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Exponential diophantine equations over function fields

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1. Introduction

In the past decade several effective finiteness results and algorithms have been proved and constructed for diophantine equations over function fields (see e.g. [2], [7], [15], [16], [20], [24]). These theorems are interesting in themselves and they have certain applications; for instance, to diophantine problems over number fields (cf. [6]). Furthermore, combining these results with analogue theorems over algebraic number fields and GYŐRY's specialization method (cf. [12], [13], [14]), similar results can be proved over finitely generated domains. To illustrate it we refer to [4], [5], [9], [12], [13] and [14].

In this paper we give an effective upper bound for the "height" of the solutions to a general class of diophantine equations over function fields, generalizing and improving some earlier related results. We apply our general theorem (Th. 1) to some special, however important equations.

2. Notation

Let k be an algebraically closed field of characteristic zero and $k(t)$ be the field of rational functions over k. Moreover, let $\mathbb{K}$ be a finite extension of $k(t)$ with genus g. The additive height of an $\alpha \in \mathbb{K}^*$ (as usual $\mathbb{K}^*$ denotes the set of non-zero elements of $\mathbb{K}$) is defined by

$$H(\alpha) = \sum_{v \in M_{\mathbb{K}}} \max\{0, v(\alpha)\},$$

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where $M_{\mathbb{K}}$ is the set of the (additive) valuations of $\mathbb{K}$ with value group $\mathbb{Z}$. In the special case when $\mathbb{K} = k(t)$ and $\alpha = \frac{p}{q}$, where $p, q \in k[t]$ are relatively prime polynomials,

$$H(\alpha) = \max\{\deg p, \deg q\}\,.$$

For a rational integer $n \geq 2$ we denote by $\mathbb{K}[X_1, \ldots, X_n]$ the polynomial ring in n variables over $\mathbb{K}$ and let $f_1, \ldots, f_n$ and g be non-zero elements of $\mathbb{K}[X_1, \ldots, X_n]$.

Consider the equation

$$\sum_{i=1}^{n} f_i(x_1, \ldots, x_n) \cdot x_i^{r_i} = g(x_1, \ldots, x_n) \tag{1}$$

in $\underline{x} = (x_1, \ldots, x_n) \in \mathbb{K}^n$ and $\underline{r} = (r_1, \ldots, r_n) \in \mathbb{Z}^n$ under the condition that

$$(*) \qquad \text{the sum } \sum_{i=1}^{n} f_i(\underline{x}) \cdot x_i^{r_i} \text{ has no proper vanishing subsum.}$$

Let R be a non-zero element of $k[t]$. In 1979 Newman and Slater [19] showed that the equation

$$x_1^r + \cdots + x_n^r = R \tag{2}$$

with $r > \deg R + n(n-1)/2$ has no non-constant solution in $k[t]$.

Later Stepanov ([24], [25]) generalized the above theorem, when the sum is weighted and the exponents are not necessary equals, under the assumption that $x_1, \ldots, x_n$ are pairwise relatively prime polynomials in $k[t]$.

By using algebraic geometry, Voloch [25] proved the following. *Let $\mathbb{K} = \mathbb{C}$ and S be a finite subset of $M_{\mathbb{K}}$ containing all the infinite valuations, that is the extensions of the degree-valuation of $k(t)$. Let $\mathcal{O}_{\mathbb{K},S}$ denote the set of S–integers in $\mathbb{K}$*[1]*. Further, let $a_1, \ldots, a_n, b$ be non-zero elements of $\mathcal{O}_{\mathbb{K},S}$. If*

$$\sum_{i=1}^{n} a_i x_i^r = b \tag{3}$$

[1]We adopt the terminology of Borevich and Shafarevich [3], an $\alpha \in \mathbb{K}^*$ is said to be an S–integer if $v(\alpha) \geq 0$ for every $v \in M_{\mathbb{K}} \setminus S$. The S–integers form a ring and the units of this ring are called S–units.

with $r > n(n-1)$ *and* $a_1x_1^r, \dots, a_nx_n^r$ *are linearly independent* (*over* $\mathbb{C}$) *then*

$$\{r - n(n-1)\} \cdot \max_i H(x_i) \le \frac{n(n-1)}{2}\{2g - 2 + \mathrm{Card}(S)\} + 2H\,,$$

where $H = H(a_1) + \dots + H(a_n) + H(b)$ *and* $\mathrm{Card}(S)$ *denotes the cardinality of set* S.

