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A characterization of midpoint-quasiaffine functions

By KAZIMIERZ NIKODEM (Bielsko-Biała) ZSOLT PÁLES* (Debrecen)

Dedicated to Professors Zoltán Daróczy and Imre Kátai on their 60th birthday

Abstract. We consider the functional inequality

$$(*)\qquad \min(f(x), f(y)) \le f\left(\frac{x+y}{2}\right) \le \max(f(x), f(y)) \quad (x, y \in X),$$

where f is a real valued function on a linear space X. This inequality is satisfied by Jensen functions (that are solutions of the Jensen functional equation) and, in the case $X = \mathbb{R}$, by monotone functions. The main result of the paper shows that, under some regularity assumptions, any solution of $(*)$ is of the form $f = g \circ \alpha$, where $\alpha : X \to \mathbb{R}$ is an additive function and $g : \mathbb{R} \to \mathbb{R}$ is monotone.

1. Introduction

Let X be a linear space and D be a convex subset of X. A function $f : D \to \mathbb{R}$ is said to be a *Jensen function* (or a *midpoint-affine function*) if it satisfies the Jensen functional equation

$$(1)\qquad f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2} \qquad (x, y \in D).$$

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Clearly, if f is a Jensen function, then

$$(2) \qquad \min(f(x), f(y)) \le f\left(\frac{x+y}{2}\right) \le \max(f(x), f(y)) \quad (x, y \in D).$$

Functions $f : D \to \mathbb{R}$ satisfying (2) will be called *midpoint-quasiaffine* (or *internal*, cf. [2], [3], [6]) *functions*. Let us observe that, if $X = \mathbb{R}$, then every monotone function $f : \mathbb{R} \to \mathbb{R}$ also satisfies (2). Moreover, it is easy to see that functions of the form

$$(3) \qquad f = g \circ \alpha,$$

where $\alpha : X \to \mathbb{R}$ is an additive function and $g : \mathbb{R} \to \mathbb{R}$ is a monotone function, are also solutions of (2). The aim of this paper is to show that solutions of the form (3) are typical. However, as it is shown by the examples below, without any additional assumptions it cannot be obtained that any solution of (2) admits the decomposition (3).

The next example shows that the domain D of f plays an essential role, namely, if $D \ne X$, then the representation (3) fails.

Example 1. Let

$$f(x_1, x_2) = \frac{x_2}{x_1} \quad \text{if} \quad (x_1, x_2) \in D := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0\}.$$

First we show that f satisfies (2). For, let $x = (x_1, x_2), y = (y_1, y_2) \in D$. Then

$$\begin{aligned} f\left(\frac{x+y}{2}\right) &= f\left(\frac{x_1+y_1}{2}, \frac{x_2+y_2}{2}\right) = \frac{x_2+y_2}{x_1+y_1} \\ &= \frac{x_1}{x_1+y_1} \cdot \frac{x_2}{x_1} + \frac{y_1}{x_1+y_1} \cdot \frac{y_2}{y_1} = \frac{x_1}{x_1+y_1} \cdot f(x) + \frac{y_1}{x_1+y_1} \cdot f(y). \end{aligned}$$

The right hand side is a convex combination of $f(x)$ and $f(y)$, hence (2) is valid. Now assume that f can be represented in the form (3). Then

$$g(\alpha(x)) = f(x) = \frac{x_2}{x_1} = \frac{rx_2}{rx_1} = f(rx) = g(\alpha(rx)) = g(r\alpha(x)),$$

if $x \in D$ and $r > 0$ is a rational number. The function f is nonconstant, therefore g is nonconstant and α is not identically zero. Assume that there exists $x \in D$ such that $\alpha(x) > 0$. By the above equality, we have that g

is constant on the set $\{r\alpha(x) \mid r > 0,\ r \in \mathbb{Q}\}$ which is a dense subset of the interval $]0, \infty[$. The function g being monotone, it must be constant on $]0, \infty[$. Similarly, if there exists $y \in D$ such that $\alpha(y) < 0$, then g is constant on the interval $]-\infty, 0[$. Thus the function $g \circ \alpha$ takes at most three values over D, which means a contradiction, since the range of f is equal to $\mathbb{R}$.

The next example shows that if we want to obtain the representation (3) for f, then also some regularity conditions on f are essential.

Example 2. Let $X = \mathbb{R}$ and $H = \{h_\gamma \mid \gamma \in \Gamma\}$ be a Hamel base for $\mathbb{R}$ over the field $\mathbb{Q}$. Let the relation $\ll$ be a well-ordering on Γ. Let

$$V := \Big\{ \sum_{i=1}^{n} r_{\gamma_i} h_{\gamma_i} \;\Big|\; n \in \mathbb{N},\ r_{\gamma_1}, \dots, r_{\gamma_n} \in \mathbb{Q},\ r_{\gamma_n} > 0, \\ \gamma_i \ll \gamma_n \ (i = 1, \dots, n-1) \Big\}.$$

It is immediate to see that V and $\mathbb{R} \backslash V$ are midpoint convex sets. Therefore the characteristic function $f = \chi_V$ satisfies (2).

