuous functions, and suppose that $g_{i,t}(B)$ has second Baire category whenever $B \subset Y$ is a subset of Y with second Baire category. Suppose that $A_i \subset X_i$ and A_i is a Baire set whenever $1 \leq i \leq n$. Then the set V of points $t \in T$ for which*

$$\bigcap_{i=1}^{n} g_{i,t}^{-1}(A_i)$$

is of second category is an open subset of T.

PROOF. The sets A_i can be written in the form $A_i = E_i \triangle M_i$, where E_i is open and M_i is of first category. Suppose that $t_0 \in V$, and let $K = \bigcap_{i=1}^{n} g_{i,t_0}^{-1}(A_i)$, $K' = K \cap \left(\bigcap_{i=1}^{n} g_{i,t_0}^{-1}(E_i)\right)$. Since $g_{i,t_0}^{-1}(M_i)$ is of first category in Y whenever $i = 1, 2, \ldots, n$, we have that

$$K \backslash K' \subset \bigcup_{i=1}^{n} g_{i,t_0}^{-1}(M_i)$$

is of first category, hence K' is of second category in Y. Let y_0 be a point of K' for which $W \cap K'$ is of second category in Y for each open neighbourhood W of y_0 (see BOURBAKI [1966], IX, §5, Exercise 3). Clearly $g_i(t_0, y_0) \in E_i$ if $i = 1, 2, \ldots, n$. Since the sets E_i are open and the functions g_i are continuous, the sets $g_i^{-1}(E_i)$ are open and contain the point (t_0, y_0), hence there exist open sets V' and W', such that $t_0 \in V'$, $y_0 \in W'$, and $V' \times W' \subset \bigcap_{i=1}^n g_i^{-1}(E_i)$. We will prove that

$$W' \cap \Big(\bigcap_{i=1}^n g_{i,t}^{-1}(A_i) \Big)$$

is of second category for each $t \in V'$. Were this not true, the sets

$$W' \backslash g_{i,t}^{-1}(A_i), \qquad i = 1, 2, \dots , n$$

would cover – except for a set of first category – the set W'. If we prove that these sets are of first category, then we have a contradiction. But this follows from the inclusion

$$W' \backslash g_{i,t}^{-1}(A_i) \subset g_{i,t}^{-1}(M_i \cap E_i),$$

which is a consequence of $W' \subset g_{i,t}^{-1}(E_i)$.

Remark. If we suppose that Y is a complete separable metric space and X_1 is metrizable then we may omit the condition that the set A_1 has the Baire property.

To prove this, let C_1 denote the set of all points $x_1 \in X_1$ such that for each neighbourhood U_1 of x_1 the set $U_1 \cap A_1$ is of second Baire category. It is known (see Bourbaki [1966], IX, §5, Exercise 3) that C_1 is a closed set and $A_1 \backslash C_1$ is of first category. Let B_1 denote the set of inner points of C_1. Then B_1 is open and $A_1 \backslash B_1$ is also of first category. As in the previous proof we obtain that $W' \backslash g_{1,t}^{-1}(B_1)$ and $W' \backslash g_{i,t}^{-1}(A_i)$, $2 \le i \le n$ are of first category. It is enough to prove that $W' \cap g_{1,t}^{-1}(A_1)$ is of second category, because then if follows that

$$W' \cap \left(\bigcap_{i=1}^n g_{i,t}^{-1}(A_i) \right)$$

cannot be of first category.

Suppose, that $W' \cap g_{1,t}^{-1}(A_1)$ is of first category. Then, using that $W' \cap g_{1,t}^{-1}(B_1)$ is an open set of second category, we obtain that

$$(W' \cap g_{1,t}^{-1}(B_1)) \backslash g_{1,t}^{-1}(A_1) = (W' \cap g_{1,t}^{-1}(B_1)) \backslash (W' \cap g_{1,t}^{-1}(A_1))$$

is a Baire set of second category. Let G be a $\mathcal{G}_\delta$ subset of second category of the set above. Then $g_{1,t}(G)$ is of second category as a subset of X_1. By Bourbaki [1966], IX, §6, Exercise 10, $g_{1,t}(G)$ is a Baire set in X_1. Clearly $g_{1,t}(G) \subset B_1 \setminus A_1$. Writing $g_{1,t}(G) = U \triangle F$ where U is open and F is of first category, we see that $U \cap B_1$ is a nonvoid open set for which the intersection with A_1 is of first category. This contradicts the definition of B_1.

In Laczkovich [1995], II.9.9 it is proved that continuous image of a Polish space in a Hausdorff space is a Baire set. This shows that it is enough to suppose that X_1 is Hausdorff.

The following lemma allows us to use derivates to verify that the conditions on the functions g_i in the previous theorem are satisfied.

