Publ. Math. Debrecen
52 / 3-4 (1998), 507–515

On a functional equation of Alsina and García-Roig

By KÁROLY LAJKÓ (Debrecen)

Dedicated to the 60th birthday of
Professors Zoltán Daróczy and Imre Kátai

Abstract. In the present paper we consider the functional equation (1) for functions $f:\mathbb{R}\to\mathbb{R}$ or $f:I\to\mathbb{R}$ respectively, supposing the existence a subset $E\subset\mathbb{R}$ (or $E\subset I$) of positive Lebesgue measure such that $f(x)\neq 0$ for all $x\in E$.

1. Introduction

In a recent paper [2] C. Alsina and J. L. García-Roig have found the continuous solutions of the following two functional equations

$$f(px+(1-p)y)f((1-p)x+py)=f(x)f(y) \tag{1}$$

$$\begin{aligned} f(x)f(px+(1-p)y)+f(y)f((1-p)x+py) \\ =f(px+(1-p)y)^2+f((1-p)x+py)^2 \end{aligned} \tag{2}$$

in the case $p=\frac{1}{3}$ and f maps $\mathbb{R}$ into $(0,\infty)$.

During the 31st ISFE (Debrecen, 1993), C. Alsina [1] asked to find continuous solution of (2) where $p\in(0,1)$ is fixed and to solve (2) on a

---

Mathematics Subject Classification: 39B22.
Key words and phrases: functional equation, general solution, k-additive symmetric functions.
Research supported by the Hungarian National Foundation for Scientific Research No. T-016846 and by the Ministry of Culture and Education of Hungary under Grant no. 0310/97 FKFP.

restricted domain. During the meeting, the first question was answered by A. Járai and Gy. Maksa [4] and an answer to the second question was given by W. Jarczyk [5].

During the 32 ISFE (Gargnano, 1994) W. Jarczyk and M. Sablik [6] presented some other results connected with the system (1), (2). They gave two results on equation (1) and (2) treated separately (see also [7]).

In connection with the functional equation (1), W. Jarczyk and M. Sablik ([6], [7]) found the following

Theorem. *Let $f : I = [a, b] \subset \mathbb{R} \to (0, \infty)$ be a solution of (1) with some $p \in (0, 1)$. Then*

(i) *if $p \in Q$ then $f(x) = Be^{a(x)} (x \in I)$, where $B > 0$ is a constant and $a : \mathbb{R} \to \mathbb{R}$ is an additive function;*

(ii) *if f is continuous then $f(x) = Be^{Ax}$ $(x \in I)$, where $A \in \mathbb{R}$ and $B > 0$ are some constans.*

The proof of this Theorem based on a result of Gy. Maksa, K. Nikodem and Zs. Páles (see [9]) connected to the t-Wright convexity.

The aim of this paper is to present the general solution of (1) for functions $f : \mathbb{R} \to \mathbb{R}$ or $f : I \to \mathbb{R}$ respectively, supposing the existence a subset $E \subset \mathbb{R}$ (or $E \subset I$) of positive Lebesgue measure such that $f(x) \neq 0$ for all $x \in E$.

2. The general solution of (1) on $\mathbb{R}$

Lemma 1. *If the function $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation (1) for all $x, y \in \mathbb{R}$ and there exists a subset $E \subset \mathbb{R}$ of positive Lebesgue measure such that $f(x) \neq 0$ for all $x \in E$, then $f(x) \neq 0$ for any $x \in \mathbb{R}$.*

Proof (see also [8]). If $p = \frac{1}{2}$, then we obtain from (1) the functional equation

$$(1') \qquad f^2\left(\frac{x+y}{2}\right) = f(x)f(y) \quad (x, y \in \mathbb{R}).$$

If there exists an $y_0 \in \mathbb{R}$ such that $f(y_0) = 0$, then (1') shows that $f(x) = 0$ for all $x \in \mathbb{R}$, which is a contradiction. Then $f(x) \neq 0$ for any $x \in \mathbb{R}$.