The case $n = 2, f_1, f_2, g$ are non-zero constants, has a special interest. There are some results connected with the equation

$$a_1x_1^{r_1} + a_2x_2^{r_2} = 1 \quad \text{in} \quad (x_1, x_2) \in \mathbb{K}^2 \tag{4}$$

and $\mathbb{K} = k(t)$, see e.g. [1], [10], [11], [18]. The proofs are based on the unique factorization property of $k[t]$ and there seems to be no way to extend them to the general algebraic function field case.

A rather general theorem on Thue's equations due to Schmidt [20] gives

$$H(x_1) + H(x_2) \le 178\{H(a_1) + H(a_2)\} + 422g$$

in the homogeneous case $r_1 = r_2$.

Using another approach, Silverman [22] shoved that

$$5 \cdot \max\{H(a_1), H(a_2)\} + 2g - 2 \ge \\ \ge \left(1 - \frac{1}{r_1} - \frac{1}{r_2} - \frac{1}{[r_1, r_2]}\right) \cdot \max\{H(x_1^{r_1}), H(x_2^{r_2})\}$$

where $[r_1, r_2]$ denotes the least common multiple of r_1 and r_2.

Mueller [17] proved that the equation

$$a_1x_1^r + a_2x_2^r = 1 \tag{5}$$

has at most *two* solutions (in $\mathbb{K}^*$), provided that $r > 30 + 20g$ and a_1 $a_2 \notin (\mathbb{K}^*)^r$, where $(\mathbb{K}^*)^r$ is the multiplicative group of the r–th powers. The proof of result based on an idea of Evertse, Győry, Stewart and Tijdeman [8]. Later, Mueller and Bombieri [2] extended this theorem to the general case $n > 2$.

3. On the height function

Following the notation of 2.§, let

$$f(X_1, \dots, X_n) = \sum_{i=1}^{m} \alpha_i X_1^{v_{i1}} \cdot \ldots \cdot X_n^{v_{in}}$$

be an element of $\mathbb{K}[X_1,\dots,X_n]$. The l*enght* of f is defined by

$$L(f)=\sum_{i=1}^{m} H(\alpha_i)\,.$$

The degree of the j–th variable (of f) is given by

$$\deg_j(f)=\sum_{i=1}^{m} v_{ij}$$

For $f_1,\dots,f_n\in\mathbb{K}[X_1,\dots,X_n]$ we put $\underline{f}=(f_1,\dots,f_n)$, $L(\underline{f})=\sum_{i=1}^{n} L(f_i)$ and $\deg_j(\underline{f})=\sum_{i=1}^{n}\deg_j(f_i)$, for $j=1,\dots,n$.

We are going to use the following known relations (cf. [15]),

$$\sum_{v\in M_{\mathbb{K}}} v(\alpha)=0\,,$$
$$\max\{H(\alpha\beta),H(\alpha+\beta)\}\le H(\alpha)+H(\beta),\quad \alpha,\beta\in\mathbb{K}^*$$
$$H(\alpha^z)=|z|\cdot H(\alpha),\quad z\in\mathbb{Z},\quad \alpha\in\mathbb{K}^*\,.$$

Let $\mathcal{H}(\alpha)$ be the cardinality of the set $\{v: v\in M_{\mathbb{K}}, v(\alpha)\neq 0\}$. One can see that $\mathcal{H}(\alpha)\le 2H(\alpha)$.