Lemma. *Let Y be an open subset of $\mathbb{R}^k$, T a topological space, D an open subset of $T \times Y$ and $(t_0, y_0) \in D$. Suppose, that the function $g : D \to \mathbb{R}^r$ is continuous and has continuous partial derivative with respect to y. If the rank of $\dfrac{\partial g}{\partial y}(t_0, y_0)$ is r, then there exist open neighbourhoods T^* and Y^* of t_0 and y_0, respectively, such that*

(i) *if B has second category in Y^*, then $g_t(B)$ has second category in $\mathbb{R}^r$ for each $t \in T^*$;*

(ii) *if A is a Baire set in $\mathbb{R}^r$, then $g_t^{-1}(A) \cap Y^*$ is a Baire set in Y for each $t \in T^*$.*

Proof. Let $q = k - r$, and let us divide the coordinates of $y = (y_1, \dots, y_k)$ into two groups $y' = (y'_1, \dots, y'_q)$ and $y'' = (y''_1, \dots, y''_r)$ such that

$$\det\left(\frac{\partial g}{\partial y''}(t_0, y_0)\right) = \det\left(\frac{\partial g}{\partial y''}(t_0, y'_0, y''_0)\right) \neq 0$$

be satisfied. Introducing the notation

$$L(t, y') = \frac{\partial g}{\partial y''}(t, y', y''_0),$$

and using the proof of the inverse function theorem (see Rudin [1964], Theorem 9.24), we have that if Y'' is an open ball in $\mathbb{R}^r$ with center y''_0, $t \in T$, $(y', y'') \in Y$ and

$$\left\| \frac{\partial g}{\partial y''}(t, y', y'') - L(t, y') \right\| < \frac{1}{2\, \|L(t, y')^{-1}\|} \tag{3}$$

for all $y'' \in Y''$, then $g_{t,y'}$ is a homeomorphism of Y'' onto an open subset $U(t,y')$ of $\mathbb{R}^r$. Now let

$$0 < \beta < \frac{1}{2\,\|L(t_0,y_0')^{-1}\|}.$$

Using the continuity of expressions in (3), we may choose an open ball Y'' with center y_0'' and open sets Y' and T^*, for which $t_0 \in T^*$, $y_0' \in Y'$, $Y^* = Y' \times Y'' \subset Y$, moreover

$$\left\| \frac{\partial g}{\partial y''}(t,y',y'') - L(t,y') \right\| < \beta$$

and

$$\beta < \frac{1}{2\,\|L(t,y')^{-1}\|}$$

whenever $t \in T^*$, $y' \in Y'$ and $y'' \in Y''$.

Suppose that there exists a subset B of $Y^* = Y' \times Y''$ of second category, and a $t \in T^*$, such that $g_t(B)$ is of first category in $\mathbb{R}^r$. Let us choose a Borel set U of first category in $\mathbb{R}^r$, for which $g_t(B) \subset U \subset g_t(Y^*)$ and let $B^* = g_t^{-1}(U) \cap Y^*$. This set B^* is a Baire set and is of second category in Y^*, but $g_t(B^*)$ is of first category in $\mathbb{R}^r$. By the Kuratowsky–Ulam theorem the set of all points $y' \in Y'$ for which $B^*_{y'}$ is of second category is a set of second category. On the other hand, by the same theorem, the set of all points $y' \in Y'$ for which $B^*_{y'}$ is not a Baire set, is of first category. From this it follows that there exists $y' \in Y'$ for which $B^*_{y'}$ is a Baire set of second category in Y''. Since $g_{t,y'}$ is a homeomorphism of Y'' onto $U(t,y')$, the set $g_{t,y'}(B^*_{y'})$ is of second category in $\mathbb{R}^r$. This is a contradiction, because $g_{t,y'}(B^*_{y'}) \subset g_t(B^*)$. Hence (1) is proved.

To prove (2) suppose that A is a Baire set in $\mathbb{R}^r$, and let us choose a Borel set B for which $A \subset B$ and $B \setminus A$ is of first category. Then

$$g_t^{-1}(A) \cap Y^* = \left(g_t^{-1}(B) \cap Y^*\right) \setminus \left(g_t^{-1}(B \setminus A) \cap Y^*\right).$$

Using that $g_t^{-1}(B)$ is a Borel set and $g_t^{-1}(B \setminus A) \cap Y^*$ has first category by (1), we have that $g_t^{-1}(A) \cap Y^*$ is a Baire set.

Now we are prepared to prove the local version of our generalization of the theorem of Piccard for a function from an open subset of $\mathbb{R}^r$ into $\mathbb{R}^m$. The condition of the following theorem means, roughly speaking, that the nullspace of the derivative is large enough and is in general position.

