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A proper standard C^*-algebra whose automorphism and isometry groups are topologically reflexive

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Dedicated to Professors Zoltán Daróczy and Imre Kátai on the occasion of their 60th birthday

Abstract. In this paper we present a proper C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ containing all finite rank operators whose automorphism and isometry groups are topologically reflexive. This solves an open problem raised in our former paper [Mol1].

1. Introduction and the main result

Let $\mathcal{X}$ be a Banach space and for any subset $\mathcal{E} \subset \mathcal{B}(\mathcal{X})$ define

$$\operatorname{ref}_{\mathrm{al}} \mathcal{E} = \{T \in \mathcal{B}(\mathcal{X})\colon Tx \in \mathcal{E}x \text{ for all } x \in \mathcal{X}\}$$

and

$$\operatorname{ref}_{\mathrm{to}} \mathcal{E} = \{T \in \mathcal{B}(\mathcal{X})\colon Tx \in \overline{\mathcal{E}x} \text{ for all } x \in \mathcal{X}\},$$

where bar denotes norm-closure. The collection $\mathcal{E}$ of transformations is called algebraically reflexive if $\operatorname{ref}_{\mathrm{al}} \mathcal{E} = \mathcal{E}$. Similarly, $\mathcal{E}$ is said to be topologically reflexive if $\operatorname{ref}_{\mathrm{to}} \mathcal{E} = \mathcal{E}$. The study of reflexive subspaces of the

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operator algebra $\mathcal{B}(\mathcal{H})$ of the Hilbert space $\mathcal{H}$ is one of the most active research areas in operator theory (see, for example, [Lar] and the references therein).

Reflexivity problems concerning sets of linear transformations acting on operator algebras rather than on Hilbert spaces were first studied by Kadison and Larson and Sourour. In [Kad] and [LaSu] it was proved that the Lie-algebra of all derivations of a von Neumann algebra, respectively that of $\mathcal{B}(\mathcal{X})$ are algebraically reflexive. As for topological reflexivity, Shul'man [Shu] proved that the derivation algebra of any C^*-algebra is topologically reflexive. Concerning automorphisms which are at least so important as derivations, the first reflexivity result of the above kind was obtained by Brešar and Šemrl. They proved in [BrSe] that for a separable infinite dimensional Hilbert space $\mathcal{H}$, the group of all automorphisms of $\mathcal{B}(\mathcal{H})$ is algebraically reflexive. Here we should emphasize that in the present paper by an automorphism we mean a merely multiplicative linear bijection, so the $*$-preserving property is not assumed. In our paper [Mol1] we proved that the group of all automorphisms as well as the group of all surjective isometries of $\mathcal{B}(\mathcal{H})$ are even topologically reflexive. Since this phenomenon seems to be rather exceptional (in fact, as it was shown in [BaMo, Theorem 5], there are even von Neumann algebras with topologically nonreflexive automorphism and isometry groups; see also [Mol2]), in [Mol1] we raised the question of the existence of a proper C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ containing all finite rank operators whose automorphism and isometry groups are topologically reflexive. The aim of this note is to answer this question.

In what follows let $\mathcal{H}, \mathcal{K}$ be infinite dimensional separable complex Hilbert spaces. Denote by $\mathcal{F}(\mathcal{H})$ the set of all finite rank operators in $\mathcal{B}(\mathcal{H})$. A subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ is called standard if it contains $\mathcal{F}(\mathcal{H})$. The ideal of all compact operators on $\mathcal{H}$ is denoted by $\mathcal{C}(\mathcal{H})$. Let $\{\mathcal{H}_1, \dots, \mathcal{H}_n\} = \{\mathcal{H}_i\}$ be a fixed finite sequence of pairwise orthogonal closed subspaces of $\mathcal{H}$ which generate $\mathcal{H}$. Let $\mathcal{B}(\{\mathcal{H}_i\})$ denote the C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ consisting of all operators $A \in \mathcal{B}(\mathcal{H})$ for which $A(\mathcal{H}_i) \subset \mathcal{H}_i$ holds true for every index $i = 1, \dots, n$. Obviously, $\mathcal{B}(\{\mathcal{H}_i\})$ is isomorphic to the direct sum $\mathcal{B}(\mathcal{H}_1) \oplus \dots \oplus \mathcal{B}(\mathcal{H}_n)$. Now, we can formulate the main result of the paper which gives affirmative answer to our question in [Mol1, p. 192]

