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A limit theorem in the theory of finite Abelian groups

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In honour of Professors Zoltán Daróczy and Imre Kátai on their 60th birthday

Abstract. In the paper a limit theorem in the sense of the weak convergence of probability measures for a Dirichlet series used in the theory of finite Abelian groups in the space of analytic functions is obtained.

1. Introduction

Let $\mathcal{G}$ be a finite Abelian group of order $|\mathcal{G}|$. Denote by $\tau(\mathcal{G})$ and $r(\mathcal{G})$ the number of subgroups of $\mathcal{G}$ and the rank of $\mathcal{G}$, respectively. Let, as usual, $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{C}$ denote the sets of real, natural, integer and complex numbers, respectively. It is known that the group $\mathcal{G}$ has rank r if

$$\mathcal{G} \cong \mathbb{Z}/m_1\mathbb{Z} \otimes \cdots \otimes \mathbb{Z}/m_r\mathbb{Z},$$

where $m_j \mid m_{j+1}$ for $j = 1, \dots, r-1$.

Let

$$t_r(m) = \sum_{|\mathcal{G}|=m,\ r(\mathcal{G})\le r} \tau(\mathcal{G}).$$

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In several recent papers the sum function

$$T(x)=\sum_{m\le x} t_2(m)$$

has been studied. Let

$$\Delta(x)=T(x)-K_1x\log^2 x-K_2x\log x-K_3x,$$

where K_1, K_2 and K_3 are effective constants. Denote by B a number bounded by a constant. G. Bhowmik and H. Menzer [3] proved that

$$\Delta(x)=Bx^{c+\varepsilon} \tag{1}$$

with $c=31/43$ for every positive ε. H. Menzer [12] improved the estimate (1) until $c=9/14$, and he also conjectured that

$$\Delta(x)=\Omega(x^{1/2}\log^2 x).$$

The latter conjecture was proved by G. Bhowmik and J. Wu in [4]. Moreover, they obtained a bound

$$\Delta(x)=Bx^{5/8}\log^4 x.$$

Finally, A. Ivič [6] investigated the mean square of the error term $\Delta(x)$ and proved that

$$\int_1^x \Delta^2(u)\,du=Bx^2(\log x)^{31/3}(\log\log x)^{28/3},$$

and

$$\int_1^x \Delta^2(u)\,du=\Omega(x^2\log^4 x).$$

These results allowed him to conjecture that

$$\int_1^x \Delta^2(u)\,du\sim Cx^2\log^4 x,\quad x\to\infty,$$

with a suitable constant $C>0$.

Let $s=\sigma+it$ be a complex variable. The Dirichlet series

$$H(s)=\sum_{m=1}^{\infty}\frac{t_2(m)}{m^s},\quad \sigma>1,$$

plays an important role in the proofs of the results mentioned above. G. Bhowmik and O. Ramaré [2], see also [4], obtained that, for $\sigma > 1/2$,

$$H(s) = \zeta^2(s)\zeta^2(2s)\zeta(2s-1)\prod_p \left(1 + \frac{1}{p^{2s}} - \frac{2}{p^{3s}}\right), \tag{2}$$

where, as usual, $\zeta(s)$ stands for the Riemann zeta-function. This gives the analytic continuation of $H(s)$ over the half-plane $\sigma > 1/2$ except for the pole of order 3 at the point $s = 1$. Our aim is to study the statistical properties of the function $H(s)$, and in this note we will prove a limit theorem in the sense of the weak convergence of probability measures in the space of analytic functions for $H(s)$. The theory of functional limit theorems for Dirichlet series was obtained by B. Bagchi in [1] (see also [8]), and we will use his ideas.