If $p \neq \frac{1}{2}$ then, under our assumptions, (1) implies that $f(u) \neq 0$ if $u \in pE+(1-p)E$. By Steinhaus's theorem, the set $pE+(1-p)E$ contains a nonvoid interval $I_1 \subset \mathbb{R}$ and $f(x) \neq 0$ for all $x \in I_1$. By the transformation

$$u = px + (1-p)y, \quad v = (1-p)x + py,$$

we get from (1) the functional equation

$$f(u)f(v) = f\left(\frac{p}{2p-1}u + \frac{p-1}{2p-1}v\right) f\left(\frac{p-1}{2p-1}u + \frac{p}{2p-1}v\right) \tag{3}$$

for all $u, v \in \mathbb{R}$. Using (3), one can easily verify that, if $f(u) \neq 0$ for all $u \in I_1$, then $f(x) \neq 0$ for all $x \in I_2$, where I_2 has the same center as I_1 and $|I_2| = \left|\frac{1}{2p-1}\right| |I_1| > |I_1|$. Now define a sequence $\langle I_n \rangle$ of intervals as folows: if I_n is given then I_{n+1} has the same center as I_n and $|I_{n+1}| = \left|\frac{1}{2p-1}\right| |I_n|$. By repeating the above argument and using induction, we find that $f(x) \neq 0$ for all $x \in I_n (n \in N)$, where $|I_n| = \left|\frac{1}{2p-1}\right|^{n-1} |I_1|$. But $\left|\frac{1}{2p-1}\right| > 1$, thus $\lim\limits_{n\to\infty} |I_n| = |I_1| \lim\limits_{n\to\infty} \left|\frac{1}{2p-1}\right|^{n-1} = +\infty$. Hence, we infer that $f(x) \neq 0$ for all $x \in \mathbb{R}$.

Lemma 2. *If the function $f : \mathbb{R} \to \mathbb{R}$, satisfies the functional equation (1) for all $x, y \in \mathbb{R}$, and $f(x) \neq 0$ $(x \in \mathbb{R})$, then $\frac{f(x)}{f(0)} > 0$ for any $x \in \mathbb{R}$.*

Proof. Let us write in (1) $x + t$ and $y + s$ for x and y, respectively, such that $(1-p)t + ps = 0$. Then we get the equation

$$f(x+t)f(y+s) = f(px + (1-p)y + pt + (1-p)s)f((1-p)x + py) \tag{4}$$

for all $x, y, t \in \mathbb{R}$, $s = \frac{p-1}{p}t$.

Since $f(x) \neq 0$ for all $x \in \mathbb{R}$, we obtain from (1) and (4) that

$$\frac{f(x+t)f(y+s)}{f(x)f(y)} = \frac{f(px + (1-p)y + pt + (1-p)s)}{f(px + (1-p)y)} \tag{5}$$

for all $x, y, t \in \mathbb{R}$, $s = \frac{p-1}{p}t$.

Replacing x, y by $x + t$, $y + v$ in (5), respectively, such that $pt + (1-p)v = 0$ (i.e. $v = \frac{p}{p-1}t$), we have

$$\frac{f(x+2t)f(y+s+v)}{f(x+t)f(y+v)} = \frac{f(px + (1-p)y + pt + (1-p)s)}{f(px + (1-p)y)} \tag{6}$$

for all $x, y, t \in \mathbb{R}$, $s = \frac{p-1}{p}t$, $v = \frac{p}{p-1}t$.

The right hand sides of (5) and (6) are equal therefore, we obtain

$$\frac{f(x+2t)(f(y+s+v)}{f(x+t)f(y+v)} = \frac{f(x+t)f(y+s)}{f(x)f(y)} \tag{7}$$

for all $x, y, t \in \mathbb{R}$, $s = \frac{p-1}{p}t$, $v = \frac{p}{p-1}t$.

Putting here $x = 0$, we get

$$\frac{f(2t)f(y+s+v)}{f(t)f(y+v)} = \frac{f(t)f(y+s)}{f(0)f(y)}$$

for all $y, t \in \mathbb{R}$, $s = \frac{p-1}{p}t$, $v = \frac{p}{p-1}t$.

This implies that, together with (7),

$$\frac{f(x+2t)}{f(0)} = \frac{f^2(x+t)f(2t)}{f(x)f^2(t)} \tag{8}$$

for all $x, t \in \mathbb{R}$.

Substitute $x = \frac{u}{2}$, $t = \frac{u}{4}$ in (8), then

$$\frac{f(u)}{f(0)} = \frac{f^2\left(\frac{3}{4}u\right)}{f^2\left(\frac{1}{4}u\right)} > 0 \quad (u \in \mathbb{R}). \tag{9}$$

This completes the proof of Lemma 2.

