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The solution of a system of functional equations related to selection probabilities

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Dedicated to Professors Z. Daróczy and I. Kátai on the occasion of their sixtieth birthdays

Abstract. In this note we solve a system of functional equations related to selection probabilities containing several unknown functions.

1. Introduction

The following notions have been introduced in Aczél–Maksa–Marley–Moszner [1]. A selection model is a pair (R,P) where R is a nonempty set, S is the set of its finite nonempty subsets and $P: R\times S \to [0,1]$. The number $P(e,E)$ ($e\in R$, $E\in S$) is called selection probability. It is convenient to think of the e as options, of E as a subset of options, R the set of all options and $P(e,E)$ the probability that the option e is chosen from E. If there exists a function $v: R\to\,]0,+\infty[$ such that

$$P(e,E)=\frac{v(e)}{\sum\limits_{d\in E} v(d)}$$

when $e\in E$, while $P(e,E)=0$ if $e\notin E$, we have a prominent example, so-called Luce's choice model (Luce [2], [3]). In [1], under natural and quite

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weak technical suppositions a wide family of selection models is described and then, making the conditions more restrictive, Luce's choice model is characterized. The functional equation technique used there leads to the following system of equations and inequalities.

$$\frac{\prod_{j=1}^{m} x_{jk}^{a_j}}{\sum_{\ell=1}^{n}\prod_{j=1}^{m} x_{j\ell}^{a_j}} = \frac{\Phi\big(F_k(x_{11},\dots,x_{1n}),\dots,F_k(x_{m1},\dots,x_{mn})\big)}{\sum_{\ell=1}^{n}\Phi\big(F_\ell(x_{11},\dots,x_{1n}),\dots,F_\ell(x_{m1},\dots,x_{mn})\big)}, \tag{1}$$

$$0 < F_k(z_1,\dots,z_n), \quad k=1,\dots,n \tag{2}$$

and

$$1 = \sum_{\ell=1}^{n} F_\ell(z_1,\dots,z_n). \tag{3}$$

Here $2 \le m$, $2 \le n$ are fixed integers, $a_j \in \mathbb{R}$ (the set of all real numbers) is fixed, $j = 1,\dots,m$, $F_k :]0,+\infty[^n \to \mathbb{R}$, $k = 1,\dots,n$ and $\Phi :]0,1[^m \to]0,+\infty[$ are unknown functions and (1), (2), and (3) hold for all $x_{jk} \in]0,+\infty[$ $(j = 1,\dots,m,\ k = 1,\dots,n)$ and $z_1,\dots,z_n \in]0,+\infty[$, respectively. Under some conditions on the diagonalization of Φ (i.e. on the function $y \to \Phi(y,\dots,y)$, $y \in]0,1[$) the system (1)–(2)–(3) has been solved recently in the following two cases

(a) $\sum_{j=1}^{m} a_j \neq 0$ and

(b) $\sum_{j=1}^{m} a_j = 0$, $a_p \neq 0$ for some $1 \le p \le m$ and

$$\Phi(y_1,\dots,y_m) = b\prod_{j=1}^{m} y_j^{a_j} \qquad (b>0 \text{ constant}).$$

(See Theorem 1, Theorem 3 and Appendix B in [1].)

In this note we solve the system (1)–(2)–(3) assuming regularity properties on Φ and assuming that $\sum_{j=1}^{m} a_j^2 > 0$.

2. The main result

For the proof of our main result the following lemma will be needed.

Lemma. *Let* $u, v :]0,1[^m \to \mathbb{R}$ *and*

$$u(x_1, \dots, x_m) = v(y_1, \dots, y_m) \tag{4}$$

for all x_j, y_j, $x_j + y_j \in]0,1[$, $j = 1, \dots, m$. *Then there exists* $c \in \mathbb{R}$ *such that*

$$u(x) = v(y) = c$$

for all $x, y \in]0,1[^m$.