Theorem. *Let X be the r-dimensional Euclidean space, and let $X_1, \dots, X_n$ be orthogonal subspaces of X with dimensions $r_1, \dots, r_n$, respectively. Suppose that $r_i \geq 1$ whenever $1 \leq i \leq n$ and $\sum_{i=1}^n r_i = r$. Let U be an open subset of X and $F : U \to \mathbb{R}^m$ a continuously differentiable function. For each $x \in U$ let N_x denote the nullspace of $F'(x)$. Let A_i be a Baire subset of X_i $(i = 1, 2, \dots, n)$, and suppose that $a \in U$ and $\dim N_a = r - m$. Let p_i denote the orthogonal projection of X onto X_i. Suppose, that $p_i(N_a) = X_i$ and $p_i(a)$ has a neighbourhood U_i such that $U_i \setminus A_i$ is of first category if $1 \leq i \leq n$. Then $F(A_1 \times \ldots \times A_n)$ is a neighbourhood of $F(a)$.*

Proof. Let $k = r - m$. Since the function $x \mapsto \operatorname{rank} F'(x)$ is lower semicontinuous and $\operatorname{rank} F'(a) = m$, we may suppose that $\operatorname{rank} F'(x) = m$ for all $x \in U$. Similarly, choosing smaller U, if necessary, we may suppose that $p_i(N_x) = X_i$ whenever $x \in U$ and $1 \leq i \leq n$.

Replacing U with a smaller subset if necessary and using the rank theorem (see Dieudonné [1971], 10.3.1), we have that there exist mappings u, p and v and an open neighbourhood V of $b = F(a)$ in $\mathbb{R}^m$ with the following properties: $F|_U = v \circ p \circ u$, u maps U onto the open cube I^r, where $I =]-1, 1[$, the mapping u is invertable, u and u^{-1} are continuously differentiable; v maps I^m onto V injectively, v and v^{-1} are continuously differentiable; p is the projection

$$p : (x_1, \dots, x_r) \mapsto (x_1, \dots, x_m)$$

of I^r onto I^m. The cube I^r can be written in the form $I^r = T \times Y$, where $T = I^m$ and $Y = I^k$. Let $u(a) = (t_0, y_0) \in T \times Y$. We will use some simple facts from differential geometry (see Dieudonné [1971], 16.8.8).

$U \cap F^{-1}(v(t))$ is a closed submanifold of U for each $t \in T$. The tangent space of this submanifold at the point $x \in U \cap F^{-1}(v(t))$ is the subspace N_x of the space X. Clearly, u^{-1} is a diffeomorphism of the closed submanifold $\{t\} \times Y$ of $T \times Y$ onto $U \cap F^{-1}(v(t))$. Let $g_i = p_i \circ u^{-1}$ if $1 \leq i \leq n$. Due to the choice of U, p_i is a submersion of $U \cap F^{-1}(v(t))$ into X_i. Hence we get that the mapping $g_{i,t} : Y \to X_i$ is also a submersion, that is, it has rank r_i whenever $y \in Y$ and $t \in T$.

By the above lemma there exist sets T^* and Y^* such that $t_0 \in T^* \subset T$, $y_0 \in Y^* \subset Y$ and $g_{i,t}(B)$ is of second category whenever $B \subset Y^*$ has second category and $t \in T^*$. Let $X_i^* = X_i$, $A_i^* = A_i$, and g_i^* be the restriction of g_i to $T^* \times Y^*$. Applying the theorem above to the sets marked by star we have that the set V^* of points t for which

$$\bigcap_{i=1}^{n} g_{i,t}^{-1}(A_i)$$

is of second category is open in T^*. Since g^*_{i,t_0} maps Y^* onto an open neighbourhood of $p_i(a)$ and $g^*(t_0, y_0) = p_i(a)$, there exists an open neighbourhood W of y_0 in Y^* such that $W \backslash g^{*-1}_{i,t_0}(A_i)$ is of first category if $1 \leq i \leq n$. This proves that $t_0 \in V^*$. Clearly $v(V^*)$ is an open neighbourhood of b in $\mathbb{R}^m$. If $z \in v(V^*)$, then $v^{-1}(z) \in V^*$, and hence the set

$$\bigcap_{i=1}^{n} g_{i,t}^{*-1}(A_i)$$

is nonvoid. If y is an element of this set then $u^{-1}(t, y) \in F^{-1}(z)$ and $x_i = p_i\left(u^{-1}(t, y)\right) \in A_i$ whenever $1 \leq i \leq n$. This implies $F(x_1, \ldots, x_n) = z$, which is enough since $v(V^*)$ is an open neighbourhood of $b = F(a)$.