Let γ be the unit circle on the complex plane $\mathbb{C}$, i.e. $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, and let

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for each prime number p. With the product topology and pointwise multiplication Ω is a compact Abelian topological group. Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S. Then there exists the probability Haar measure m_H on $(\Omega, \mathcal{B}(\Omega))$. This yields a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Let $\omega(p)$ stand for the projection of $\omega \in \Omega$ to the coordinate space γ_p. Setting

$$\omega(k) = \prod_{p^\alpha \| k} \omega^\alpha(p),$$

where $p^\alpha \| k$ means that $p^\alpha \mid k$ but $p^{\alpha+1} \nmid k$, we obtain an extension of $\omega(p)$ to the set $\mathbb{N}$ as a completely multiplicative unimodular function.

Let G be a region on $\mathbb{C}$. Denote by $H(G)$ and $M(G)$ the spaces of analytic and meromorphic functions on G, respectively, equipped with the topology of uniform convergence on compacta. Let $D = \{s \in \mathbb{C} : \sigma > 3/4\}$. Now we may define an $H(D)$-valued random element on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$:

$$H(s,\omega) = \prod_p \left(1 - \frac{\omega(p)}{p^s}\right)^{-2} \prod_p \left(1 - \frac{\omega^2(p)}{p^{2s}}\right)^{-2} \prod_p \left(1 - \frac{\omega^2(p)}{p^{2s-1}}\right)^{-1}$$
$$\times \prod_p \left(1 + \frac{\omega^2(p)}{p^{2s}} - \frac{2\omega^3(p)}{p^{3s}}\right), \quad \omega \in \Omega,\ s \in D.$$

The second and the last products in this formula converge uniformly for $\sigma > 3/4$, and therefore, they define $H(D)$-valued random elements. It is easy to check similarly as in [8] that the first and the third products converge uniformly for almost all $\omega \in \Omega$ on every compact subsets of D. Hence they define $H(D)$-valued random elements. Thus $H(s,\omega)$ is an $H(D)$-valued random element defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$.

Let $w(t)$ be a positive function of bounded variation on $[T_0,\infty)$, $T_0 > 0$, such that its variation $V_a^b w$ on $[a,b]$ satisfies the inequality $V_a^b w \le cw(a)$ with some $c > 0$ for all $b \ge a \ge T_0$. Moreover, let

$$U = U(T,w) = \int_{T_0}^{T} w(t)\,dt,$$

and suppose that $\lim_{T\to\infty} U(T,w) = \infty$. In addition we assume that the function $w(t)$ satisfies some special condition related to the ergodic theory. Let $X(\tau,\omega)$ be an ergodic process with $E|X(\tau,\omega)| < \infty$, and let its sample paths be integrable almost surely in the Riemann sense over every finite interval. Here EX denotes the mean of the random variable X. Then we suppose that

$$\frac{1}{U}\int_{T_0}^{T} w(\tau)X(t+\tau,\omega)\,d\tau = EX(0,\omega) + \mathrm{o}(1+|t|)^{\alpha} \tag{3}$$

almost surely for all $t \in \mathbb{R}$ with some $\alpha > 0$ as $T \to \infty$. The latter equality is a generalization of the classical Birkhoff–Khinchin theorem which asserts that

$$\lim_{T\to\infty} \frac{1}{T}\int_{0}^{T} X(\tau,\omega)\,d\tau = EX(0,\omega) \tag{4}$$

almost surely. Thus (3) with $w(t) \equiv 1$ implies (4). Some examples of functions satisfying (3) can be found in [11].

Let $D_1 = \{s \in \mathbb{C} : 3/4 < \sigma < 1\}$, $D_2 = \{s \in \mathbb{C} : \sigma > 1\}$, and let I_A denote the indicator function of the set A. Define two probability measures

$$P_{j,T,w}(A) = \frac{1}{U}\int_{T_0}^{T} w(\tau) I_{\{\tau : H(s+i\tau)\in A\}} d\tau, \quad A \in \mathcal{B}(H(D_j)),\ j = 1,2.$$

Let P_ξ denote the distribution of an $H(D)$-valued random element ξ, and let $P_{j,\xi}$ be the restriction of P_ξ to $(H(D_j), \mathcal{B}(H(D_j)))$, $j = 1,2$.