Theorem. *The automorphism group and the isometry group of the C^*-algebra $\mathcal{C}(\mathcal{H}) + \mathcal{B}(\{\mathcal{H}_i\})$ are topologically reflexive.*

Let $\mathcal{A}$ be a C^*-algebra. We say that the bounded linear map $\Phi:\mathcal{A}\to\mathcal{A}$ is an approximately local automorphism of $\mathcal{A}$ if for every $A \in \mathcal{A}$ there exists a sequence (Φ_n) of automorphisms such that $\Phi(A) = \lim_n \Phi_n(A)$. The definition of approximately local surjective isometries should now be self-explanatory. Our main result can be reformulated by saying that every approxiamtely local automorphism of $\mathcal{C}(\mathcal{H})+\mathcal{B}(\{\mathcal{H}_i\})$ is an automorphism and similar statement holds true for isometries as well.

It seems to be a natural idea to try to obtain our Theorem from our former reflexivity results [Mol1, Theorem 2 and Theorem 3]. In fact, one might think that every approximately local automorphism of the algebra $\mathcal{C}(\mathcal{H}) + \mathcal{B}(\{\mathcal{H}_i\})$ is, by restriction, an approximately local automorphism of the direct sum $\mathcal{B}(\mathcal{H}_1)\oplus\ldots\oplus\mathcal{B}(\mathcal{H}_n)$ and then apply the mentioned result in [Mol1]. However, the starting point of this argument is false. Namely, it is easy to give an automorphism of $\mathcal{C}(\mathcal{H}\oplus\mathcal{H})+\mathcal{B}(\{\mathcal{H},\mathcal{H}\})$ whose restriction to $\mathcal{B}(\{\mathcal{H},\mathcal{H}\})$ is not an automorphism of $\mathcal{B}(\{\mathcal{H},\mathcal{H}\})$. For example, let $0 \neq P$ be a finite rank projection on $\mathcal{H}$ and let $S, T \in \mathcal{B}(\mathcal{H})$ be such that $\ker S = \ker T = \operatorname{rng} P$, $\operatorname{rng} S = \operatorname{rng} T = \operatorname{rng}(I-P)$ and $ST = TS = I-P$. Using elementary computations, one can check that the map

$$\begin{bmatrix} A & K \\ C & B \end{bmatrix} \longmapsto \begin{bmatrix} P & T \\ S & P \end{bmatrix} \begin{bmatrix} A & K \\ C & B \end{bmatrix} \begin{bmatrix} P & T \\ S & P \end{bmatrix}$$

is an automorphism of the algebra $\mathcal{C}(\mathcal{H} \oplus \mathcal{H}) + \mathcal{B}(\{\mathcal{H},\mathcal{H}\})$ which does not leave $\mathcal{B}(\{\mathcal{H},\mathcal{H}\})$ invariant. Therefore, we have to look for a different approach to verify our main result.

2. Proof

We reach the proof of the Theorem via a series of auxiliary statements. First, in what follows we need the concept of Jordan homomorphisms. A linear map $\mathcal{J}$ between algebras $\mathcal{A}$ and $\mathcal{B}$ is called a Jordan homomorphism if

$$\mathcal{J}(A)^2 = \mathcal{J}(A^2) \qquad (A \in \mathcal{A}).$$

Observe that linearizing the previous equality, i.e. replacing A by $A + B$ we can deduce that $\mathcal{J}$ satisfies

$$\mathcal{J}(AB + BA) = \mathcal{J}(A)\mathcal{J}(B) + \mathcal{J}(B)\mathcal{J}(A) \qquad (A, B \in \mathcal{A}).$$

Our main objectives are standard C^*-algebras. The structures of all Jordan automorphisms, automorphisms, antiautomorphism (i.e. linear bijections reversing the order of multiplication) as well as surjective isometries of these algebras are easy to describe as we see in the following proposition.