Let us observe now that every additive function $\alpha : \mathbb{R} \to \mathbb{R}$ which is bounded below (or above) on V must be identically zero (cf. Kuczma [5, Theorem IX.3.4, p. 213] and Ger [4]). Indeed, if α is not identically zero, then there is $\gamma_0 \in \Gamma$ such that $\alpha(h_{\gamma_0}) \neq 0$. Let $\gamma_1 \gg \gamma_0$. Then $rh_{\gamma_0} + h_{\gamma_1} \in V$ for all $r \in \mathbb{Q}$. However,

$$\alpha(rh_{\gamma_0} + h_{\gamma_1}) = r\alpha(h_{\gamma_0}) + \alpha(h_{\gamma_1})$$

which is not a bounded function of $r \in \mathbb{Q}$. Hence α is also not bounded below and above on V.

Suppose now that f is of the form (3). The function f takes only the values 0 and 1, thus the sets

$$A := \{x \in \mathbb{R} \mid g(x) = 0\} \quad \text{and} \quad B := \{x \in \mathbb{R} \mid g(x) = 1\}$$

cover the range of α. The function g is monotone, hence A and B are convex subsets of $\mathbb{R}$. They are also nonempty and disjoint sets, therefore there exists a constant $c_0 \in \mathbb{R}$ such that either $\sup A \le c_0 \le \inf B$ or $\inf A \ge c_0 \ge \sup B$. In the first and in the second case, we get

$$\inf_{x \in V} \alpha(x) \ge c_0 \quad \text{and} \quad \sup_{x \in V} \alpha(x) \le c_0,$$

respectively. Therefore, α is bounded below or above on V. Hence α must be identically zero and f must be constant. The contradiction obtained shows the impossibility of the decomposition of f in the form (3).

In order to motivate the additional assumptions on f, observe that Jensen functions satisfy a stronger version of the inequality (2). Namely, for $x, y \in D$,

$$\min(f(x), f(y)) < f\left(\frac{x+y}{2}\right) < \max(f(x), f(y)) \quad \text{if } f(x) \neq f(y).$$

A midpoint-quasiaffine function that also satisfies (4) will be called *strictly midpoint-quasiaffine*. It is well known (cf. [5]) that Jensen functions also satisfy the equation

$$f(rx + (1-r)y) = rf(x) + (1-r)f(y) \quad x, y \in D,\ r \in [0,1] \cap \mathbb{Q}. \tag{5}$$

Therefore, Jensen functions have the following *radial continuity property*

$$\lim_{\substack{r \to 0^+ \\ r \in \mathbb{Q}}} f(rx + (1-r)y) = f(y).$$

If a function $f : D \to \mathbb{R}$ satisfies the apparently weaker condition

$$\limsup_{\substack{r \to 0^+ \\ r \in \mathbb{Q}}} f(rx + (1-r)y) \leq f(y)$$

for all $x, y \in D$, then we say that f is $\mathbb{Q}$-*radially upper semicontinuous on* D.

In the main results of the paper, we will show that strictly midpoint-quasiaffine and $\mathbb{Q}$-radially upper semicontinuous functions can be represented in the form (3) and this representation is unique up to a natural transformation.

In order to accomplish the above aim, we study the relationship between (strictly) midpoint-quasiaffine functions and (strictly) $\mathbb{Q}$-quasiaffine functions in the next section. As we have noted above, (5) is equivalent to (1). Therefore the functional inequality

$$\begin{gathered} \min(f(x), f(y)) \leq f(rx + (1-r)y) \leq \max(f(x), f(y)) \\ x, y \in D,\ r \in [0,1] \cap \mathbb{Q} \end{gathered} \tag{6}$$

is closely related to (2). Solutions of (6) will be called $\mathbb{Q}$-*quasiaffine functions*. If $f(x) \neq f(y)$ and (6) holds with strict inequalities for $r \neq 0, 1$, then f is called a *strictly* $\mathbb{Q}$-*quasiaffine function*. In the next section, we show that if $D = X$, then (strict) midpoint-quasiaffinity and (strict) $\mathbb{Q}$-quasiaffinity are equivalent properties.

The main results of the paper will be obtained in Section 3 using a version of the Hahn–Banach separation theorem due to Páles [8], [10] that was originally developed for the characterization of quasideviation means. The structure theorem obtained for strictly midpoint-quasiaffine and $\mathbb{Q}$-radially upper semicontinuous functions gives some explanation for the irregularity properties in the nonmonotone case discussed by Császár [2], [3] and Marcus [6].