Theorem. *The measures $P_{j,T,w}$ converge weakly to $P_{j,H}$ as $T \to \infty$.*

Denote by meas$\{A\}$ the Lebesgue measure of the set A, and set, for $T > 0$,

$$\nu_T^\tau(\cdots) = \frac{1}{T}\,\mathrm{meas}\{\tau \in [0,T], \dots\},$$

where instead of dots we write a condition satisfied by τ. Now let

$$P_{j,T}(A) = \nu_T^\tau\big(H(s+i\tau) \in A\big), \quad A \in \mathcal{B}(H(D_j)),\ j = 1,2.$$

Corollary. *The measures $P_{j,T}$ converge weakly to $P_{j,H}$ as $T \to \infty$.*

Thus, the limit measure in the Theorem is independent of the function $w(t)$. This is a consequence of (3).

Note that the Theorem can be used for the investigation of the universality of the function $H(s)$.

Since the case $j = 2$ is simpler and similar to that of $j = 1$, we will consider the case $j = 1$ only.

2. Auxiliary results

For the proof of the theorem we will apply the fact that each multiplier in (2) has limit distribution. Let

$$Q_T^{(1)}(A) = \frac{1}{U}\int_{T_0}^{T} w(\tau) I_{\{\tau:\zeta(s+i\tau)\in A\}} d\tau, \quad A \in \mathcal{B}(H(D_1)),$$

and

$$\zeta_1(s,\omega) = \prod_p \left(1 - \frac{\omega(p)}{p^s}\right)^{-1}, \quad \omega \in \Omega,\ s \in D.$$

Lemma 1. *The measure $Q_T^{(1)}$ converges weakly to P_{1,ζ_1} as $T \to \infty$.*

Proof. The lemma is a Theorem from [11] with such a difference that in [11] $D_1 = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$.

Now let

$$Q_T^{(2)}(A) = \frac{1}{U}\int_{T_0}^{T} w(\tau) I_{\{\tau:\zeta(2(s+i\tau))\in A\}} d\tau, \quad A \in \mathcal{B}(H(D_1)),$$

and

$$\zeta_2(s,\omega)=\prod_p\left(1-\frac{\omega^2(p)}{p^{2s}}\right)^{-1},\quad \omega\in\Omega,\ s\in D.$$

Lemma 2. *The measure $Q_T^{(2)}$ converges weakly to P_{1,ζ_2} as $T\to\infty$.*

Proof. It coincides with that of Lemma 1. We note only that the Dirichlet series for $\zeta(2s)$ is

$$\sum_{m=1}^{\infty}\frac{a_m}{m^s},$$

where

$$a_m=\begin{cases}1 & \text{if } m=k^2,\\ 0 & \text{otherwise.}\end{cases}$$

Thus in this case

$$\zeta_2(s,\omega)=\sum_{m=1}^{\infty}\frac{a_m\omega(m)}{m^s}=\sum_{m=1}^{\infty}\frac{\omega(m^2)}{m^{2s}}=\sum_{m=1}^{\infty}\frac{\omega^2(m)}{m^{2s}}$$
$$=\prod_p\left(1-\frac{\omega^2(p)}{p^{2s}}\right)^{-1}.$$

Now we set

$$V_T^{(3)}(A)=\frac{1}{U}\int_{T_0}^{T}w(\tau)I_{\{\tau:\zeta(2(s+i\tau)-1)\in A\}}d\tau,\quad A\in\mathcal{B}(H(D_1)),$$

and

$$\xi_3(s,\omega)=\prod_p\left(1-\frac{\omega^2(p)}{p^{2s-1}}\right)^{-1},\quad \omega\in\Omega,\ s\in D.$$

Lemma 3. *The measure $V_T^{(3)}$ converges weakly to P_{1,ξ_3} as $T\to\infty$.*

Proof. It repeats the arguments of that of Lemmas 1 and 2. The Dirichlet series for $\zeta(2s-1)$ is

$$\sum_{m=1}^{\infty}\frac{a_m}{m^s},$$

where

$$a_m = \begin{cases} \sqrt{m} & \text{if } m = k^2, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, we have

$$\begin{aligned} \xi_3(s,\omega) &= \sum_{m=1}^{\infty} \frac{a_m \omega(m)}{m^s} = \sum_{m=1}^{\infty} \frac{m\omega(m^2)}{m^{2s}} \\ &= \sum_{m=1}^{\infty} \frac{\omega^2(m)}{m^{2s-1}} = \prod_p \left(1 - \frac{\omega^2(p)}{p^{2s-1}}\right)^{-1} \end{aligned}$$

for almost all $\omega \in \Omega$.