Proof. For $q \in \mathbb{N}$ (the set of all positive integers) define $a_q = v\big(\overset{1}{\overbrace{\tfrac{1}{q+1}}}, \overset{\dots}{\dots}, \overset{m}{\overbrace{\tfrac{1}{q+1}}}\big)$. Since $\frac{1}{q+1} + \frac{1}{2}\frac{q}{q+1} < 1$ and $\frac{1}{2}\frac{q}{q+1} + \frac{1}{q+2} < 1$, applying (4) two times, we have

$$\begin{aligned} a_q = v\left(\frac{1}{q+1}, \dots, \frac{1}{q+1}\right) &= u\left(\frac{1}{2}\,\frac{q}{q+1}, \dots, \frac{1}{2}\,\frac{q}{q+1}\right) \\ = v\left(\frac{1}{q+2}, \dots, \frac{1}{q+2}\right) &= a_{q+1} \end{aligned}$$

for all $q \in \mathbb{N}$. Therefore $a_q = c$ for some $c \in \mathbb{R}$ and for all $q \in \mathbb{N}$. On the other hand let now $x = (x_1, \dots, x_m) \in]0,1[^m$ be arbitrary. Then there exists $q \in \mathbb{N}$ such that $x_j + \frac{1}{q+1} < 1$ for all $j \in \{1, \dots, m\}$. Thus, by (4), $u(x) = a_q = c$. Furthermore, if $y = (y_1, \dots, y_m) \in]0,1[^m$, then there exists $x = (x_1, \dots, x_m) \in]0,1[^m$ such that $x_j + y_j < 1$ for all $j \in \{1, \dots, m\}$. Therefore, again by (4), $v(y) = u(x) = c$. $\square$

The solutions of system (1)–(2)–(3) strictly depend on the fact whether $n > 2$ or $n = 2$. In the following theorem we deal with the case $n > 2$.

Theorem 1. *Let* $2 \le m$ *and* $2 < n$ *be fixed integers,* $a_1, \dots, a_m \in \mathbb{R}$, $\sum\limits_{j=1}^{m} a_j = a$, $\sum\limits_{j=1}^{m} a_j^2 > 0$, $\Phi :]0,1[^m \to]0,+\infty[$ *and* $F_k :]0,+\infty[^n \to \mathbb{R}$, $k = 1, \dots, n$. *Suppose that* Φ *is continuous and strictly monotonic in each variable. Then* (1), (2) *and* (3) *hold simultaniously if and only if there* *exist a continuous and strictly monotonic function* $\varphi :]0,1[\to]0,+\infty[$ *and* $c, c_1, \dots, c_m \in]0,+\infty[$ *such that*

$$\Phi(y_1,\dots,y_m) = c\prod_{j=1}^{m}\varphi(y_j)^{a_j}, \quad (y_1,\dots,y_m) \in]0,1[^m, \tag{5}$$

$$c_r^a = 1, \qquad r = 1,\dots,n, \tag{6}$$

furthermore, for all $(x_1,\dots,x_n) \in]0,+\infty[^n$ *there exits the unique solution* $x = L(x_1,\dots,x_n) \in]0,+\infty[$ *of the equation*

$$\sum_{k=1}^{n}\varphi^{-1}(c_k x_k x) = 1 \tag{7}$$

with the property

$$\varphi^{-1}\big(c_k x_k L(x_1,\dots,x_n)\big) > 0, \quad k = 1,\dots,n \tag{8}$$

and

$$F_k(x_1,\dots,x_n) = \varphi^{-1}\big(c_k x_k L(x_1,\dots,x_n)\big), \quad k = 1,\dots,n. \tag{9}$$

Proof. First suppose that (1), (2) and (3) hold. Since $\sum\limits_{j=1}^{m} a_j^2 > 0$, there exists $p \in \{1,\dots,m\}$ such that $a_p \neq 0$. For $k \in \{1,\dots,n\}$ define the function φ_k on $]0,1[$ by

$$\varphi_k(t) = \Phi\big(F_k(1,\overset{1}{\smile}\!\!\dots,1),\dots,\overset{p}{\underset{\smile}{t}},\dots,F_k(1,\overset{m}{\smile}\!\!\dots,1)\big)^{\frac{1}{a_p}}. \tag{10}$$