4. Results

Our main results is

Theorem 1. *If* $\underline{x}=(x_1,\dots,x_n)\in\mathbb{K}^n$, $\underline{r}=(r_1,\dots,r_n)\in\mathbb{Z}^n$ *is a solution of the equation* (1) *satisfying the condition* ($*$) *then*

$$\sum_{i=1}^{n}|r_i|\cdot H(x_i)\le c_1(n)\mathcal{G}+c_2(n)L(\underline{f})+c_3(n)L(g)+$$
$$+\sum_{i=1}^{n}\Big(2c_1(n)+c_2(n)\deg_i(\underline{f})+c_3(n)\deg_i(g)\Big)\cdot H(x_i)\,,$$

where $c_1(n)=\frac{1}{2}n^2(n-1)$, $c_2(n)=n^2(n-1)+1$, $c_3(n)=n^2(n-1)+n$ *and* $\mathcal{G}=\max(2g-2,0)$.

Remark 1. In the special case when the polynomials $f_1,\dots,f_n,g$ are constans, that is $f_i\equiv a_i,\ g\equiv b,\ a_i,b\in\mathbb{K}^*$ for $i=1,\dots,n$, Theorem 1 yields

$$\sum_{i=1}^{n}|r_i|\cdot H(x_i)\le\frac{1}{2}n^2(n-1)\mathcal{G}+\{n^2(n-1)+1\}\cdot\sum_{i=1}^{n}H(a_i)+ \\ +\{n^2(n-1)+n\}H(b)+n^2(n-1)\sum_{i=1}^{n}H(x_i)\,.$$

It is generalization of the above-mentioned result of Voloch with a weaker condition which is necessary and cannot be made weaker. However, Voloch's bound is sharper because of the generality of Theorem 1 and the method that we used.

Remark 2. For an arbitrary $x\in\mathbb{K}$ the triple $(x^{r_2},x^{r_1},1)\in\mathbb{K}^3$ is a solution of the equation

$$x_1^{r_1}-x_2^{r_2}+x_3^{r_3}=1$$

showing that the condition $(*)$ is necessary. It seems to be a harder problem to characterize all the solutions without condition $(*)$.

We are going to mention two simple consequences of Theorem 1.

Corollary 1. *Under the conditions of Theorem 1, if*

$$|r_i|>2c_1(n)+c_2(n)\deg_i(\underline{f})+c_3\deg_i(g)$$

then

$$\sum_{i=1}^{n}H(x_i)\le c_1(n)\mathcal{G}+c_2(n)L(\underline{f})+c_3(n)L(g)\,.$$

(The constants are defined in Theorem 1).

Corollary 2. *The equation* (1) *with the above conditions has no solution* $(\underline{x},\underline{r})\in\mathbb{K}^n\times\mathbb{Z}^n$ *with* $x_i\in\mathbb{K}\setminus k\ \ (i=1,\dots n)$ *and*

$$\sum_{i=1}^{n}|r_i|>c_1(n)\{2n+\mathcal{G}\}+c_2(n)\left\{\sum_{i=1}^{n}\deg_i(\underline{f})+L(\underline{f})\right\}+ \\ +c_3(n)\left\{\sum_{i=1}^{n}\deg_i(g)+L(g)\right\}\,.$$

The ground field k is algebraically closed, hence the conditions $x_i\in\mathbb{K}\setminus k$ are necessary.

Remark 3. In the case $n = 2$, $\min(r_1, r_2) \geq 5$, our inequalities imply

$$(r_1 - 4)H(x_1) + (r_2 - 4)H(x_2) \leq 10 \cdot \max\{H(a_1), H(a_2)\} + 2\mathcal{G}$$

improving and generalizing a related theorem of SCHMIDT [5].

The next theorems are further applications of Theorem 1.

