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On equivalence of coefficient conditions with assumptions of monotonicity

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Dedicated to Professors Z. Daróczy and I. Kátai on their 60th birthday

Abstract. Among others we give two different types of sufficient conditions ensuring the equivalence of the conditions

$$\sum_{m=1}^{\infty} \lambda_m \Big(\sum_{n=m}^{\infty} c_n^q \Big)^{p/q} < \infty$$

and

$$\sum_{n=1}^{\infty} \Big(\sum_{k=1}^{n} \lambda_k \Big)^{q/p} c_n^q < \infty,$$

where $\lambda_n \geq 0$, $c_n \geq 0$ and $0 < p < q$.

1. Introduction

Several families of coefficient conditions play a very important role in analysis and especially in the theory of orthogonal series. Recently some papers have studied the relations and equivalences of these conditions. For itemized references regarding the relevant results we refer to [5].

In the same paper we proved the following twin theorems.

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Theorem A. *Let $0 < p < q$, $\{\lambda_n\}$ and $\{c_n\}$ be sequences of nonnegative numbers, furthermore let $\Lambda_n := \sum_{k=1}^n \lambda_k$. The condition*

$$\sum_{m=1}^{\infty} \lambda_m \Big(\sum_{n=m}^{\infty} c_n^q \Big)^{p/q} < \infty \tag{1.1}$$

holds if and only if there exists a nondecreasing sequence $\{\mu_n\}$ of positive numbers satisfying the conditions

$$\sum_{n=1}^{\infty} c_n^q \mu_n < \infty \tag{1.2}$$

and

$$\sum_{n=1}^{\infty} \lambda_n \Big(\frac{\Lambda_n}{\mu_n} \Big)^{p/(q-p)} < \infty. \tag{1.3}$$

Theorem B. *Let $0 < p < q$, $\{\beta_n\}$ and $\{c_n\}$ be sequences of nonnegative numbers, $\sum_{n=1}^{\infty} \beta_n < \infty$, furthermore let $B_n := \sum_{k=n}^{\infty} \beta_k$. The condition*

$$\sum_{m=1}^{\infty} \beta_m \Big(\sum_{n=1}^{m} c_n^q \Big)^{p/q} < \infty \tag{1.4}$$

holds if and only if there exists a nonincreasing sequence $\{\mu_n\}$ of positive numbers satisfying the conditions (1.2) *and*

$$\sum_{n=1}^{\infty} \beta_n \left(\frac{B_n}{\mu_n} \right)^{p/(q-p)} < \infty. \tag{1.5}$$

It is also known (see [8], Theorem 3) that the condition (1.1) is equivalent to the condition (1.2) without the additional condition (1.3) if and only if the three sequences

$$\{\mu_n\}, \quad \{1/\mu_n\} \qquad \text{and} \qquad \{\Lambda_n\}$$

are bounded. Then all of these conditions claim that

$$0 < \sum_{n=1}^{\infty} c_n^q < \infty.$$

Obviously then the condition (1.3) is also satisfied.

These results show that generally the conditions (1.1) and (1.2) are not equivalent without additional conditions on $\{\mu_n\}$ and $\{\lambda_n\}$. It is also evident that then these sequences are interrelated. So if e.g. $\mu_n := \Lambda_n^{\frac{q}{p}+\varepsilon}$, $\varepsilon > 0$, then an easy calculation shows that (1.3) is satisfied, consequently, by Theorem A, then (1.1) and (1.2) are equivalent. If $\mu_n := \Lambda_n^{q/p}$ then (1.3) reduces to

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{\Lambda_n} < \infty, \tag{1.6}$$

and thus

$$\sum_{n=1}^{\infty} \Lambda_n^{q/p} c_n^q < \infty \tag{1.7}$$

is equivalent to (1.1).

However, if (1.6) does not hold true, then we cannot state the equivalence of the conditions (1.1) and (1.7) for every sequence $\{c_n\}$. It seems to be natural to ask whether then the equivalence of (1.1) and (1.7) holds for certain strongly monotone or only "block-monotone" sequences $\{c_n\}$.

The aim of the present work is to analyze this problem. Among others we ask: If $\sum_{n=1}^{\infty} \frac{\lambda_n}{\Lambda_n} = \infty$, or equivalently $\sum_{n=1}^{\infty} \lambda_n = \infty$, what monotonicity assumptions on the sequence $\{c_n\}$ imply the equivalence of the conditions (1.1) and (1.7)?

According to the rate of decrease of the "tails" of the sequence $\{c_n\}$ we can give two different types of sufficient conditions ensuring the equivalence of (1.1) and (1.7).