Then $\varphi_k :]0,1[\to]0,+\infty[$ is continuous and strictly monotonic for all $k \in \{1,\dots,n\}$. On the other hand let $x_1,\dots,x_n \in]0,+\infty[$ and $x_{jk} = 1$ if $j \neq p$ and $x_{pk} = x_k$ in (1) ($k = 1,\dots,n$, $j = 1,\dots,m$). Then, by (10), we get

$$\frac{x_k^{a_p}}{\sum\limits_{\ell=1}^{n} x_\ell^{a_p}} = \frac{\varphi_k\big(F_k(x_1,\dots,x_n)\big)^{a_p}}{\sum\limits_{\ell=1}^{n}\varphi_\ell(F_\ell(x_1,\dots,x_n))^{a_p}}, \quad k = 1,\dots,n. \tag{11}$$

Introducing the function L by

$$L(x_1,\dots,x_n)=\left(\frac{\sum\limits_{\ell=1}^{n}\varphi_\ell\big(F_\ell(x_1,\dots,x_n)\big)^{a_p}}{\sum\limits_{\ell=1}^{n}x_\ell^{ap}}\right)^{\frac{1}{a_p}},\ (x_1,\dots,x_n)\in\,]0,+\infty[^{\,n},$$

(11) implies that

$$\varphi_k\big(F_k(x_1,\dots,x_n)\big)=x_kL(x_1,\dots,x_n),$$

that is,

$$F_k(x_1,\dots,x_n)=\varphi_k^{-1}\big(x_kL(x_1,\dots,x_n)\big) \tag{12}$$

for all $(x_1,\dots,x_n)\in\,]0,+\infty[^{\,n}$ and $k\in\{1,\dots,n\}$. It follows from (12), (3), (2) and the strict monotonicity of the function $x\to\sum\limits_{k=1}^{n}\varphi_k^{-1}(x_kx)$, $x\in\,]0,+\infty[$ ($x_1,\dots,x_n$ are fixed positive numbers) that $x=L(x_1,\dots,x_n)$ is the unique solution of the equation

$$\sum_{k=1}^{n}\varphi_k^{-1}(x_kx)=1 \tag{13}$$

and this solution has the property

$$\varphi_k^{-1}\big(x_kL(x_1,\dots,x_n)\big)>0,\quad k=1,\dots,n. \tag{14}$$

Let now $y_{jk}\in\,]0,+\infty[$ with $\sum\limits_{k=1}^{n}y_{jk}=1$ ($j=1,\dots,m$, $k=1,\dots,n$). Then

$$1=\sum_{k=1}^{n}y_{jk}=\sum_{k=1}^{n}\varphi_k^{-1}\big(\varphi_k(y_{jk})\cdot 1\big).$$

Therefore 1 is the unique solution of (13) with fixed $1\le j\le m$ and $x_k=\varphi_k(y_{jk})$. Thus $L(\varphi_1(y_{j1}),\dots,\varphi_n(y_{jn}))=1$ and, by (12),

$$F_k\big(\varphi_1(y_{j1}),\dots,\varphi_n(y_{jn})\big)=\varphi_k^{-1}\big(\varphi_k(y_{jk})\big)=y_{jk}$$

for all $j \in \{1, \dots, m\}$ and $k \in \{1, \dots, n\}$. Taking this equality into consideration, equation (1) with $x_{ik} = \varphi_k(y_{ik})$, implies that

$$\frac{\prod_{j=1}^{m} \varphi_k(y_{jk})^{a_j}}{\sum_{\ell=1}^{n} \prod_{j=1}^{m} \varphi_\ell(y_{j\ell})^{a_j}} = \frac{\Phi(y_{1k}, \dots, y_{mk})}{\sum_{\ell=1}^{n} \Phi(y_{1\ell}, \dots, y_{m\ell})} \tag{15}$$

holds for all $y_{jk} \in\,]0, +\infty[$ with $\sum_{k=1}^{n} y_{jk} = 1$ $(j = 1, \dots, m,\ k = 1, \dots, n)$.