Proposition 1. *Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a standard C^*-algebra. Then every Jordan automorphism of $\mathcal{A}$ is either an automorphism or an antiautomorphism. In the first case we have an invertible bounded linear operator T on $\mathcal{H}$ such that Φ is of the form*

$$\Phi(A) = TAT^{-1} \quad (A \in \mathcal{A}).$$

In the second case we have an invertible bounded linear operator S on $\mathcal{H}$ such that Φ is of the form

$$\Phi(A) = SA^{tr}S^{-1} \quad (A \in \mathcal{A}),$$

where ^{tr} denotes the transpose with respect to an arbitrary but fixed complete orthonormal sequence in $\mathcal{H}$. This latter assertion is equivalent to saying that there is an invertible bounded conjugate-linear operator S' on $\mathcal{H}$ such that

$$\Phi(A) = S'A^*{S'}^{-1} \quad (A \in \mathcal{A}).$$

If $\mathcal{A}$ contains I and $\Psi : \mathcal{A} \to \mathcal{A}$ is a surjective linear isometry, then there are unitary operators U, V on $\mathcal{H}$ such that Φ is either of the form

$$\Psi(A) = UAV \qquad (A \in \mathcal{A})$$

or of the form

$$\Psi(A) = UA^{tr}V \qquad (A \in \mathcal{A}).$$

Proof. It is a well-known theorem of Herstein [Her] (see also [Pal, Theorem 6.3.7]) that every Jordan homomorphism onto a prime algebra is either a homomorphism or an antihomomorphism. Since every standard operator algebra is prime, we have the first assertion. It is a classical theorem of Kadison [KaRi, 7.6.17, 7.6.18] that every surjective linear isometry of a unital C^*-algebra is a Jordan $*$-automorphism (i.e. a Jordan automorphism preserving the $*$-operation) followed by multiplication by a fixed unitary element. Now, our statement follows from folk results on the forms of automorphisms, antiautomorphisms, $*$-automorphisms and

$*$-antiautomorphisms of standard operator algebras (cf. [Che] and [Sem]). $\square$

Let $\Phi : \mathcal{A} \to \mathcal{A}$ be an approximately local automorphism. We are now interested in the question of when it follows that Φ is a Jordan homomorphism. A possible solution is given in the following proposition.

Proposition 2. *Let $\mathcal{A}, \mathcal{B} \subset \mathcal{B}(\mathcal{H})$ be closed $*$-subalgebras and suppose that for every self-adjoint element A of $\mathcal{A}$, the spectral measure of any Borel subset of $\sigma(A)$ which is bounded away 0 belongs to $\mathcal{A}$. If $\Phi : \mathcal{A} \to \mathcal{B}$ is a continuous linear map which sends projections to idempotents, then Φ is a Jordan homomorphism.*

Proof. Let $P, Q \in \mathcal{A}$ be mutually orthogonal projections. Then $\Phi(P) + \Phi(Q)$ is an idempotent and since it is the sum of two idempotents, we have $\Phi(P)\Phi(Q) = \Phi(Q)\Phi(P) = 0$. If $\lambda_1, \ldots, \lambda_n$ are real numbers and $P_1, \ldots, P_n \in \mathcal{A}$ are mutually orthogonal projections, then we infer

$$\begin{aligned}\left(\Phi\left(\sum_{k=1}^{n} \lambda_k P_k\right)\right)^2 &= \left(\sum_{k=1}^{n} \lambda_k \Phi(P_k)\right)^2 = \sum_{k=1}^{n} \lambda_k^2 \Phi(P_k) \\ &= \Phi\left(\left(\sum_{k=1}^{n} \lambda_k P_k\right)^2\right).\end{aligned}$$