Lemma 3. *If the function $f : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ satisfies* (1) *for all $x, y \in \mathbb{R}$, then the function*

$$F : \mathbb{R} \to \mathbb{R}, \qquad F(x) = \ln \frac{f(x)}{f(0)} \tag{10}$$

satisfies the functional equation

$$\Delta_t^3 F(x) = F(x+3t) - 3F(x+2t) + 3F(x+t) - F(x) = 0 \tag{11}$$

for all $x, t \in \mathbb{R}$.

Proof. From the previous lemma, it follows that (8) is satisfied. Let us write in (8) $x + t$ for x, then we get the equation

$$\frac{f(x+3t)}{f(0)} = \frac{f^2(x+2t)f(2t)}{f(x+t)f^2(t)} \quad (x, t \in \mathbb{R}). \tag{12}$$

Dividing this equation by (8), we obtain

$$f(x+3t)f^3(x+t) = f^3(x+2t)f(x) \quad (x,t \in \mathbb{R}). \tag{13}$$

Dividing (13) by $f^4(0)$ and using that $\frac{f(x)}{f(0)} > 0\,(x \in \mathbb{R})$, we obtain that the function F defined by (10) satisfies the functional equation (11) for all $x, t \in \mathbb{R}$.

Lemma 4 (Djokovič, Székelyhidi). *The function $F : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation* (11) *for all $x, t \in \mathbb{R}$ if and only if there exist k-additive symmetric functions $A_k : \mathbb{R}^k \to \mathbb{R}$ $(k = 0, 1, 2)$ such that*

$$F(x) = A_2(x,x) + A_1(x) + A_0, \quad x \in \mathbb{R} \tag{14}$$

where $A_0 \in \mathbb{R}$ constant.

Proof. See [3], [11].

This results allow us to formulate the following theorem which gives the general solution of our problem for function $f : \mathbb{R} \to \mathbb{R}$.

Theorem 1. *If the function $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation* (1) *for all $x, y \in \mathbb{R}$ and there exists a subset $E \subset \mathbb{R}$ of positive Lebesgue measure such that $f(x) \neq 0$ for all $x \in E$, then f has the form*

$$f(x) = B\exp(A_2(x,x) + A_1(x)), \tag{15}$$

where $A_k : \mathbb{R}^k \to \mathbb{R}$ is k-additive symmetric function $(k = 1, 2)$, B is a real constant and

$$A_2(pt, (1-p)t) = 0 \qquad (t \in \mathbb{R}) \tag{16}$$

holds.

Proof. By Lemma 1, $f(x) \neq 0$ for all $x \in \mathbb{R}$. Then Lemma 2 implies that $\frac{f(x)}{f(0)} > 0$ for any $x \in \mathbb{R}$ and we obtain that, by Lemma 3, F satisfies the functional equation (11) for all $x, t \in \mathbb{R}$. Due to Lemma 4, we have the form (14) for F. Thus, from (10), (15) follows for the function f (since $F(0) = 0$, $A_0 = 0$).

An easy calculation shows that the function (15) satisfies the functional equation (1) if and only if (16) holds.

Corollary. *Under the conditions of Theorem 1*

(i) *if $p \in Q$ then $f(x) = Be^{a(x)}(x \in \mathbb{R})$, where $0 \neq B \in \mathbb{R}$ is a constant and $a : \mathbb{R} \to \mathbb{R}$ is an additive function;*

(ii) *if f is continuous then $f(x) = Be^{Ax}(x \in R)$, where A and $B \neq 0$ are some constants in $\mathbb{R}$.*

Proof. If $p \in Q$, then

$$A_2(pt, (1-p)t = p(1-p)A_2(t,t) = 0$$

gives that $A_2(t,t) = 0$ for all $t \in \mathbb{R}$ and so (15) reduces to (i) with $B = f(0) \neq 0$ and with $a(x) = A_1(x)\,(x \in \mathbb{R})$.

To prove (ii), observe that A_2 and A_1 are continuous if f is, therefore

$$A_1(t) = At, \quad A_2(t,t) = Ct^2 \qquad (t \in \mathbb{R}).$$

Then (16) gives again that $C = 0$, and we obtain (ii).