Similar problem will be treated in connection with Theorem B.

In the paper K will denote positive constants, not necessarily the same in their occurances; they may depend on parameters of the problem concerned, but are independent of the sequence $\{c_n\}$.

2. Theorems

We prove the following theorems.

Theorem 1. *Let* $0 < p < q$, $\varepsilon > 0$, $0 \le \lambda_n \le 1$, $\sum_{n=1}^{\infty} \lambda_n = \infty$ *and* $\Lambda_n := \sum_{k=1}^{n} \lambda_k$, *furthermore let* s *be a positive integer. Denote* p_m *the smallest natural number such that* $\Lambda_{p_m} \ge m$; *furthermore let* $q_m := p_{2^m}$. *If the sequence* $\{c_n\}$ *of nonnegative numbers satisfies either*

$$\sum_{k=p_m}^{\infty} c_k^q \le K m^{q/p} (\lambda_m/\Lambda_m)^{\frac{q-p}{p}} \Lambda_m^{-\varepsilon} \sum_{k=p_{m-s}}^{p_{m+s}} c_k^q \tag{2.1}$$

or

$$\Big(\sum_{k=q_m}^{\infty} c_k^q\Big)^{p/q} \le K 2^{m(\frac{q}{p}-1)} \sum_{m=q_{m-s}}^{q_{m+s}} c_n^q \tag{2.2}$$

for all $m > s$, *then the conditions* (1.1) *and* (1.7) *are equivalent.*

We remark that the implication (1.1) $\Rightarrow$ (1.7) holds without any additional assumption on the block-monotonicity of $\{c_n\}$. The conditions (2.1) or (2.2) are required only to the proof of (1.7) $\Rightarrow$ (1.1).

We also mention that in (2.1) the factor $(\lambda_m/\Lambda_m)^{\frac{q-p}{p}} \Lambda_m^{-\varepsilon}$ can be replaced by any factor ρ_m satisfying the condition

$$\sum_{m=1}^{\infty} \rho_m^{p/(q-p)} < \infty, \tag{2.3}$$

namely in the proof we shall use only this property of the factors, and it is obvious that

$$\sum_{m=1}^{\infty} \lambda_m \Lambda_m^{-1-\frac{\varepsilon p}{q-p}} < \infty. \tag{2.4}$$

Finally we observe that at the proof of (1.7) $\Rightarrow$ (1.1) the assumption (2.1) can be replaced by

$$\sum_{k=q_m}^{\infty} c_k^q \le K 2^{mq/p} \rho_m \sum_{k=q_{m-s}}^{q_{m+s}} c_k^q, \tag{2.5}$$

where $\{\rho_m\}$ satisfies (2.3).

The following theorem gives sufficient conditions for the equivalence of (1.4) and

$$\sum_{n=1}^{\infty} B_n^{q/p} c_n^q < \infty. \tag{2.6}$$

Theorem 2. *Let* $0 < p < q$, $\varepsilon > 0$, $\beta_n \geq 0$, $B_n := \sum_{k=n}^{\infty} \beta_k < \infty$ *and* $\beta_n \leq B_{n+1}$, *furthermore let* s *be a positive integer. Denote* r_m *the largest natural number such that* $B_{r_m} \geq 2^{-m}$. *If the sequence* $\{c_n\}$ *of nonnegative numbers satisfies either*

$$\sum_{k=1}^{r_m} c_k^q \leq K m^{1-\frac{q}{p}-\varepsilon} \sum_{k=r_{m-s}}^{r_{m+s}} c_k^q \tag{2.7}$$

or

$$\Big(\sum_{k=1}^{r_m} c_k^q\Big)^{p/q} \leq K 2^{m(1-\frac{p}{q})} \sum_{k=r_{m-s}}^{r_{m+s}} c_k^q \tag{2.8}$$

for all $m > s$, *then the conditions* (1.4) *and* (2.6) *are equivalent.*

Now we can mention that only the implication (2.6) $\Rightarrow$ (1.4) requires the additional assumptions (2.7) or (2.8); furthermore that in (2.7) the factor $m^{1-\varepsilon-\frac{q}{p}}$ can be replaced by any factor ρ_m satisfying (2.3).

Afterwards we present two applications of our results. First we show that a general orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x), \tag{2.9}$$

under certain monotonicity conditions of the coefficients c_n, converge for all orthonormal system $\{\varphi_n(x)\}$ almost every in the interval of orthogonality (a, b) if and only if it is also convergent unconditionally almost everywhere, i.e. at every arrangement of its terms. These conditions are the following two ones:

$$c_n \geq c_{n+1} (\geq 0) \tag{2.10}$$

and

$$\sum_{n=2^m}^{\infty} c_n^2 \le Km(\log m)^{-1-\varepsilon} \sum_{n=2^{m-s}}^{2^{m+s}} c_n^2. \tag{2.11}$$

More precisely we prove the following theorem.