For all $\sigma > 1/2$ let

$$U(s) = \sum_{m=1}^{\infty} \frac{u_m}{m^s} = \prod_p \left(1 + \frac{1}{p^{2s}} - \frac{2}{p^{3s}}\right),$$

the latter Dirichlet series being absolutely convergent for $\sigma > 1/2$. Moreover, let

$$V_T^{(4)}(A) = \frac{1}{U} \int_{T_0}^{T} w(\tau) I_{\{\tau : U(s+i\tau) \in A\}} d\tau, \quad A \in \mathcal{B}(H(D_1)),$$

and

$$\xi_4(s,\omega) = \prod_p \left(1 + \frac{\omega^2(p)}{p^{2s}} - \frac{2\omega^3(p)}{p^{3s}}\right), \quad \omega \in \Omega, \ s \in D.$$

Lemma 4. *The measure $V_T^{(4)}$ converges weakly to P_{1,ξ_4} as $T \to \infty$.*

Proof. We give only a sketch of the proof because of its similarity to the proof of a Theorem from [9]. Let

$$\begin{aligned} U_n(s) &= \sum_{m=1}^{n} \frac{u_m}{m^s}, \\ U_n(s,\omega) &= \sum_{m=1}^{n} \frac{u_m \omega(m)}{m^s}, \quad \omega \in \Omega. \end{aligned}$$

Then Lemma 2 of [11] asserts that the probability measures

$$\frac{1}{U}\int_{T_0}^{T} w(\tau) I_{\{\tau: U_n(s+i\tau)\in A\}} d\tau, \quad A\in\mathcal{B}(H(D_1)),$$

and

$$\frac{1}{U}\int_{T_0}^{T} w(\tau) I_{\{\tau: U_n(s+i\tau,\omega)\in A\}} d\tau, \quad A\in\mathcal{B}(H(D_1)),$$

converge weakly to the same measure as $T\to\infty$. Using this and the relations

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{U}\int_{T_0}^{T} w(\tau)\sup_{s\in K}\left|U(s+i\tau)-U_n(s+i\tau)\right| d\tau = 0,$$

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{U}\int_{T_0}^{T} w(\tau)\sup_{s\in K}\left|\xi_4(s+i\tau,\omega)-U_n(s+i\tau,\omega)\right| d\tau = 0$$

which are valid for any compact subset K of D_1, we obtain that the measures $V_T^{(4)}$ and

$$\frac{1}{U}\int_{T_0}^{T} w(\tau) I_{\{\tau: \xi_4(s+i\tau,\omega)\in A\}} d\tau, \quad A\in\mathcal{B}(H(D_1)),$$

converge weakly to some measure V as $T\to\infty$ simultaneously. The last step of the proof consists of the checking that V coincides with P_{1,ξ_4}. The arguments used to show this are similar to those of [9], and involve elements of the ergodic theory.

Let S and S_1 be two metric spaces, and let $h: S\to S_1$ be a measurable function. Then every probability measure P on $(S,\mathcal{B}(S))$ induces a unique probability measure Ph^{-1} on $(S_1,\mathcal{B}(S_1))$ defined by $Ph^{-1}(A)=P(h^{-1}A)$, $A\in\mathcal{B}(S_1)$.

Lemma 5. *Let $h: S\to S_1$ be a continuous function, and let P_n and P be probability measures on $(S,\mathcal{B}(S))$. Suppose that P_n converges weakly to P as $n\to\infty$. Then $P_n h^{-1}$ converges weakly to Ph^{-1} as $n\to\infty$.*

Proof. This is a special case of Theorem 5.1 of [5].