3. The general solution of (1) on intervals

Let $I \subset \mathbb{R}$ be a real interval and suppose that the function $f : I \to \mathbb{R}$ satisfies the functional equation (1) for all $x, y \in I$.

There is an analogue of Lemma 1 for function $f : I \to \mathbb{R}$ satisfying (1). We have

Lemma 5. *If the function $f : I \to \mathbb{R}$ satisfies the functional equation (1) for all $x, y \in I$ and there exits a subset $E \subset I$ of positive Lebesgue measure such that $f(x) \neq 0$ for all $x \in E$, then $f(x) \neq 0$ for any $x \in I$.*

Proof. If $p = \frac{1}{2}$, the we obtain from (1) the functional equation (1') for all $x, y \in I = [a, b]$. Suppose that $f(y_0) = 0$ for some $y_0 \in I$. Then we infer from equation $f^2\left(\frac{x+y_0}{2}\right) = f(x)f(y_0) = 0$ $(x \in \mathbb{R})$ that $f(x) = 0$ for all $x \in \left[\frac{a+y_0}{2}, \frac{y_0+b}{2}\right]$. By induction, we get for any $n \in N$ that $f(x) = 0$ for all $x \in \left[\frac{(2^n-1)a+y_0}{2^n}, \frac{y_0+(2^n-1)b}{2^n}\right]$. Then, because of $\lim\limits_{n\to\infty} \frac{(2^n-1)a+y_0}{2^n} = a$ and $\lim\limits_{n\to\infty} \frac{y_0+(2^n-1)b}{2^n} = b$, we obtain that $f(x) = 0$ for all $x \in (a, b)$, which is a contradiction. Thus $f(x) \neq 0$ for any $x \in I$.

In case $p \neq \frac{1}{2}$ let the sequence of intervals $\langle I_n \rangle$ be defined as in the proof of Lemma 1 and let $I'_n = I_n \cap I$. Using induction again, we find that

$f(x) \neq 0$ for all $x \in I_n'$ $(n \in N)$. Because of $\lim\limits_{n\to\infty} |I_n| = +\infty$, we get that there exists an $n_0 \in N$ such that $I'_{n_0} = I$ and thus we have $f(x) \neq 0$ for all $x \in I$. The proof for noncompact interval is similar.

Lemma 6 (Páles)**.** *Let $(G, \cdot)$ be an abelian semigroup, $\varphi_i : G \to G$ are homomorphisms with $\varphi_i \circ \varphi_j = \varphi_j \circ \varphi_i$ for all $i, j = 1, \dots, n$ (i.e. pairwise commuting). If the function $f : I \to G$ satisfies the functional equation*

$$f(x) = \prod_{i=1}^{n} \varphi_i(f(\lambda_i x + (1 - \lambda_i)y)) \tag{17}$$

for all $x, y \in I$, where $\lambda_i \in [0,1)$ $(i = 1, \dots, n)$, then there exists a function $\bar{f} : \mathbb{R} \to G$ such that $\bar{f}$ satisfies the functional equation (17) *all $x, y \in \mathbb{R}$ and $\bar{f} |_I = f$.*

Proof. See [10].

Lemma 7. *If the function $f : I \to \mathbb{R} \setminus \{0\}$ satisfies the functional equation* (1) *for all $x, y \in I$, then there exists a function $\bar{f} : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ such that $\bar{f}$ satisfies the functional equation* (1) *for all $x, y \in \mathbb{R}$ and $\bar{f} |_I = f$.*

Proof. Since $f(x) \neq 0$ for all $x \in I$, we infer from (1) the functional equation

$$f(x) = \frac{f(px + (1-p)y) f((1-p)x + p(y))}{f(y)} \tag{18}$$

for all $x, y \in I$.

If $\varphi_1(f) = f$, $\varphi_2(f) = f$ and $\varphi_3(f) = \frac{1}{f}$, then $(\varphi_1 \circ \varphi_2)(f) = (\varphi_2 \circ \varphi_1)(f) = f$, $(\varphi_1 \circ \varphi_3)(f) = (\varphi_3 \circ \varphi_1)(f) = \frac{1}{f}$, $(\varphi_2 \circ \varphi_3)(f) = (\varphi_3 \circ \varphi_2)(f) = \frac{1}{f}$ (i.e. φ_1, φ_2, φ_3 are pairwise commuting homomorphisms). Further, if $\lambda_1 = p$, $\lambda_2 = 1 - p$, $\lambda_3 = 0$, then $\lambda_i \in [0, 1)$ $(i = 1, 2, 3)$.