Theorem 2. *Let $a_1, a_2 \in \mathbb{K}$ with $a_1 \neq 0$. If $(x_1, x_2) \in \mathbb{K}^2$, $(r_1, r_2) \in \mathbb{Z}^2$ satisfy*

$$a_1 \cdot \frac{x_1^{r_1} - 1}{x_1 - 1} = x_2^{r_2} + a_2 \tag{6}$$

with $x_1 \neq 1 - \dfrac{a_1}{a_2}$ in the case $a_2 \neq 0$, then

$$|r_1| \cdot H(x_1) + |r_2| \cdot H(x_2) \leq 2\mathcal{G} + 11H(a_1) + 12H(a_2) + 15H(x_1) + 4H(x_2)\,.$$

If $a_2 \neq 0$ and $x_1 = 1 - \dfrac{a_1}{a}$ then we cannot give upper bound for $H(x_2)$, r_1 and r_2. Indeed, let $a_2 = -1$, $e \in \mathbb{N}$ and $r_1 = r_2 e$, say. Then $x_1 = a_1 + 1$ and $x_2 = (a_1 + 1)^e$ satisfy (6), and $H(x_2) = eH(a_1 + 1) \geq e$, provided that $a_1 \notin k$.

Corollary 3. *The equation (6) has no solution satisfying $x_i \in \mathbb{K} \setminus k$, $i = 1, 2$, $|r_1| > 15$, $|r_2| > 4$ and*

$$|r_1| + |r_2| > 19 + 2\mathcal{G} + 11H(a_1) + 12H(a_2)\,.$$

Theorem 3. *Let $a_1 \in \mathbb{K}^*$ and $a_2 \in \mathbb{K}$. Then all solutions $(x_1, x_2) \in \mathbb{K}^2$, $(r_1, r_2) \in \mathbb{Z}^2$ of the equation*

$$a_1 \frac{x_1^{r_1} - 1}{x_1 - 1} = \frac{x_2^{r_2} - 1}{x_2 - 1} + a_2 \tag{7}$$

with

$$\frac{x_1 - 1}{x_2 - 1} \neq a_1 + a_2(x_1 - 1)$$

satisfy

$$|r_1| \cdot H(x_1) + |r_2| \cdot H(x_2) \leq 2\mathcal{G} + 22H(a_1) + 24H(a_2) + 21H(x_1) + 21H(x_2).$$

Corollary 4. *The equation* (7) *has no solution $(x_1, x_2) \in \mathbb{K}^2$, $(r_1, r_2) \in \mathbb{Z}^2$ satisfying $x_i \in \mathbb{K} \setminus k$, $i = 1, 2$, $\min\{|r_1|, |r_2|\} > 21$ and*

$$|r_1| + |r_2| > 42 + 2\mathcal{G} + 22H(a_1) + 24H(a_2)\,.$$

Remark 4. In the number field case, equations (6) and (7) are related number systems and investigated by several authors. For further details we refer to SHOREY and TIJDEMAN [21] (Ch. 11–12.).

5. Proofs

The proof of our main result is based on a theorem of Brownawell and Masser [7] concerning S–unit equations in several variables. Similar inequality had been proved by Mason [16] with weaker constants.

Lemma 1. (Brownawell and Masser [7]). *Let S be a finite subset of $M_{\mathbb{K}}$ containing all the infinite valuations. Furthermore, let $u_1, \dots, u_l$ be S–units, for which*

(i) $$u_1 + \cdots + u_l = 0$$

(ii) *there is no proper vanishing subsum in (i)*

Then

$$\max\left\{H\left(\frac{u_2}{u_1}\right), \dots, H\left(\frac{u_l}{u_1}\right)\right\} \le \frac{1}{2}(l-1)(l-2)\{\mathrm{Card}\, S + \mathcal{G}\}\,.$$

where $\mathcal{G} = \max(2g-2, 0)$.

Proof of Theorem 1. Let $(\underline{x}, \underline{r}) \in \mathbb{K}^n \times \mathbb{Z}^n$ be an arbitrary but fixed solution with condition $(*)$. The valuation set S_1 is defined by

$$S_1 = \left\{ v : v \in M_{\mathbb{K}}, \left(\sum_{i=1}^{n} v^2(x_i) + \sum_{i=1}^{n} v^2(f_i(\underline{x})) \right) + v^2(g(\underline{x})) > 0 \right\}.$$

It is clear that

$$\mathrm{Card}(S_1) \le \sum_{i=1}^{n} \mathcal{H}(x_i) + \sum_{i=1}^{n} \mathcal{H}(f_i(\underline{x})) + \mathcal{H}(g(\underline{x}))\,. \tag{8}$$