Let $1 < r \le n$ be fixed, $(x_1, \dots, x_m)$, $(y_1, \dots, y_m) \in\,]0, +\infty[^{\,m}$ such that $x_j + y_j < 1$ for all $1 \le j \le m$ and

$$y_{jr} = y_j, \quad y_{j1} = x_j \quad \text{and} \quad y_{jk} = \frac{1 - x_j - y_j}{n-2} \quad \text{if} \quad 1 < k \le n,\ k \ne r.$$

Then $y_{jk} \in\,]0, +\infty[$ and $\sum_{k=1}^{n} y_{jk} = 1$. (This is the first step in the proof when the condition $n > 2$ is used). Thus (15), with $k = r$ and next with $k = 1$, implies

$$\frac{\prod_{j=1}^{m} \varphi_r(y_j)^{a_j}}{\sum_{\ell=1}^{n} \prod_{j=1}^{m} \varphi_\ell(y_{j\ell})^{aj}} = \frac{\Phi(y_1, \dots, y_m)}{\sum_{\ell=1}^{n} \Phi(y_{1\ell}, \dots, y_{m\ell})}$$

and

$$\frac{\prod_{j=1}^{m} \varphi_1(x_j)^{a_j}}{\sum_{\ell=1}^{n} \prod_{j=1}^{m} \varphi_\ell(y_{j\ell})^{aj}} = \frac{\Phi(x_1, \dots, x_m)}{\sum_{\ell=1}^{n} \Phi(y_{1\ell}, \dots, y_{m\ell})},$$

respectively. Dividing the first equation by the second one we have that

$$\frac{\prod_{j=1}^{m} \varphi_r(y_j)^{a_j}}{\prod_{j=1}^{m} \varphi_1(x_j)^{a_j}} = \frac{\Phi(y_1, \dots, y_m)}{\Phi(x_1, \dots, x_m)},$$

that is,

$$\frac{\Phi(y_1,\dots,y_m)}{\prod\limits_{j=1}^{m}\varphi_r(y_j)^{a_j}} = \frac{\Phi(x_1,\dots,x_m)}{\prod\limits_{j=1}^{m}\varphi_1(x_j)^{a_j}}. \tag{16}$$

Applying our Lemma we obtain that

$$\Phi(x_1,\dots,x_m) = c\prod_{j=1}^{m}\varphi_1(x_j)^{a_j} \tag{17}$$

with some $c>0$ and for all $(x_1,\dots,x_m)\in\,]0,1[^m$. It follows from (16) and (17) that

$$\prod_{j=1}^{m}\varphi_1(y_j)^{a_j} = \prod_{j=1}^{m}\varphi_r(y_j)^{a_j} \tag{18}$$

for all $(y_1,\dots,y_m)\in\,]0,1[^m$ and $r\in\{1,\dots,n\}$. Let $t\in\,]0,1[$, $y_j=\frac{1}{2}$ if $j\in\{1,\dots,m\}\setminus\{p\}$ and $y_p=t$. Then, by (18), $\varphi_r(t)^{a_p}\varphi_r\left(\frac{1}{2}\right)^{a-a_p} = \varphi_1(t)^{a_p}\varphi_1\left(\frac{1}{2}\right)^{a-a_p}$, that is,

$$c_r\varphi_r(t) = \varphi_1(t) \tag{19}$$

where $c_r = \left(\frac{\varphi_r\left(\frac{1}{2}\right)}{\varphi_1\left(\frac{1}{2}\right)}\right)^{\frac{a-a_p}{a_p}} > 0$, $r=1,\dots,n$.

Finally, let $\varphi=\varphi_1$. Then $\varphi: \,]0,1[\,\to\,]0,+\infty[$ is continuous strictly monotonic and (5) and (6) follow from (17) and (18), (19), respectively. Furthermore, by (13) and (19), the equation

$$1 = \sum_{k=1}^{n}\varphi_k^{-1}(x_k x) = \sum_{k=1}^{n}\varphi^{-1}(c_k x_k x)$$

has the unique solution $x=L(x_1,\dots,x_n)$ for all $(x_1,\dots,x_n)\in\,]0,+\infty[^n$, i.e. (7) is satisfied. Moreover, since $\varphi: \,]0,1[\,\to\,]0,+\infty[$, (8) holds, too. (9) is a simple consequence of (12) and (19).

The converse is a simple computation and it is valid also for $n=2$. $\square$

In the following theorem we cover the case $n=2$.