As application, we consider *quasi-additive functions* in the sense of Tabor [11], [12], (see also Baran [1]) and we obtain some information on the structure of such functions. This structure theorem explains the irregularity properties of noncontinuous quasi-additive functions. In the last section we consider Jensen-convex functions and prove that, under a weak condition, they can be represented as the composition of a continuous convex function and an additive function.

2. Midpoint-quasiaffine and $\mathbb{Q}$-quasiaffine functions

If $f : D \subset X \to \mathbb{R}$ then we define the upper and lower level sets of f by

$$A_c = A(f, c) = \{x \in D \mid f(x) < c\},$$
$$\overline{A}_c = \overline{A}(f, c) = \{x \in D \mid f(x) \leq c\},$$

and

$$B_c = B(f, c) = \{x \in D \mid f(x) > c\},$$
$$\overline{B}_c = \overline{B}(f, c) = \{x \in D \mid f(x) \geq c\}.$$

The midpoint-convexity property of these sets is related to the functional inequality (2) by the following lemma.

Lemma 1. *Let D be a convex subset of the linear space X. Then $f : D \to \mathbb{R}$ is a midpoint-quasiaffine function if and only if, for all $c \in \mathbb{R}$, the level sets A_c, $\overline{A}_c$, B_c, and $\overline{B}_c$ are midpoint-convex.*

The proof of this lemma is elementary, therefore, it is omitted.

Analogously, we have

Lemma 2. *Let D be a convex subset of the linear space X. Then $f : D \to \mathbb{R}$ is a $\mathbb{Q}$-quasiaffine function if and only if, for all $c \in \mathbb{R}$, the level sets A_c, $\overline{A}_c$, B_c, and $\overline{B}_c$ are $\mathbb{Q}$-convex.*

In order to obtain the equivalence of the functional inequalities (2) and (6), we shall need the following result on the equivalence of midpoint-convexity and $\mathbb{Q}$-convexity.

Lemma 3 (Páles [7, Lemma]). *If $A \subset X$ is a midpoint-convex set such that its complement $X \setminus A$ is also midpoint-convex, then A is also $\mathbb{Q}$-convex.*

Theorem 1. *Let $f : X \to \mathbb{R}$ be a midpoint-quasiaffine function. Then it is also $\mathbb{Q}$-quasiaffine. Moreover, if f is a strictly midpoint-quasiaffine function, then it is also strictly $\mathbb{Q}$-quasiaffine.*

Proof. If f is midpoint-quasiaffine, then, by Lemma 1, all the level sets A_c, $\overline{A}_c$, B_c, and $\overline{B}_c$ are midpoint-convex. However, $X \setminus A_c = \overline{B}_c$. Therefore, the complement of A_c is also midpoint-convex. Thus, by Lemma 3, A_c is $\mathbb{Q}$-convex, too. Analogously, $\overline{A}_c$, B_c, and $\overline{B}_c$ are $\mathbb{Q}$-convex for all $c \in \mathbb{R}$. Therefore, due to Lemma 2, f is $\mathbb{Q}$-quasiaffine.

Assume now that f is strictly midpoint-quasiaffine. Then, by induction, we can get that

$$\min(f(x), f(y)) < f(dx + (1-d)y) < \max(f(x), f(y)) \tag{7}$$

if $f(x) \neq f(y)$ and $d \in]0,1[$ is a diadic rational number, that is $d = k/2^n$, where $k, n \in \mathbb{N}$, $0 < k < 2^n$. Let $r \in]0,1[\cap \mathbb{Q}$ be arbitrary and $f(x) \neq f(y)$. There exists diadic rational numbers d', d'' such that $0 < d' < r < d'' < 1$. Then the element $rx + (1-r)y$ is a $\mathbb{Q}$-convex combination of $d'x + (1-d')y$ and $d''x + (1-d'')y$. Therefore, by the $\mathbb{Q}$-quasiaffinity of f, we have

$$\begin{aligned} &\min(f(d'x + (1-d')y), f(d''x + (1-d'')y)) \\ &\leq f(rx + (1-r)y) \leq \max(f(d'x + (1-d')y), f(d''x + (1-d'')y)). \end{aligned}$$

On the other hand, we have (7) with $d = d'$ and $d = d''$. These inequalities together with the previous one yield

$$\min(f(x), f(y)) \leq f(rx + (1-r)y) \leq \max(f(x), f(y)).$$

Hence f is strictly $\mathbb{Q}$-quasiaffine. $\square$

3. Main results

In this section we derive the desired decomposition (3) of strictly midpoint-quasiaffine and $\mathbb{Q}$-radially upper semicontinuous functions defined on the whole of X. In order to accomplish this aim, we investigate first the connection between the additional regularity assumptions and the corresponding properties of the level sets of the given function.