Let $f(X_1, \dots, X_n) = \sum\limits_{i=1}^{m} \alpha_i X_1^{v_{i1}} \cdot \dots \cdot X_n^{v_{in}}$, with $\alpha_i \in \mathbb{K}$ and $v_{ij} \in \mathbb{Z}$, $v_{ij} \ge 0$ for $i = 1, \dots, n$; $j = 1, \dots, m$. Then

$$\mathcal{H}(f(\underline{x})) \le 2\cdot H(f(\underline{x})) \le 2\cdot L(f) + 2\cdot H(x_1)\cdot \deg_1(f) + \cdots + 2\cdot H(x_n)\cdot \deg_n(f)\,.$$

Combining this inequality with (8) we have

$$\begin{aligned} \mathrm{Card}(S_1) \le{}& 2\sum_{i=1}^{n} H(x_i) + 2L(\underline{f}) + 2L(g) + 2H(x_1)\cdot \deg_1(\underline{f}) + \dots \\ &\dots + 2H(x_n)\cdot \deg_n(\underline{f}) + 2H(x_1)\cdot \deg_1(g) + \cdots + 2H(x_n)\cdot \deg_n(g)\,. \end{aligned} \tag{9}$$

The relation

$$\sum_{i=1}^{n} f_i(\underline{x}) \cdot x_i^{r_i} - g(\underline{x}) = 0$$

can be considered as an S_1–unit equation and there is no proper vanishing subsum because of condition $(*)$. The inequality of Brownawell and Masser (Lemma 1) yields

$$
\begin{aligned}
|r_i|\cdot H(x_i) = H(x_i^{r_i}) \le \\
&\le \frac{1}{2}n(n-1)\{\mathrm{Card}(S_1)+\mathcal{G}\} + H(g(\underline{x})) + H(f_i(\underline{x})) \le \\
&\le \frac{1}{2}n(n-1)\{\mathrm{Card}(S_1)+\mathcal{G}\} L(g) + L(f_i) + \\
&\quad + \sum_{j=1}^{n} \deg_j(g)\cdot H(x_j) + \sum_{j=1}^{n} \deg_j(f_i)\cdot H(x_j)
\end{aligned} \tag{10}
$$

for $i = 1, 2, \dots, n$.

By taking the sum of these inequalities and applying (9) we obtain Theorem 1.

Proof of Theorem 2. Let $(x_1, x_2) \in \mathbb{K}^2$, $(r_1, r_2) \in \mathbb{Z}^2$ be a solution of (6). In the trivial case $x_1 = 0$ or $x_2 = 0$ one can verify the theorem. In the sequel we assume that $x_1x_2 \ne 0$. If $a_2 = 0$ then

$$a_1x_1^{r_1} - (x_1-1)x_2^{r_2} - a_1 = 0$$

and $a_1x_1x_2(x_1-1) \ne 0$, moreover if $a_2 \ne 0$ then

$$a_1x_1^{r_1} - (x_1-1)x_2^{r_2} - a_2(x_1-1) - a_1 = 0$$

and because of $x_1 \ne 1 - \dfrac{a_1}{a_2}$ there is no proper vanishing subsum. In both cases Theorem 1 implies Theorem 2.

Proof of Theorem 3. We may assume again that $x_1x_2 \ne 0$. We have two cases to distinguish: either

$$a_1(x_2-1)x_1^{r_1} - (x_1-1)x_2^{r_2} - a_1(x_2-1) + (x_1-1) = 0$$

or

$$a_1(x_2-1)x_1^{r_1} - (x_1-1)x_2^{r_2} - a_1(x_2-1) + (x_1-1) - a_2(x_2-1)(x_1-1) = 0,$$

with $a_2 \ne 0$. The condition

$$a_1(x_2-1) - (x_1-1) + a_2(x_2-1)(x_1-1) \ne 0$$

provides that there is no proper vanishing subsum and applying Theorem 1, a simple calculation gives Theorem 3.

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