By the spectral theorem and the continuity of Φ this implies that $\Phi(A)^2 = \Phi(A^2)$ holds true for every self-adjoint element $A \in \mathcal{A}$. Linearizing this equality, we immediately get $\Phi(AB + BA) = \Phi(A)\Phi(B) + \Phi(B)\Phi(A)$ for every self-adjoint $A, B \in \mathcal{A}$. Finally, if $T \in \mathcal{A}$ is arbitrary, then it can be written in the form $T = A + iB$ with self-adjoint $A, B \in \mathcal{A}$ and the previous equalities result in $\Phi(T)^2 = \Phi(T^2)$. $\square$

For any idempotents $P, Q \in \mathcal{B}(\mathcal{H})$ we write $P \leq Q$ if $QP = PQ = P$.

Lemma 3. *Let Q_n be a bounded sequence of idempotents in $\mathcal{B}(\mathcal{H})$ such that $Q_n \leq Q_{n+1}$ $(n \in \mathbb{N})$. Then (Q_n) converges strongly to an idempotent $Q \in \mathcal{B}(\mathcal{H})$.*

Proof. It follows from the proof of [Mol1, Lemma 2, p. 186]. $\square$

Proposition 4. *Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a standard C^*-algebra. Suppose that the linear subspace generated by the set of all projections in $\mathcal{A}$ is norm-dense and that for every closed nontrivial ideal $\mathcal{I}$ of $\mathcal{A}$, the quotient algebra $\mathcal{A}/\mathcal{I}$ contains uncountably many pairwise orthogonal projections. Let $\Phi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ be a continuous Jordan homomorphism. If (P_n) is a maximal family of rank-one projections in $\mathcal{B}(\mathcal{H})$, then the idempotent $E = \sum_n \Phi(P_n)$ is well-defined (we mean that it does not depend on the particular choice of (P_n)), E commutes with the range of Φ and we have $\Phi(.) = \Phi(.)E$.*

Proof. The assertions that E is well-defined and commutes with the range of Φ follow easily from the proof of [Mol1, Lemma 2, p. 187]. As for the remaining statement $\Phi(.) = \Phi(.)E$, observe that the map

$$\Psi : A \longmapsto \Phi(A)(I - E)$$

is a continuous Jordan homomorphism and it is easy to see that Ψ vanishes on every finite-rank projection. The kernel $\mathcal{I}$ of Ψ is a closed Jordan ideal of $\mathcal{A}$. It is well-known that every closed Jordan ideal in a C^*-algebra is an associative ideal as well [CiYo, Theorem 5.3]. Therefore, $\mathcal{I}$ is a closed associative ideal in $\mathcal{A}$. If $\mathcal{I} \neq \mathcal{A}$, then by our assumption on $\mathcal{A}$ it follows that the range of Ψ contains an uncountable family of pairwise orthogonal nonzero idempotents. Since this contradicts the separability of $\mathcal{K}$, we have $\Psi = 0$. Thus we have $\Phi(.) = \Phi(.)E$. □

Corollary 5. *Let $\mathcal{A}$ be as in Proposition 4 above. If $\Phi, \Phi' : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ are continuous Jordan homomorphisms which coincide on $\mathcal{F}(\mathcal{H})$, then we have $\Phi = \Phi'$.*

Proof. Let (P_n) be a maximal family of pairwise orthogonal rank-one projections in $\mathcal{B}(\mathcal{H})$. Let $A \in \mathcal{A}$ be arbitrary. By Proposition 4 we infer

$$\begin{aligned}
2\Phi(A) &= \sum_n (\Phi(A)\Phi(P_n) + \Phi(P_n)\Phi(A)) \\
&= \sum_n \Phi(AP_n + P_nA) = \sum_n \Phi'(AP_n + P_nA) \\
&= \sum_n (\Phi'(A)\Phi'(P_n) + \Phi'(P_n)\Phi'(A)) = 2\Phi'(A). \qquad \square
\end{aligned}$$