Now let

$$V_T^{(1)}(A) = \frac{1}{U}\int_{T_0}^{T} w(\tau) I_{\{\tau:\zeta^2(s+i\tau)\in A\}} d\tau, \quad A \in \mathcal{B}(H(D_1)),$$

$$V_T^{(2)}(A) = \frac{1}{U}\int_{T_0}^{T} w(\tau) I_{\{\tau:\zeta^2(2(s+i\tau))\in A\}} d\tau, \quad A \in \mathcal{B}(H(D_1)),$$

and

$$\xi_1(s,\omega) = \prod_p \left(1 - \frac{\omega(p)}{p^s}\right)^{-2}, \quad \omega \in \Omega,\ s \in D_1,$$

$$\xi_2(s,\omega) = \prod_p \left(1 - \frac{\omega^2(p)}{p^{2s}}\right)^{-2}, \quad \omega \in \Omega,\ s \in D_1.$$

Lemma 6. *The measures $V_T^{(1)}$ and $V_T^{(2)}$ converge weakly to the measures P_{1,ξ_1} and P_{1,ξ_2}, respectively, as $T \to \infty$.*

Proof. The assertion of the lemma immediately follows from Lemmas 1, 2 and 5.

3. A fourdimensional limit theorem

In the previous section we have seen that the functions $\zeta^2(s)$, $\zeta^2(2s)$, $\zeta(2s-1)$ and $U(s)$ have a limit distribution in the space $H(D_1)$. In this section we will prove a joint limit theorem for these functions. We will use the following notation. We put

$$\xi_1(s) = \zeta^2(s), \qquad \xi_2(s) = \zeta^2(2s),$$
$$\xi_3(s) = \zeta(2s-1), \qquad \xi_4(s) = U(s),$$

and denote by $v_j(m)$ the coefficients of the Dirichlet series of $\xi_j(s)$, $j = 1, \dots, 4$. Moreover, let

$$\Phi(s) = \big(\xi_1(s), \dots, \xi_4(s)\big),$$

$$\Phi(s,\omega) = \big(\xi_1(s,\omega), \dots, \xi_4(s,\omega)\big), \quad \omega \in \Omega, s \in D_1,$$

where

$$\xi_j(s,\omega)=\sum_{m=1}^{\infty} v_j(m)\frac{\omega(m)}{m^s}.$$

Thus $\Phi(s,\omega)$ is an $H^4(D_1)$-valued random element, where $H^4(D_1)$ denotes the Cartesian product of $H(D_1)\times H(D_1)\times H(D_1)\times H(D_1)$. Let as above P_Φ stand for the distribution of $\Phi(s,\omega)$, and define the probability measure

$$P_{T,w}^{(4)}(A)=\frac{1}{U}\int_{T_0}^{T} w(\tau)I_{\{\tau:\Phi(s+i\tau)\in A\}}d\tau, \quad A\in\mathcal{B}(H^4(D_1)).$$

Proposition. *The measure $P_{T,w}^{(4)}$ converges weakly to P_Φ as $T\to\infty$.*

We divide the proof of the proposition into three parts, and we state two first parts as individual lemmas. Before that we recall that the family of probability measures $\{P\}$ is relatively compact if every sequence of elements of $\{P\}$ contains a weakly convergent subsequence, and the family $\{P\}$ is tight if for an arbitrary $\varepsilon>0$ there exists a compact set K such that $P(K)>1-\varepsilon$ for all P from $\{P\}$.

