Publ. Math. Debrecen
52 / 3-4 (1998), 559–562

Continuous solutions of a functional equation

By JANUSZ MATKOWSKI (Zielona Góra)

Dedicated to the 60th birthday of
Professors Zoltán Daróczy and Imre Kátai

Abstract. Under some general assumptions on the given functions M and N, we show that every continuous at a point solution f of the functional equation $f(M(x,y)) = N(x,y,f(x),f(y))$ must be continuous everywhere.

It is well known (see e.g. KUCZMA [2]) that if $f : \mathbb{R} \to \mathbb{R}$ is continuous at a least at one point and satisfies the Cauchy or Jensen functional equation then it is continuous everywhere. The aim of this note is to show that this property remains true for a more general functional equation. The main result reads as follows.

Theorem 1. *Let (X, d) and (V, ρ) be metric spaces, and suppose that the functions $M : X^2 \to X$; $N : X^2 \times V^2 \to V$ satisfy the following conditions*

(1) *for every $y \in X$, the function $M(\cdot, y) : X \to X$ is a homeomorphism of X onto $M(X, y)$;*
(2) *for all $y \in X$, and $v \in V$, the function $N(\cdot, y, \cdot, v) : X \times V \to V$ is continuous.*

If $f : X \to V$ is continuous at a point $z_0 \in X$ and satisfies the functional equation

$$f(M(x,y)) = N(x,y,f(x),f(y)), \quad x, y \in X, \tag{1}$$

Mathematics Subject Classification: Primary: 39B22, 39B52.
Key words and phrases: functional equation, continuous solutions, uniqueness theorem.

then f is continuous on the set $M(z_0, X)$. If, moreover,

(3) *for every $x \in X$ there exist $n \in \mathbb{N} \cup \{0\}$ and a sequence $z_1, \dots, z_n$ such that*

$$z_i \in M(z_{i-1}, X), \quad i = 1, \dots, n, \text{ and } x \in M(z_n, X),$$

then the function f is continuous everywhere on X.

Proof. Let $x \in M(z_0, X)$. Then there exists a point $y \in X$ such that

$$x = M(z_0, y). \tag{2}$$

Take an arbitrary sequence (x_n), $x_n \in X$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} x_n = x.$$

By assumption 1, the function $M(\cdot, y)$ is a homeromorphism of X onto $M(X, y)$, and $x \in M(X, y)$, so there exists $n_0 \in \mathbb{N}$ such that

(a) $x_n \in M(X, y)$ for all $n \ge n_0$;
(b) for every $n \ge n_0$, there is $z_n \in X$ such that $M(z_n, y) = x_n$;
(c) $\lim_{n\to\infty} z_n = z_0$.

Now, applying in turn: (b), equation (1), the continuity of f at z_0, again equation (1), and (2), we get

$$\begin{aligned}\lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} f(M(z_n, y)) = \lim_{n\to\infty} N(z_n, y, f(z_n), f(y)) \\ &= N(z_0, y, f(z_0), f(y)) = f(M(z_0, y)) = f(x),\end{aligned}$$

which proves the continuity of f at the point x. Thus, we have proved that f is continuous at the points of $M(z_0, X)$. Repeating this argument, we get that f is continuous at the points of $M(M(z_0, X), X)$, etc. Hence, by assumption 3, f is continuous on X. This completes the proof.

Remark 1. Let (X, d) and (V, ρ) be metric spaces, and suppose that the functions $M : X^2 \to X$; $N : X^2 \times V^2 \to V$ satisfy the conditions of Theorem 1. If a function $f : X \to V$ satisfying equation (1) is discontinuous at a point then it is discontinuous everywhere.