Theorem 3. *If $\varepsilon > 0$ and s is a positive integer, furthermore the conditions* (2.10) *and* (2.11) *hold, then the series* (2.9) *converges for all orthonormal system $\{\varphi_n(x)\}$ in (a,b) almost everywhere if and only if it also converges unconditionally.*

Secondly, among others, we show that the condition

$$\sum_{n=1}^{\infty} nc_n^2 < \infty \tag{2.12}$$

implies the absolute convergence of (2.9) if the sequence $\{c_n\}$ satisfies an additional condition of type (2.1).

Namely we prove the following result.

Theorem 4. *If $0 \le \alpha < 1/2$, $\varepsilon > 0$ and s is a positive integer, furthermore the condition*

$$\sum_{n=m^{2/(1-2\alpha)}}^{\infty} c_n^2 \le Km^{1-\varepsilon} \sum_{n=(m-s)^{2/(1-2\alpha)}}^{(m+s)^{2/(1-2\alpha)}} c_n^2 \tag{2.13}$$

holds, then the condition

$$\sum_{n=1}^{\infty} n^{1-2\alpha} c_n^2 < \infty \tag{2.14}$$

implies the absolute $|C,\alpha|$-summability of (2.9) *for all orthonormal system $\{\varphi_n(x)\}$ in (a,b) almost everywhere.*

If $\alpha = 0$ then Theorem 4 conveys our assertion stated above.

3. Proofs

Proof of Theorem 1. We start with the proof of the implication (1.1) $\Rightarrow$ (1.7). In the course of the following consideration first we make blocks, use the monotonicity of Λ_n, then, since $p < q$, can apply the so-called power-sum inequality (see e.g. [1], p. 28), and in the last steps we utilize the definition of the numbers q_m and its consequences. Thus we obtain that

$$\begin{aligned}\Big(\sum_{n=q_2}^{\infty}\Lambda_n^{q/p}c_n^q\Big)^{p/q} &\le \Big(\sum_{m=2}^{\infty}\sum_{n=q_m}^{q_{m+1}}\Lambda_n^{q/p}c_n^q\Big)^{p/q}\\ &\le \Big(\sum_{m=2}^{\infty}\Lambda_{q_{m+1}}^{q/p}\sum_{n=q_m}^{q_{m+1}}c_n^q\Big)^{p/q}\\ &\le \sum_{m=2}^{\infty}\Lambda_{q_{m+1}}\Big(\sum_{n=q_m}^{q_{m+1}}c_n^q\Big)^{p/q}\\ &\le 8\sum_{m=2}^{\infty}\sum_{k=q_{m-1}}^{q_m}\lambda_k\Big(\sum_{n=k}^{\infty}c_n^q\Big)^{p/q}\\ &\le 8\sum_{k=q_1}^{\infty}\lambda_k\Big(\sum_{n=k}^{\infty}c_n^q\Big)^{p/q}.\end{aligned}\tag{3.1}$$

Hereby we have proved the implication (1.1) $\Rightarrow$ (1.7) without using the additional assumptions (2.1) and (2.2).

Secondly we verify the implication (1.7) $\Rightarrow$ (1.1) assuming (2.1). As in (3.1) we get

$$\begin{aligned}\sum_{k=p_1}^{\infty}\lambda_k\Big(\sum_{n=k}^{\infty}c_n^q\Big)^{p/q} &\le \sum_{m=1}^{\infty}\sum_{k=p_m}^{p_{m+1}}\lambda_k\Big(\sum_{n=p_m}^{\infty}c_n^q\Big)^{p/q}\\ &\le 3\sum_{m=1}^{\infty}\Big(\sum_{n=p_m}^{\infty}c_n^q\Big)^{p/q} =: S_1.\end{aligned}\tag{3.2}$$

Now utilizing (2.1) we obtain that

$$S_1 \le K\sum_{m=s+1}^{\infty} m(\lambda_m/\Lambda_m)^{(q-p)/q}\Lambda_m^{-\varepsilon p/q}\Big(\sum_{k=p_{m-s}}^{p_{m+s}}c_k^q\Big)^{p/q}.$$