Corollary. *Let U be an open subset of $\mathbb{R}^r \times \mathbb{R}^r$, and let $F : (x, y) \mapsto F(x, y)$ be a continuously differentiable mapping of U into $\mathbb{R}^r$. Suppose, that $A, B \subset \mathbb{R}^r$ and A, B are Baire sets. If $(a, b) \in U$,*

$$\det \frac{\partial F}{\partial x}(a, b) \neq 0, \qquad \det \frac{\partial F}{\partial y}(a, b) \neq 0$$

and there exist a neighbourhoods U and V of a and b respectively such that $U \backslash A$ and $V \backslash B$ is of first category, then $F(A, B)$ contains a neighbourhood of $F(a, b)$.

Proof. By the theorem above we only have to prove that $p_1(N_{a,b}) = \mathbb{R}^r$ and $p_2(N_{a,b}) = \mathbb{R}^r$, where $N_{a,b}$ is the nullspace of $F'(a, b)$. Let $(x, y) \in N_{a,b}$. If $p_1(x, y) = 0$ then $x = 0$. Hence

$$0 = F'(a, b)(x, y) = \frac{\partial F}{\partial y}(a, b)(y).$$

But $\det \dfrac{\partial F}{\partial y}(a, b) \neq 0$, hence $y = 0$. This proves that $p_1 : N_{a,b} \to \mathbb{R}^r$ is one-to-one, that is $p_1(N_{a,b}) = \mathbb{R}^r$. Similarly, $p_2(N_{a,b}) = \mathbb{R}^r$.

The above theorem can be formulated in the following global form:

Theorem. *Let X be the r-dimensional Euclidean space, and let $X_1, \dots, X_n$ be orthogonal subspaces of X with dimensions $r_1, \dots, r_n$, respectively. Let p_i denote the orthogonal projection of X onto X_i. Suppose that $r_i \geq 1$ whenever $1 \leq i \leq n$ and $\sum_{i=1}^{n} r_i = r$. Let U be an open subset of X and $F : U \to \mathbb{R}^m$ a continuously differentiable function. For each $x \in U$ let N_x denote the nullspace of $F'(x)$. If $\dim N_x = r - m$ and $p_i(N_x) = X_i$ whenever $x \in U$ and $i = 1, 2, \dots, n$, moreover $A_1 \times \dots \times A_n \subset U$ and A_i is a Baire set having second category if $1 \leq i \leq n$, then $F(A_1 \times \dots \times A_n)$ contains a nonvoid open set.*

Proof. A Baire set A_i can be written in the form $U_i \triangle F_i$ where U_i is a nonvoid open set and F_i is of first category. For any $a \in U_1 \times U_2 \times \dots \times U_n$ we may apply the previous theorem.

Remark. Using the previous Remark, we may omit the condition that A_1 is a Baire set.

References

[1] N. Bourbaki [1966], Elements of mathematics. General topology, *Addison–Wesley, Reading, Mass.*, *Palo Alto, London, Don Mills, Ont.*, 1966.

[2] J. Dieudonné [1971], Grundzüge der modernen Analysis I–IX., *VEB Deutscher Verlag der Wissenschaften*, *Berlin*, 1971, 1971–1988.

[3] K.-G. Grosse-Erdmann [1989], Regularity properties of functional equations and inequalities, *Aequationes Math.* **37** (1989), 233–251.

[4] A. Járai [1986], On regular solutions of functional equations, *Aequationes Math.* **30** (1986), 21–54.

[5] A. Járai [1993], A Steinhaus type theorem, *Publ. Math. Debrecen* **47** (1995), 1–13.

[6] Z. Kominek [1973], Some generalization of the theorem of S. Piccard., *Prace Naukowe Uniwersytetu Ślaskiego w Katowicach* **37** (1973), 31–33.

[7] M. Laczkovich [1995], Valós függvénytan, Egyetemi jegyzet, *Eötvös Loránd University, Budapest*, 1995.

[8] J. C. Oxtoby [1971], Mass und Kategorie, *Springer*, *Berlin, Heidelberg, New York*, 1971.

[9] S. Piccard [1939], Sur les ensembles de distances des ensembles de points d'un espace euclidien, Secrétariat de l'Université Neuchâtel, vol. 13, *Mém. Univ. Neuchâtel*, 1939.

[10] W. Rudin [1964], Principles of mathematical analysis, 2nd edn, *McGraw-Hill*, *New York*, 1964.

[11] W. Sander [1976], Verallgemeinerungen eines Satzes von H. Steinhaus, *Manuscripta Math.* **18** (1976), 25–42; Errata hierzu, *Manuscripta Math.* **20** (1977), 101–103.

[12] W. Sander [1978], Regularitätseigenschaften von Funktionalgleichungen, *Glas. Math. Ser. III* **13** no. 33 (1978), 237–247.

[13] W. Sander [1979], A generalization of a theorem of S. Piccard., *Proc. Am. Math. Soc.* **73** (1979), 281–282.

[14] W. Sander [1981], Ein Beitrag zur Baire-Kategorie-Theorie, *Manuscripta Math.* **34** (1981), 71–83.

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