Lemma 7. *The family of probability measures $P_{T,w}^{(4)}$ is relatively compact.*

Proof. By Lemmas 3, 4 and 6 we have that the probability measures $V_T^{(l)}$ converge weakly to the measures P_{1,ξ_l}, respectively, as $T\to\infty$, $l=1,\dots,4$. Consequently, the family of probability measures $\{V_T^{(l)}\}$ is relatively compact, $l=1,\dots,4$. The space of analytic functions $H(D_1)$ is a complete separable space. Hence it follows by the Prokhorov theorem (see, for example, [5], Theorem 6.2) that the family $\{V_T^{(l)}\}$ is tight, $l=1,\dots,4$. Thus for an arbitrary $\varepsilon>0$ there exists a compact set $K_l\in H(D_1)$ such that

$$V_T^{(l)}\big(H(D_1)\backslash K_l\big)<\frac{\varepsilon}{4}, \quad l=1,\dots,4. \tag{5}$$

Let η_T be a random variable on $(\widetilde{\Omega},\mathcal{F},\mathbb{P})$ such that

$$\mathbb{P}(\eta_T\in A)=\frac{1}{U}\int_{T_0}^{T} w(t)I_A\,dt, \quad A\in\mathcal{B}(\mathbb{R}),$$

and let

$$\xi_{l,T}(s) = \xi_l(s + i\eta_T), \quad l = 1, \dots, 4,$$
$$\Phi_T(s) = \big(\xi_{1,T}(s), \dots, \xi_{4,T}(s)\big).$$

Then in view of (5) the definition of $V_T^{(l)}$ yields

$$\mathbb{P}\big(\xi_{l,T}(s) \in H(D_1)\backslash K_l\big) < \frac{\varepsilon}{4}, \quad l = 1, \dots, 4. \tag{6}$$

Now let us take $K = K_1 \times \cdots \times K_4$. Then K is a compact set of the space $H^4(D_1)$, and in virtue of (6)

$$\begin{aligned} P_{T,w}^{(4)}\big(H^4(D_1)\backslash K\big) &= \mathbb{P}\big(\Phi_T(s) \in H^4(D_1)\backslash K\big) \\ &= \mathbb{P}\Big(\bigcup_{l=1}^{4} \big(\xi_{l,T}(s) \in H(D_1)\backslash K_l\big)\Big) \\ &\le \sum_{l=1}^{4} \mathbb{P}\big(\xi_{l,T}(s) \in H(D_1)\backslash K_l\big) < \varepsilon. \end{aligned}$$

This shows that the family of probability measures $\{P_{T,w}^{(4)}\}$ is tight. Hence by the Prokhorov theorem (Theorem 6.1 of [5]) it is relatively compact.

Now let $s_1, \dots, s_n$ be arbitrary points on D_1, and we set

$$\sigma_1 = \min_{1 \le m \le n} \operatorname{Re} s_m, \quad \sigma_2 = \max_{1 \le m \le n} \operatorname{Re} s_m.$$

Then we have $\sigma_1 > 3/4$. Moreover, let $\sigma_3 = 3/4 - \sigma_1 < 0$, $\sigma_4 = 1 - \sigma_2 > 0$ and $D_3 = \{s \in \mathbb{C} : \sigma_3 < \sigma < \sigma_4\}$. We take arbitrary complex numbers u_{lm}, and let the function $h : H^4(D_1) \to H(D_3)$ be given by the formula

$$h(f_1, \dots, f_4) = \sum_{l=1}^{4} \sum_{m=1}^{n} u_{lm} f_l(s_m + s),$$

where $s \in D_3$, $f_l \in H(D_1)$, $l = 1, \dots, 4$. We set

$$W(s) = h\big(\xi_1(s), \dots, \xi_4(s)\big),$$

and denote the convergence in distribution by $\xrightarrow{\mathcal{D}}$.

Lemma 8. *The relation*

$$W(s+i\eta_T) \underset{T\to\infty}{\overset{\mathcal{D}}{\to}} h(\Phi)$$

holds.

For the proof of Lemma 8 we need the following result.