Hence, using Hölder's inequality, we get

$$(3.3)\qquad \begin{aligned} S_1 &\le K\left\{\sum_{m=s+1}^{\infty} m^{q/p}\sum_{k=p_{m-s}}^{p_{m+s}} c_k^q\right\}^{p/q}\\ &\times\left\{\sum_{m=1}^{\infty}\lambda_m\Lambda_m^{-1-\frac{\epsilon p}{q-p}}\right\}^{\frac{p}{q}} =: KS_2\cdot S_3.\end{aligned}$$

By the definition of Λ_m it is clear that the second factor S_3 in (3.3) is finite, furthermore by the definition of p_m

$$\begin{aligned}\sum_{m=s+1}^{\infty} m^{q/p}\sum_{k=p_{m-s}}^{p_{m+s}} c_k^q &\le K(s)\sum_{m=1}^{\infty} m^{q/p}\sum_{k=p_m}^{p_{m+s}} c_k^q\\ &\le K(s)\sum_{m=1}^{\infty}\Lambda_{p_m}^{q/p}\sum_{k=p_m}^{p_{m+s}} c_k^q \le K_1(s)\sum_{k=1}^{\infty}\Lambda_k^{q/p} c_k^q,\end{aligned}$$

i.e. the factor S_2 in (3.3) by (1.7) is also finite, whence, by (3.2) and (3.3), the implication (1.7) $\Rightarrow$ (1.1) follows.

Finally we prove the implication (1.7) $\Rightarrow$ (1.1) assuming (2.2). Likewise as in the previous proof with q_m in place of p_m we get

$$\begin{aligned}\sum_{k=q_{s+1}}^{\infty}\lambda_k\Big(\sum_{n=k}^{\infty} c_n^q\Big)^{p/q} &\le \sum_{m=s+1}^{\infty}\sum_{k=q_m}^{q_{m+1}}\lambda_k\Big(\sum_{n=q_m}^{\infty} c_n^q\Big)^{p/q}\\ &\le 4\sum_{m=s+1}^{\infty} 2^m\Big(\sum_{n=q_m}^{\infty} c_n^q\Big)^{p/q} =: S_4.\end{aligned}$$

By (2.2) we obtain that

$$\begin{aligned} S_4 &\le K\sum_{m=s+1}^{\infty} 2^{m\frac{q}{p}}\sum_{n=q_{m-s}}^{q_{m+s}} c_n^q\\ &\le K\sum_{m=s+1}^{\infty}\Lambda_{q_m}^{q/p}\sum_{n=q_{m-s}}^{q_{m+s}} c_n^q\end{aligned}$$

$$\le K(s)\sum_{m=1}^{\infty}\Lambda_{q_m}^{q/p}\sum_{n=q_m}^{q_{m+s}}c_n^q$$

$$\le K_1(s)\sum_{n=q_1}^{\infty}\Lambda_n^{q/p}c_n^q.$$

These inequalities yield that (1.7) implies (1.1).

The proof is complete.

Proof of Theorem 2. Without loss of generality we can assume that $B_1 = 1$. On the other hand the assumption $\beta_n \le B_{n+1}$ implies that $r_{m+1} \ge r_m + 1$ holds for every m.

Now we launch the proof of (1.4) $\Rightarrow$ (2.6). In our consideration we make blocks, use the monotonicity of B_n and the definition of r_m, furthermore we use again the power-sum inequality. Thus we get

$$\Big(\sum_{n=r_\circ+1}^{\infty}B_n^{q/p}c_n^q\Big)^{p/q}=\Big(\sum_{m=1}^{\infty}\sum_{n=r_{m-1}+1}^{r_m}B_n^{q/p}c_n^q\Big)^{p/q}$$

$$\le 2\sum_{m=1}^{\infty}2^{-m}\Big(\sum_{n=r_{m-1}+1}^{r_m}c_n^q\Big)^{p/q}$$

$$\le 4\sum_{m=1}^{\infty}\sum_{k=r_m}^{r_{m+1}}\beta_k\Big(\sum_{n=1}^{r_m}c_n^q\Big)^{p/q}$$

$$\le 8\sum_{k=r_1}^{\infty}\beta_n\Big(\sum_{n=1}^{k}c_n^q\Big)^{p/q}.$$

These inequalities plainly prove the implication (1.4) $\Rightarrow$ (2.6).