Theorem 2. *Let $2 \le m$ be fixed integer, $a_1, \dots, a_m \in \mathbb{R}$, $\sum_{j=1}^{m} a_j = a$, $\sum_{j=1}^{m} a_j^2 > 0$, $\Phi :]0,1[^m \to]0,+\infty[\,$, $F_1, F_2 :]0,+\infty[^2 \to \mathbb{R}$. Suppose that Φ is continuous and strictly monotonic in each variable. Then (1), (2) and (3) hold simultaniously with $n = 2$ if and only if there exist a continuous function $T :]0,1[^m \to]0,+\infty[\,$, continuous and strictly monotonic functions $\varphi_1, \varphi_2 :]0,1[\to]0,+\infty[$ and $\alpha > 0$ such that*

$$(20)\qquad T(y_1, \dots, y_m) = T(1-y_1, \dots, 1-y_m), \quad (y_1, \dots, y_m) \in]0,1[^m,$$

$$(21)\qquad \varphi_1(1-y)\varphi_1(y) = \alpha\varphi_2(1-y)\varphi_2(y), \quad y \in]0,1[,$$

$$(22)\qquad \alpha^a = 1,$$

$$(23)\qquad \Phi(y_1, \dots, y_m) = T(y_1, \dots, y_m)\sqrt{\prod_{j=1}^{m} \varphi_1(y_j)^{a_j}\varphi_2(y_j)^{a_j}}$$

for all $(y_1, \dots, y_m) \in]0,1[^m$. Furthermore, for all $x_1, x_2 \in]0,+\infty[$ there exists the unique solution $x = L(x_1, x_2) \in]0,+\infty[$ of the equation

$$(24)\qquad \varphi_1^{-1}(x_1 x) + \varphi_2^{-1}(x_2 x) = 1$$

with the property

$$(25)\qquad \varphi_k^{-1}\big(x_k L(x_1, x_2)\big) > 0, \quad k = 1, 2$$

and

$$(26)\qquad F_k(x_1, x_2) = \varphi_k^{-1}\big(x_k L(x_1, x_2)\big), \quad k = 1, 2.$$

Proof. First we suppose that (1), (2) and (3) hold for $n = 2$. Everything goes as in the proof of Theorem 1 until (15) which is still valid. Let now $y_j \in]0,1[$ and $y_{j1} = y_j$, $y_{j2} = 1 - y_j$, $j = 1, \dots, m$. Then applying (15) first to $k = 1$ and next to $k = 2$ we obtain

$$\frac{\prod_{j=1}^{m} \varphi_1(y_j)^{a_j}}{\prod_{j=1}^{m} \varphi_1(y_j)^{a_j} + \prod_{j=1}^{m} \varphi_2(1-y_j)^{a_j}} = \frac{\Phi(y_1, \dots, y_m)}{\Phi(y_1, \dots, y_m) + \Phi(1-y_1, \dots, 1-y_m)}$$

and

$$\frac{\prod_{j=1}^{m} \varphi_2(1-y_j)^{a_j}}{\prod_{j=1}^{m} \varphi_1(y_j)^{a_j} + \prod_{j=1}^{m} \varphi_2(1-y_j)^{a_j}} = \frac{\Phi(1-y_1,\ldots,1-y_m)}{\Phi(y_1,\ldots,y_m)+\Phi(1-y_1,\ldots,1-y_m)}.$$

Dividing the first equation by the second one we have

$$\frac{\prod_{j=1}^{m} \varphi_1(y_j)^{a_j}}{\prod_{j=1}^{m} \varphi_2(1-y_j)^{a_j}} = \frac{\Phi(y_1,\ldots,y_m)}{\Phi(1-y_1,\ldots,1-y_m)}. \tag{27}$$

Write here $1-y_j$ instead of y_j $(j=1,\ldots,m)$ and multiply (27) by the equation so obtained to get

$$\frac{\prod_{j=1}^{m} \varphi_1(1-y_j)^{a_j}}{\prod_{j=1}^{m} \varphi_2(y_j)^{a_j}} = \frac{\prod_{j=1}^{m} \varphi_2(1-y_j)^{a_j}}{\prod_{j=1}^{m} \varphi_1(y_j)^{a_j}}. \tag{28}$$

Let now $y \in \,]0,1[$ and $y_p = y$, $y_j = \frac{1}{2}$ if $j \neq p$, $j=1,\ldots,m$ in (28). Then we have (21) with

$$\alpha = \left(\frac{\varphi_2\left(\frac{1}{2}\right)}{\varphi_1\left(\frac{1}{2}\right)}\right)^{\frac{2(a-a_p)}{a_p}} > 0.$$