Thus f satisfies the conditions of Lemma 6, which implies that there exists a function $\bar{f} : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ such that $\bar{f}$ satisfies (18) and so (1) for all $x, y \in \mathbb{R}$ and $\bar{f} |_I = f$.

Now, we can formulate the main result of this part of the paper.

Theorem 2. *If the function $f : I \to \mathbb{R}$ satisfies the functional equation (1) for all $x, y \in I$ and there exists a subset $E \subset I$ of positive Lebesgue measure such that $f(x) \neq 0$ for all $x \in E$, then f has the form*

$$f(x) = B \exp(A_2(x,x) + A_1(x)), \quad x \in I, \tag{19}$$

where $A_k : \mathbb{R}^k \to \mathbb{R}$ is k-additive symmetric function $(k = 1, 2)$, $B \neq 0$ is an arbitrary real constant and

$$A_2(pt, (1-p)t) = 0 \qquad (t \in \mathbb{R}) \tag{20}$$

holds.

Proof. By Lemma 5, $f(x) \neq 0$ for any $x \in I$. Then Lemma 7 implies that there exists a function $\bar{f} : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ such that $\bar{f}$ satisfies the functional equation (1) for all $x, y \in \mathbb{R}$ and $\bar{f}\,|_I = f$.
Now, using Theorem 1, we have

$$\bar{f}(x) = B \exp(A_2(x,x) + A_1(x)), \quad x \in \mathbb{R},$$

where $A_k : \mathbb{R}^k \to \mathbb{R}$ is k-additive symmetric function $(k = 1, 2)$ and (20) holds.

Finally, $\bar{f}\,|_I = f$ gives (19).

It is easy to see that the function (19) satisfies (1) if (20) holds.

From Theorem 2, similarly as in the proof of our corollary, we get the result of Jarczyk and Sablik.

References

[1] C. Alsina, 1. Problem, Report of the 31st ISFE, August 22–28, 1993, Debrecen, Hungary, *Aeq. Math.* **47** (1994), 302.

[2] C. Alsina and J. L. García-Roig, On two functional equations related to a result of Grégorie de Saint Vincent, *Results in Math.* **28** (1995), 33–39.

[3] D. Ž. Djokovič, A representation theorem for $(X_1 - 1) \dots (X_n - 1)$ and its applications, *Ann. Polon. Math.* **22** (1969), 189–198.

[4] A. Járai and Gy. Maksa, 2. Remark, Report of the 31st ISFE, August 22–28, 1993, Debrecen, Hungary, *Aeq. Math.* **47** (1994), 302.

[5] W. Jarczyk, 3. Remark, Report of the 31st ISFE, August 22–28, 1993, Debrecen, Hungary, *Aeq. Math.* **47** (1994), 303.

[6] W. Jarczyk and M. Sablik, Duplicating the cube and functional equations, Report of the 32nd ISFE, June 12–19, 1994, Gargnano, Italy, *Aeq. Math.* **49** (1995), 179–180.

[7] W. Jarczyk and M. Sablik, Duplicating the cube and functional equations, *Results in Math.* **26** (1994), 324–335.

[8] K. Lajkó, 8. Remark , Report of the 32nd ISFE, June 12–19, 1994, Gargnano, Italy, *Aeq. Math.* **49** (1995), 186–187.

[9] Gy. Maksa, K. Nikodem and Zs. Páles, Results on t-Wright convexity, *C.R. Math. Rep. Acad. Sci. Canada* **13** (1991), 274–278.

[10] Zs. Páles, Extension theorems for functional equations with bisymmetric operations, (*preprint*).

[11] L. Székelyhidi, An extension theorem for a functional equation, *Publ. Math., Debrecen* **28** (1981), 257–259.

[12] L. Székelyhidi, Convolution Type Functional Equations on Toplogical Abelian Groups, *World Sci.*, *Singapur, New Jersey, London, Hong Kong*, 1991.

KÁROLY LAJKÓ
INSTITUTE OF MATHEMATICS AND INFORMATICS
LAJOS KOSSSUTH UNIVERSITY
H–4010 DEBRECEN, P.O. BOX 12
HUNGARY

(Received October 28, 1997)