Proposition 6. *Let $\Phi : \mathcal{C}(\mathcal{H}) \to \mathcal{C}(\mathcal{K})$ be a continuous Jordan homomorphism. Then the second adjoint Φ^{**} of Φ defines a weak*-continuous Jordan homomorphism from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{K})$ which extends Φ.*

Proof. It is well-known that the dual space of $\mathcal{C}(\mathcal{H})$ is the Banach algebra $\mathcal{T}(\mathcal{H})$ of all trace-class operators on $\mathcal{H}$ and the dual space of $\mathcal{T}(\mathcal{H})$ is $\mathcal{B}(\mathcal{H})$. The dualities in question are given by the pair

$$\langle A, B \rangle = \operatorname{tr} AB$$

where $A \in \mathcal{C}(\mathcal{H}), B \in \mathcal{T}(\mathcal{H})$, respectively $A \in \mathcal{T}(\mathcal{H}), B \in \mathcal{B}(\mathcal{H})$. Here, tr denotes the usual trace-functional.

Now, if $K \in \mathcal{C}(\mathcal{H})$ and $T \in \mathcal{T}(\mathcal{K})$, then we compute

$$\operatorname{tr} \Phi^{**}(K)T = \operatorname{tr} K\Phi^*(T) = \operatorname{tr} \Phi(K)T.$$

This apparently gives us that Φ^{**} is an extension of Φ. Let $P \in \mathcal{B}(\mathcal{H})$ be an arbitrary projection and let (P_n) be a monotone increasing sequence of finite rank projections which converges strongly to P. We then have $\operatorname{tr} P_n T \to \operatorname{tr} PT$ for every trace-class operator T and we infer

$$\operatorname{tr} \Phi(P_n)T = \operatorname{tr} P_n \Phi^*(T) \longrightarrow \operatorname{tr} P\Phi^*(T) = \operatorname{tr} \Phi^{**}(P)T.$$

This implies that $\Phi(P_n)$ converges weakly to $\Phi^{**}(P)$. On the other hand, by Lemma 3 it follows that $\Phi(P_n)$ converges strongly to an idempotent. Hence, $\Phi^{**}(P)$ is an idempotent whenever P is a projection. Using Proposition 2 we obtain that Φ^{**} is a Jordan homomorphism. $\square$

Proposition 7. *Let $\mathcal{A}$ be as in Proposition 4. If, in addition, $\mathcal{A}$ contains I, and $\Phi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ is a unital, continuous Jordan homomorphism which preserves the rank-one operators, then there is an invertible bounded linear operator $T : \mathcal{H} \to \mathcal{K}$ so that Φ is either of the form*

$$\Phi(A) = TAT^{-1} \qquad (A \in \mathcal{A})$$

or of the form

$$\Phi(A) = TA^{tr}T^{-1} \qquad (A \in \mathcal{A}).$$

Proof. Let Ψ be the restriction of Φ onto $\mathcal{C}(\mathcal{H})$. Clearly, $\operatorname{rng} \Psi \subset \mathcal{C}(\mathcal{H})$. By Proposition 6, Ψ^{**} is a weak*-continuous Jordan homomorphism which preserves the rank-one operators. Moreover, by Corollary 5 we have $\Psi^{**}_{|\mathcal{A}} = \Phi$ and hence $\Psi^{**}(I) = I$. We now apply a result of Hou [Hou, Theorem 1.3] on the form of rank-one preservers. This says that either there are continuous linear operators $T : \mathcal{H} \to \mathcal{K}$ and $S : \mathcal{K} \to \mathcal{H}$ so that Ψ^{**} is of the form

$$\Psi^{**}(A) = TAS \qquad (A \in \mathcal{B}(\mathcal{H}))$$

or there are bounded conjugate-linear operators $T' : \mathcal{H} \to \mathcal{K}$ and $S' : \mathcal{K} \to \mathcal{H}$ so that Φ is of the form