Applying Theorem 1 and the *uniqueness theorem* on functional equations (cf. J. Aczél and J. Dhombres [1], p. 243, Theorem 4) we obtain

Corollary 1. *Let $I \subset \mathbb{R}$ be an interval. Suppose that $M : I^2 \to I$ and $N : I^2 \times \mathbb{R}^2 \to \mathbb{R}$ satisfy the conditions:*

(1) *M is continuous and internal, that is*

$$\min(x, y) < M(x, y) < \max(x, y), \quad x \neq y \ (x, y \in I);$$

(2) *for every $y \in I$, the function $M(\cdot, y) : I \to I$ is injective;*
(3) *for all $y \in I$ and $v \in \mathbb{R}$, the function $N(\cdot, y, \cdot, v) : I \times \mathbb{R} \to \mathbb{R}$ is continuous;*
(4) *either for all fixed $x, y \in I$, $v \in \mathbb{R}$, the function $N(x, y, \cdot, v) : \mathbb{R} \to \mathbb{R}$, or, for all fixed $x, y \in I$, $u \in \mathbb{R}$, the function $N(x, y, u, \cdot) : \mathbb{R} \to \mathbb{R}$, is injective;*
(5) *there exists $z_0 \in I$ such that: either the function $M(z_0, \cdot) : I \to I$ is bijective, or, for every $x \in I$, there are $n \in \mathbb{N}$ and a sequence $z_1, \dots, z_n$ such that*

$$z_i \in M(z_{i-1}, I), \quad i = 1, \dots, n, \ \text{and } x \in M(z_n, I).$$

Further, let $f_1, f_2 : I \to \mathbb{R}$ be solutions of equation (1) *such that each of them is continuous at the point z_0, and for some $\alpha, \beta \in I$,*

$$f_1(\alpha) = f_2(\alpha), \qquad f_1(\beta) = f_2(\beta).$$

Then

$$f_1(x) = f_2(x), \quad \text{for all } x \in I.$$

In the next corollary, the functions f_1 and f_2 need not be continuous at the same point, therefore Condition 5 of Corollary 1 must be strenghtened.

Corollary 2. *Let $I \subset \mathbb{R}$ be an interval. Suppose that $M : I^2 \to I$ and $N : I^2 \times \mathbb{R}^2 \to \mathbb{R}$ satisfy the Conditions 1–4 of Corollary 1, and for every $z_0 \in I$ either $M(z_0, \cdot) : I \to I$ is bijective, or, for every $x \in I$, there are $n \in \mathbb{N}$ and a sequence $z_1, \dots, z_n$ such that*

$$z_i \in M(z_{i-1}, I), \quad i = 1, \dots, n, \quad \text{and } x \in M(z_n, I).$$

If $f_1, f_2 : I \to \mathbb{R}$ are solutions of equation (1) *such that each of them is continuous at least at one point, and for some $\alpha, \beta \in I$,*

$$f_1(\alpha) = f_2(\alpha), \quad f_1(\beta) = f_2(\beta),$$

then

$$f_1(x) = f_2(x), \quad \text{for all } x \in I.$$

Remark 2. The assumption of the continuity of the function f_i $(i = 1, 2)$ at a point in Corollary 2 can be replaced by its *quasi-monotonicity* at each point of a set $J \setminus A$ where J is an open subinterval of I and A at most countable set (cf. [2], Corollary 1). Recall that a real function f defined on an interval J is *quasi-increasing at a point* $x_0 \in J$, iff

$$\limsup_{x \to x_0 -} f(x) \leq f(x_0) \leq \liminf_{x \to x_0 +} f(x),$$

f is *quasi-decreasing at a point* $x_0 \in J$, iff

$$\liminf_{x \to x_0 -} f(x) \geq f(x_0) \geq \limsup_{x \to x_0 +} f(x).$$

References

[1] J. Aczél and J. Dhombres, Functional equations in several variables, *Encyclopedia of Mathematics and its Applications, Cambridge University Press*, 1989.

[2] M. Kuczma, An introduction to the theory of functional equations and inequalities, *PWN, Warszawa, Krakow, Katowice*, 1985.

[3] J. Matkowski and T. Świątkowski, Quasi-monotonicity, subadditive bijections of R_+, and characterization of L^p-norm, *J. Math. Anal. Appl.* **154** (1991), 493–506.

JANUSZ MATKOWSKI
INSTITUTE OF MATHEMATICS
PEDAGOGICAL UNIVERSITY
PLAC SŁOWIAŃSKI 9
PL–65069 ZIELONA GÓRA
POLAND

(Received November 12, 1997)