Lemma 9. *Let for $\sigma > \sigma_0 + 1/2$ the function $f(s)$ be given by an absolutely convergent Dirichlet series*

$$\sum_{m=1}^{\infty} \frac{a_m}{m^s},$$

such that $\sum_{m\le n} |a_m|^2 = Bn^{2\sigma_0}$. Suppose that $f(s)$ is a meromorphic function in the half-plane $\sigma > \sigma_0$, all poles in this region are included in a compact set, and for $\sigma \ge \sigma_0$

$$f(\sigma+it) = B|t|^{\delta}$$

with some positive δ. Moreover, suppose that the functions $w(\tau)$ and $f(s)$ satisfy the estimate

$$\int_{T_0}^{T} w(\tau)|f(\sigma+it+i\tau)|\,d\tau = BU(1+|t|)^{\beta}$$

with some positive β for all $\sigma > \sigma_0$ and all $t \in \mathbb{R}$. Then the probability measure

$$\frac{1}{U}\int_{T_0}^{T} w(\tau) I_{\{\tau: f(s+i\tau)\in A\}} d\tau, \quad A \in \mathcal{B}(M(D_0)),$$

where $D_0 = \{s \in \mathbb{C} : \sigma > \sigma_0\}$, converges weakly to the distribution of the random element

$$\sum_{m=1}^{\infty} \frac{a_m \omega(m)}{m^s}, \quad \omega \in \Omega,\ s \in D_0,$$

as $T \to \infty$.

Proof. The lemma is a Theorem of [7] proved for the Matsumoto zeta-function which satisfies all conditions of the lemma. The general case of the lemma is identic to that of [7].

Proof of Lemma 8. For $\sigma > \sigma_3 + 1/4$ we have

$$W(s) = \sum_{l=1}^{4} \sum_{m=1}^{n} u_{lm} \xi_l(s_m + s).$$

Suppose that in this region

$$\xi_l(s) = \sum_{m=1}^{\infty} \frac{v_l(m)}{m^s}.$$

Thus

$$W(s) = \sum_{k=1}^{\infty} \frac{w_k}{k^s},$$

where

$$w_k = \sum_{l=1}^{4} \sum_{m=1}^{n} \frac{u_{lm} v_l(k)}{k^{s_m}}.$$

By Lemma 4 of [10]

$$\int_{T_0}^{T} w(\tau) |\zeta(\sigma + it + i\tau)|^2 \, d\tau = BU(1 + |t|)^2$$

for $\sigma > 1/2$ and for all $t \in \mathbb{R}$. Consequently, the well-known properties of the Riemann zeta-function as well as of $U(s)$ and Lemma 9 with $\sigma_0 = 3/4$ yield the weak convergence of the probability measure

$$\frac{1}{U} \int_{T_0}^{T} w(\tau) I_{\{\tau : W(s+i\tau) \in A\}} d\tau, \quad A \in \mathcal{B}(H(D_3)), \tag{7}$$

to the measure P_W as $T \to \infty$, where P_W is the distribution of the random element

$$W(s, \omega) = \sum_{k=1}^{\infty} \frac{w_k \omega(k)}{k^s}, \quad \omega \in \Omega, \ s \in D_3.$$

On the other hand,

$$W(s, \omega) = \sum_{l=1}^{4} \sum_{m=1}^{n} u_{lm} \sum_{k=1}^{\infty} \frac{w_k \omega(k)}{k^{s_m + s}} =$$

$$= \sum_{l=1}^{4} \sum_{m=1}^{n} u_{lm} \xi_l(s_m + s, \omega) = h(\Phi(s, \omega)).$$

Thus the measure (7) converges weakly to P_W as $T \to \infty$. This proves the lemma.

Proof of the Proposition. By Lemma 7 there exists a sequence $T_1 \to \infty$ such that the measure $P^{(4)}_{T_1,w}$ converges weakly to some probability measure P as $T_1 \to \infty$. Suppose that P is the distribution of an $H^4(D_1)$-valued random element

$$\Phi_1(s) = \big(\xi_{11}(s), \dots, \xi_{14}(s)\big).$$

Then, clearly,

$$\Phi_{T_1} \xrightarrow[T_1\to\infty]{\mathcal{D}} \Phi_1. \tag{8}$$

Taking into account the continuity of the function h, hence and from Lemma 5 we deduce that