$$\Psi^{**}(A) = T'A^*S' \qquad (A \in \mathcal{B}(\mathcal{H})).$$

In fact, Hou's theorem was formulated for weak-continuous maps but an inspection of the proof shows that this condition was used only to prove the continuity of T, S and to show that if the above formula is valid on $\mathcal{F}(\mathcal{H})$, then it holds true on $\mathcal{B}(\mathcal{H})$ as well. Obviously, in both places weak*-continuity can play the same role. Going further in our proof, let us suppose that Ψ^{**} is of the first form. Since Ψ^{**} is a Jordan homomorphism, it preserves the idempotents. This yields that $\langle x, y\rangle = 1$ implies $\langle Tx, S^*y\rangle = 1$ for every $x, y \in \mathcal{H}$. We have $\langle STx, y\rangle = \langle x, y\rangle$ $(x, y \in \mathcal{H})$ which gives $ST = I$. On the other hand, Ψ^{**} is unital and hence we infer $TS = I$. Therefore $S = T^{-1}$. If Ψ^{**} is of the second form above, one can follow the same argument. $\square$

After this preparation we now are in a position to prove our main result.

Proof of Theorem. Suppose that Φ is a continuous linear map which is an approximately local automorphism of the C^*-algebra $\mathcal{A} = \mathcal{C}(\mathcal{H}) + \mathcal{B}(\{\mathcal{H}_i\})$. We first show that Φ is a Jordan homomorphism. Observe that $\mathcal{A}$ does not fulfil the condition in Proposition 2, so we have to invent a more sophisticated argument. Clearly, the restrictions $\Phi_{|\mathcal{C}(\mathcal{H})}$ and $\Phi_{|\mathcal{B}(\{\mathcal{H}_i\})}$ send idempotents to idempotents. To these maps Proposition 2 applies and we obtain that they are Jordan homomorphisms. By the local property of Φ and the form of automorphisms of standard operator algebras it is apparent that Φ maps $\mathcal{C}(\mathcal{H})$ into itself. Let $\Psi = \Phi^{**}_{|\mathcal{C}(\mathcal{H})}$. Proposition 6 tells us that Ψ is a Jordan homomorphism on $\mathcal{B}(\mathcal{H})$. For every i we define linear maps $\Phi_i, \Psi_i : \mathcal{B}(\mathcal{H}_i) \to \mathcal{B}(\mathcal{H})$ by

$$\Phi_i(A) = \Phi(\hat{A}) \quad \text{and} \quad \Psi_i(A) = \Psi(\hat{A}),$$

where $\hat{A}$ is the element of $\mathcal{B}(\{\mathcal{H}_i\})$ which coincide with A on $\mathcal{H}_i$ and with 0 on $\mathcal{H}_i^\perp$. The maps Φ_i, Ψ_i are Jordan homomorphisms which are equal on $\mathcal{C}(\mathcal{H}_i)$. Observe that for any infinite dimensional separable Hilbert space $\mathcal{K}$, the algebra $\mathcal{B}(\mathcal{K})$ satisfies the conditions in Proposition 4. This is because of the well-known facts that the only nontrivial closed ideal in $\mathcal{B}(\mathcal{K})$ is $\mathcal{C}(\mathcal{K})$ and that the Calkin algebra $\mathcal{B}(\mathcal{K})/\mathcal{C}(\mathcal{K})$ has uncountably many pairwise orthogonal idempotents (see e.g. the proof of [KaRi, 10.4.11. Proposition]). Using Corollary 5 we deduce $\Phi_i(A) = \Psi_i(A)$ $(A \in \mathcal{B}(\mathcal{H}_i))$. After summation we conclude that $\Phi(A) = \Psi(A)$ holds true for every $A \in \mathcal{B}(\{\mathcal{H}_i\})$. Since Ψ is a Jordan homomorphism which extends $\Phi_{|\mathcal{C}(\mathcal{H})}$, we compute

$$\begin{aligned}\Phi(A)\Phi(K) + \Phi(K)\Phi(A) &= \Psi(A)\Psi(K) + \Psi(K)\Psi(A)\\ &= \Psi(AK + KA) = \Phi(AK + KA)\end{aligned}$$

for every $A \in \mathcal{B}(\{\mathcal{H}_i\}), K \in \mathcal{C}(\mathcal{H})$. The fact that Φ is a Jordan homomorphism now follows from the equality

$$\begin{aligned}(\Phi(K + A))^2 &= \Phi(A)^2 + \Phi(A)\Phi(K) + \Phi(K)\Phi(A) + \Phi(K)^2\\ &= \Phi(A^2 + AK + KA + K^2) = \Phi((K + A)^2).\end{aligned}$$