(22) follows from (21) and (28). Define the function T on $]0,+\infty[^m$ by

$$T(y_1,\ldots,y_m) = \frac{\Phi(y_1,\ldots,y_m)}{\sqrt{\prod_{j=1}^{m} \varphi_1(y_j)^{a_j}\varphi_2(y_j)^{a_j}}}.$$

Then T is continuous and we obtain (23). (20) follows from (27), (28) and the definition of T. Finally, (12), (13) and (14) with $n=2$ imply (26), (24) and (25), respectively.

The proof of the converse is purely and simply computation. $\square$

3. Remark and examples

Remark. If $a \neq 0$ then, by (6) and (22), $c_1 = \cdots = c_n = 1$ in Theorem 1 and $\alpha = 1$ in Theorem 2, respectively. If $a = 0$ then $c_1, \dots, c_n$ and α are arbitrary positive numbers.

Example 1. Let $\varphi(y) = y$, $y \in \,]0,1[$ in Theorem 1. Then equation (7) has the unique solution $x = L(x_1, \dots, x_n) = \left(\sum_{k=1}^{n} c_k x_k\right)^{-1}$ with the property (8). Thus, by (9),

$$F_k(x_1, \dots, x_n) = \frac{c_k x_k}{\sum_{\ell=1}^{n} c_\ell x_\ell} \qquad (k = 1, \dots, n). \tag{29}$$

The choice model with selection probabilities given by (29) is called the "beta model" (see [2]). If in addition $a \neq 0$ then, by (6), $c_1 = \cdots = c_n = 1$ and (29) becomes

$$F_k(x_1, \dots, x_n) = \frac{x_k}{\sum_{\ell=1}^{n} x_\ell} \qquad (k = 1, \dots, n) \tag{30}$$

which are the selection probabilities of Luce's choice model. We can construct other models by taking a continuous and stricly monotonic function $\varphi : \,]0,1[\, \to \,]0,+\infty[$, finding the unique solution of (7) with (8) and defining F_k by (9). (See the example $\varphi(y) = \frac{1}{4}(\sqrt{1+4y} - 1)^2$, $y \in \,]0,1[$, $\varphi^{-1}(t) = t + \sqrt{t}$, $t > 0$ in [1].

Example 2. Let $\varphi_1(y) = e^{y-\frac{1}{2}}(e^y - 1)$, $\varphi_2(y) = e^y - 1$ and $T(y_1, \dots, y_m) = 1 + \sum_{j=1}^{m} |1 - 2y_j|$; $y, y_1 \dots, y_m \in \,]0,1[$ in Theorem 2. Then (20) and (21) with $\alpha = 1$ is satisfied. Furthermore

$$\varphi_1^{-1}(z) = \ln \frac{1 + \sqrt{1 + 4\sqrt{e}\, z}}{2} \quad \text{and} \quad \varphi_2^{-1}(z) = \ln(z+1), \qquad z > 0$$

and (24) reduces to an algebraic equation of degree 3 which has exactly one solution $x = L(x_1, x_2) > 0$ for all $x_1, x_2 \in \,]0,+\infty[$. Since φ_1^{-1} and φ_2^{-1} are positive functions (25) holds, too. Define the functions Φ and F_k $(k = 1, 2)$ by (23) and (26), respectively. Then it follows from Theorem 2 that the triple (Φ, F_1, F_2) is a solution of the system (1)–(2)–(3) with $n = 2$. This solution is not of the form given by (5) and (9) with $n = 2$ in Theorem 1. (Otherwise

$$T(y_1, \dots, y_m) = c \prod_{j=1}^{m} \left(\frac{\varphi(y_j)}{\sqrt{\varphi_1(y_j)\varphi_2(y_j)}} \right)^{a_j}, \qquad (y_1, \dots, y_m) \in \,]0,1[^m$$

would follow for some continuous and strictly monotonic $\varphi : \,]0,1[\, \to \,]0,+\infty[$ and for some $c > 0$, which is impossible.) This example shows that there are solutions in the case $n = 2$ which are not solutions of (1)–(2)–(3) for $n > 2$.

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