$$h(\Phi_{T_1}) \xrightarrow[T_1\to\infty]{\mathcal{D}} h(\Phi_1).$$

Therefore, by the definition of W

$$W(s+i\eta_T) \xrightarrow[T_1\to\infty]{\mathcal{D}} h(\Phi_1). \tag{9}$$

By Lemma 8

$$W(s+i\eta_T) \xrightarrow[T_1\to\infty]{\mathcal{D}} h(\Phi).$$

Hence, and from (9)

$$h(\Phi) \overset{\mathcal{D}}{=} h(\Phi). \tag{10}$$

Now let $h_1 : H(D_3) \to \mathbb{C}$ be defined by the formula

$$h_1(f) = f(0), \quad f \in H(D_3).$$

This function, clearly, is measurable. Therefore (10) implies the relation

$$h(\Phi)(0) \overset{\mathcal{D}}{=} h(\Phi_1)(0).$$

Thus by the definition of h we find that

$$\sum_{l=1}^{4}\sum_{m=1}^{n} u_{lm}\xi_l(s_m) \overset{\mathcal{D}}{=} \sum_{l=1}^{4}\sum_{m=1}^{n} u_{lm}\xi_{1l}(s_m) \tag{11}$$

for arbitrary complex numbers u_{lm}. The hyperplanes in the space $\mathbb{R}^{8n}$ form a determining class (see [5]). Therefore, the hyperplanes also form a determining class in the space $\mathbb{C}^{4n}$. Thus, taking into account (11), we obtain that $\mathbb{C}^{4n}$-valued random elements $\xi_l(s_m)$ and $\xi_{1l}(s_m)$, $l = 1, \dots, 4$, $m = 1, \dots, n$, have the same distribution.

Now let K be a compact subset of the strip D_1, and let $f_1, \dots, f_4 \in H(D_1)$. For an arbitrary $\varepsilon > 0$ we set

$$G = \Big\{(g_1, \dots, g_4) \in H^4(D_1) : \sup_{s\in K} |g_l(s) - f_l(s)| \le \varepsilon,\ l = 1, \dots, 4\Big\},$$

and we choose a sequence $\{s_m\}$ to be dense in K. Moreover, let

$$G_n = \Big\{(g_1, \dots, g_4) \in H^4(D_1) : |g_l(s_m) - f_l(s_m)| \le \varepsilon,\\ l = 1, \dots, 4,\ m = 1, \dots, n\Big\}.$$

Thus the above mentioned properties of the random elements $\xi_l(s_m)$ and $\xi_{1l}(s_m)$ show that

$$m_H\big(\omega \in \Omega : \Phi(s, \omega) \in G_n\big) = P(\Phi_1(s) \in G_n). \tag{12}$$

Since the sequence $\{s_m\}$ is dense in K, we have $G_n \to G$ as $n \to \infty$. Thus, letting $n \to \infty$ in (12), we find

$$m_H\big(\omega \in \Omega : \Phi(s, \omega) \in G\big) = P(\Phi_1(s) \in G). \tag{13}$$

The space $H^4(D_1)$ is separable. Therefore, finite intersections of the spheres form a determining class (see [5]). Hence we obtain from (12) and (13) that

$$\Phi \overset{\mathcal{D}}{=} \Phi_1.$$

This and (8) yield

$$\Phi_{T_1} \xrightarrow[T_1\to\infty]{\mathcal{D}} \Phi. \tag{14}$$

This means that the measure $P^{(4)}_{T,w}$ converges weakly to the distribution of the random element Φ as $T_1 \to \infty$. Now the assertion of the Proposition follows from Lemma 7, since the random element Φ in (14) is independent of the choice of the sequence T_1.

4. Proof of the Theorem

Let the function $h : H^4(D_1) \to H(D_1)$ be given by the formula

$$h(f_1, f_2, f_3, f_4) = f_1 f_2 f_3 f_4, \quad f_1, \dots , f_4 \in H(D_1).$$

Since the latter function is continuous, the assertion of the Theorem follows from the Proposition and Lemma 5.

The Corollary is the Theorem with $w(\tau) \equiv 1$.

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