Next, we assert that Φ preserves the rank-one operators. Indeed, as a consequence of the local property of Φ we obtain that Φ sends every rank-one operator to an operator having rank at most one. Furthermore, if the image of a rank-one operator under Φ is 0, then the kernel of Φ is nontrivial. Since this kernel is a closed Jordan ideal and hence an associative ideal as well, it follows that Φ vanishes on $\mathcal{C}(\mathcal{H})$. Moreover, since the C^*-algebra $\mathcal{A} = \mathcal{C}(\mathcal{H}) + \mathcal{B}(\{\mathcal{H}_i\})$ satisfies the conditions in Proposition 4 (see the reference to the Calkin algebra above), by Corollary 5 we infer $\Phi = 0$. But, apparently, $\Phi(I) = I$ which is a contradiction. This shows that Φ preserves the rank-one operators. Now, from Proposition 7 it follows that there is an invertible operator $T \in \mathcal{B}(\mathcal{H})$ such that Φ is either of the form

$$\Phi(A) = TAT^{-1} \qquad (A \in \mathcal{A})$$

or of the form

$$\Phi(A) = TA^{tr}T^{-1} \qquad (A \in \mathcal{A}).$$

Suppose that Φ is of this latter form. Let $\mathcal{H}_i$ be an infinite dimensional subspace from our collection $\{\mathcal{H}_i\}$. Pick an operator $U \in \mathcal{A}$ which is a unilateral shift on $\mathcal{H}_i$ and the identity on $\mathcal{H}_i^{\perp}$. Obviously, U has a left inverse in $\mathcal{A}$ but it does not have a right one. Clearly, the same must hold true for the image of U under any automorphism of $\mathcal{A}$. Let (Φ_n) be a sequence of automorphisms of $\mathcal{A}$ for which $\Phi(U) = \lim_n \Phi_n(U)$. Since the set of all elements which have right inverse in $\mathcal{A}$ is open, we deduce that $\Phi(U)$ has no right inverse. On the other hand, we can compute

$$\Phi(U)\Phi(U^*) = \Phi(U^*U) = \Phi(I) = I.$$

Thus, we have arrived at a contradiction and, consequently, it follows that $\Phi(A) = TAT^{-1}$ for every $A \in \mathcal{A}$.