Next we turn to the proof of (2.6) $\Rightarrow$ (1.4) assuming (2.7). Some elementary calculations give that

$$\sum_{n=r_\circ+1}^{\infty}\beta_n\Big(\sum_{k=1}^{n}c_k^q\Big)^{p/q}\le\sum_{m=1}^{\infty}\sum_{n=r_{m-1}+1}^{r_m}\beta_n\Big(\sum_{k=1}^{r_m}c_k^q\Big)^{p/q}$$
(3.4)
$$\le 2\sum_{m=1}^{\infty}2^{-m}\Big(\sum_{k=1}^{r_m}c_k^q\Big)^{p/q}.$$

Now applying (2.7) and Hölder's inequality we get

$$\begin{aligned}\sum_{m=s+1}^{\infty} 2^{-m}\Big(\sum_{k=1}^{r_m} c_k^q\Big)^{p/q} &\le K \sum_{m=s+1}^{\infty} 2^{-m}\Big(\sum_{k=r_{m-s}}^{r_{m+s}} c_k^q\Big)^{p/q} m^{\frac{p}{q}-1-\frac{\epsilon p}{q}} \\ &\le K \left\{\sum_{m=s+1}^{\infty} 2^{-\frac{mq}{p}} \sum_{k=r_{m-s}}^{r_{m+s}} c_k^q\right\}^{p/q} \left\{\sum_{m=1}^{\infty} m^{-1-\frac{\epsilon q}{q-p}}\right\}^{1-\frac{p}{q}}\end{aligned} \tag{3.5}$$

By the definition of r_m, for $k \le r_{m+s}$

$$2^{-m-s} \le B_k$$

holds, thus

$$\begin{aligned}&\sum_{m=s+1}^{\infty} 2^{-\frac{mq}{p}} \sum_{k=r_{m-s}}^{r_{m+s}} c_k^q \\ &\le K(s,p,q) \sum_{k=1}^{\infty} B_k^{q/p} c_k^q.\end{aligned} \tag{3.6}$$

The estimations (3.4)–(3.6) yield the implication (2.6) ⇒ (1.4).

The proof of the implication (2.6) ⇒ (1.4) under the assumption (2.8) is very easy.

Using the estimation given in (3.4) and the assumption (2.8) we have again that

$$\begin{aligned}&\sum_{n=r_\circ+1}^{\infty} \beta_n \Big(\sum_{k=1}^{n} c_k^q\Big)^{p/q} \\ &\le K \sum_{m=s+1}^{\infty} 2^{-\frac{mp}{q}} \sum_{k=r_{m-s}}^{r_{m+s}} c_k^q,\end{aligned}$$

whence, by (3.6), the implication (2.6) ⇒ (1.4) follows.

The poof is complete.

Proof of Theorem 3. It is clearly enough to prove that if the series (2.9) converges for all orthonormal system then it is also convergent unconditionally, namely the converse is obvious. In order to verify this we utilize the special case $p = 1$, $q = 2$ and $\lambda_n = \frac{1}{n}$ of Theorem 1 with assumption (2.10), which states that if (2.11) holds then

$$\sum_{n=2}^{\infty} c_n^2 \log^2 n < \infty \tag{3.7}$$

and

$$\sum_{m=1}^{\infty} \frac{1}{m} \left\{ \sum_{n=m}^{\infty} c_n^2 \right\}^{1/2} < \infty \tag{3.8}$$

are equivalent.

On the other hand K. Tandori [6] proved that if (2.10) holds, then the condition (3.7) is not only sufficient but also necessary in order that the series (2.9) for all orthonormal system should be convergent almost everywhere in (a,b). Consequently, if the assumptions (2.10) and (2.11) are fulfilled and the series (2.9) converges for all orthonormal system, then (3.7) and (3.8) hold simultaneously. However, the condition (3.8) implies the unconditional convergence of (2.9) by the results of K. Tandori [7] and L. Leindler [3].

The proof is thus completed.

Proof of Theorem 4. The proof is similar to that of Theorem 3. By the special case $p = 1$, $q = 2$ and $\lambda_n = n^{-\alpha-\frac{1}{2}}$ of Theorem 1 with assumption (2.13) the condition (2.14) is equivalent to

$$\sum_{m=1}^{\infty} m^{-\alpha-1/2} \Big(\sum_{n=m}^{\infty} c_n^2 \Big)^{1/2} < \infty, \tag{3.9}$$

and by Theorem 2.1 of [4] (see the case (2.4) with $\gamma = -\alpha - 1/2$, $\alpha = \beta = 0$, $\epsilon = 1$) the condition (3.9) is equivalent to

$$\sum_{m=1}^{\infty} 2^{m(\frac{1}{2}-\alpha)} \left\{ \sum_{n=2^m+1}^{2^{m+1}} c_n^2 \right\}^{1/2} < \infty.$$

Since, this last condition, by Theorem 2 of [2], implies the absolute $|C, \alpha|$-summability of (2.9), thus we have completed the proof of Theorem 4.

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