So, Φ is a homomorphism on $\mathcal{A}$. Since Φ maps $\mathcal{A}$ into $\mathcal{A}$, we have $TAT^{-1} \in \mathcal{A}$ for every $A \in \mathcal{A}$. We claim that this implies that $T^{-1}AT \in \mathcal{A}$ ($A \in \mathcal{A}$) which then will give us the surjectivity of Φ. Consider the matrix representations of the elements of $\mathcal{A}$ corresponding to the subspaces $\{\mathcal{H}_i\}$. Let $T = [T_{ij}]$ and $T^{-1} = S = [S_{ij}]$. Let the index i_0 be fixed for a moment and pick any operator $A_{i_0} \in \mathcal{B}(\mathcal{H}_{i_0})$. By $T\mathcal{A}T^{-1} \subset \mathcal{A}$ we obtain that the off-diagonal elements of the matrix $[T_{ii_0}A_{i_0}S_{i_0j}]$ are all compact operators. So, for any $i \neq j$ we have $T_{ii_0}\mathcal{B}(\mathcal{H}_{i_0})S_{i_0j} \subset \mathcal{C}(\mathcal{H}_j, \mathcal{H}_i)$. It is well-known that a bounded linear operator is compact if and only if its range does not contain an infinite dimensional closed subspace. Using this characterization, from $T_{ii_0}\mathcal{B}(\mathcal{H}_{i_0})S_{i_0j} \subset \mathcal{C}(\mathcal{H}_j, \mathcal{H}_i)$ we can infer that either T_{ii_0} or S_{i_0j} must be compact. Let us remove those rows and colums from the matrices of T and S which correspond to finite dimensional subspaces but hold on the numbering of the entries. Denote the matrices obtained in this way by $\tilde{T}$ and $\tilde{S}$, respectively. Obviously, we still have the property that, considering the ith column of $\tilde{T}$ and the ith row of $\tilde{S}$, from any pair of entries sitting in different positions, one of them is compact. We show that in every row and column of $\tilde{T}$ there is exactly one non-compact entry and the same holds true for $\tilde{S}$. To see this, consider the ith column of $\tilde{T}$. If every entry of it is compact, then by $ST = I$ it follows that the identity on $\mathcal{H}_i$ is compact which implies that $\mathcal{H}_i$ is finite dimensional and this is a contradiction. Next, suppose that there are two non-compact entries in the column in question. Then it easily follows that the ith row of $\tilde{S}$ consists of compact entries. Using $ST = I$ just as above, we arrive at a contradiction again. Therefore, there is exactly one non-compact entry in every column of $\tilde{T}$. Suppose that there is a row in $\tilde{T}$ which contains two non-compact elements. Then we necessarily have another row of $\tilde{T}$ whose entries are all compact. But by $TS = I$ this is untenable. Hence, we have proved that every row and column of $\tilde{T}$ contain exactly one non-compact entry. Clearly, this implies that $\tilde{S}$ has the same property. In fact, there is a non-compact element in position ij in $\tilde{T}$ if and only if there is a non-compact element in position ji in $\tilde{S}$. Now, it is apparent that the off-diagonal elements in $[S_{ii_0}A_{i_0}T_{i_0j}]$ are all compact. This gives us the desired inclusion $S\mathcal{A}T \subset \mathcal{A}$ and we obtain the surjectivity of Φ. Consequently, the automorphism group of $\mathcal{C}(\mathcal{H}) + \mathcal{B}(\{\mathcal{H}_i\})$ is topologically reflexive.

Let us now prove the topological reflexivity of the isometry group. By Proposition 1, every surjective isometry of $\mathcal{A}$ preserves the unitary group. Plainly, if Φ is a continuous linear map which is an approximately local surjective isometry, then Φ has the same preserver property. But the structure of unitary group preservers on C^*-algebras is well-known. In fact, [RuDy, Corollary] gives us that there is a unital Jordan $*$-homomorphism Ψ on $\mathcal{A}$ and a unitary element $U \in \mathcal{A}$ so that $\Phi(A) = U\Psi(A)$ $(A \in \mathcal{A})$. Obviously, we may suppose that $U = I$. Similarly to the case when our map was an approximately local automorphism, one can verify that Φ preserves the rank-one operators. Therefore, by Proposition 7 we infer that there is an invertible operator (in fact, a unitary one in the case of Jordan $*$-homomorphisms) T such that Φ is either of the form

$$\Phi(A) = TAT^{-1} \qquad (A \in \mathcal{A})$$

or of the form

$$\Phi(A) = TA^{tr}T^{-1} \qquad (A \in \mathcal{A}).$$

This latter form can be rewritten as $\Phi(A) = T'A^*{T'}^{-1}$ with some invertible bounded conjugate-linear operator T'. The proof can now be completed as in the case of the automorphism group. □

To conclude the paper we note that it seems to be an exciting question to investigate the reflexivity of the automorphism and isometry groups of the C^*-algebra $\mathcal{C}(\mathcal{H})+\mathcal{B}(\{\mathcal{H}_i\})$ in the case when the set $\{\mathcal{H}_i\}$ of subspaces is infinite. We feel that this problem is much more difficult than what we have treated here and the solution needs a completely different approach.

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