Lemma 4. *Let $D \subset X$ be a convex set, $f : D \to \mathbb{R}$ be a strictly $\mathbb{Q}$-quasiaffine function. Then, for all $c \in \mathbb{R}$,*

$$rA_c + (1-r)\overline{A}_c \subset A_c \quad \text{and} \quad rB_c + (1-r)\overline{B}_c \subset B_c \tag{8}$$

if $r \in]0,1[\cap \mathbb{Q}$.

Proof. To prove the first inclusion in (8), let $c \in \mathbb{R}$. The function f is $\mathbb{Q}$-quasiaffine, hence, by Lemma 2, A_c is $\mathbb{Q}$-convex, i.e. $rA_c+(1-r)A_c \subset A_c$ for all $r \in [0,1] \cap \mathbb{Q}$. In order to prove the statement, it suffices to show that if $0 < r < 1$, $x \in A_c$, and $y \in \overline{A}_c \setminus A_c$, then $rx + (1-r)y \in A_c$. Indeed, in this case $f(x) < c$ and $f(y) = c$. By the strict $\mathbb{Q}$-quasiaffinity of f, we have

$$f(rx + (1-r)y) < \max(f(x), f(y)) = f(y) = c.$$

Hence $rx + (1-r)y \in A_c$. The proof of the second inclusion in (8) is analogous. $\square$

A subset $A \subset D$ will be called $\mathbb{Q}$-*algebraically open* in D if, for all $x \in D$ and $y \in A$ there exists $\rho \in]0,1[\cap \mathbb{Q}$ such that $rx + (1-r)y \in A$ whenever $r \in [0,\rho] \cap \mathbb{Q}$. If A is $\mathbb{Q}$-convex, then it is $\mathbb{Q}$-algebraically open if and only if, for all $x \in D$, $y \in A$, there exists $\rho \in]0,1[\cap \mathbb{Q}$ such that $\rho x + (1-\rho)y \in A$.

Lemma 5. *Let $D \subset X$ be a convex set and $f : D \to \mathbb{R}$ be $\mathbb{Q}$-radially upper semicontinuous on D. Then, for all $c \in \mathbb{R}$, A_c is $\mathbb{Q}$-algebraically open in D.*

Proof. Let $x \in D$ and $y \in A_c$. Then $f(y) < c$. By the $\mathbb{Q}$-radial upper semicontinuity of f on D, we have

$$\limsup_{\substack{r \to 0^+ \\ r \in \mathbb{Q}}} f(rx + (1-r)y) \le f(y) < c.$$

Therefore, there exists $\rho \in]0,1[\cap \mathbb{Q}$ such that

$$f(rx + (1-r)y) < c \quad \text{if} \quad r \in]0,\rho] \cap \mathbb{Q},$$

that is,

$$rx + (1-r)y \in A_c \quad \text{if} \quad r \in]0,\rho] \cap \mathbb{Q}.$$

□

Remark 1. If $f : D \to \mathbb{R}$ is a $\mathbb{Q}$-quasiaffine function, then it is easy to check that the $\mathbb{Q}$-radial upper semicontinuity of f is equivalent to the following weaker property: for all $x, y \in D$,

$$\liminf_{\substack{r \to 0^+ \\ r \in \mathbb{Q}}} f(rx + (1-r)y) \leq f(y).$$

Therefore, this property also yields the $\mathbb{Q}$-algebraic openness of the level sets A_c if f is $\mathbb{Q}$-quasiaffine.

Our next result is a version of the Hahn–Banach separation theorem in vector spaces over the field $\mathbb{Q}$.

Lemma 6. *Let A and B be nonempty disjoint $\mathbb{Q}$-convex subsets of X such that A is $\mathbb{Q}$-algebraically open in X. Then there exists an additive function $\alpha : X \to \mathbb{R}$ and a constant $\gamma \in \mathbb{R}$ such that*

$$\alpha(a) < \gamma \quad (a \in A) \quad \text{and} \quad \gamma \leq \alpha(b) \quad (b \in B).$$

Proof. The main idea to prove this separation theorem is to deduce it from a separation theorem for disjoint subsemigroups of abelian semigroups developed by Páles [8].

Define two subsets of $X^* := \mathbb{R} \times X$ by

$$A^* := \{(r, ra) : a \in A,\, r > 0,\, r \in \mathbb{Q}\},$$
$$B^* := \{(r, ra) : a \in B,\, r > 0,\, r \in \mathbb{Q}\}.$$

Then X^* is a group and the multiplication by rational numbers can be defined in X^* in a natural way. Being A and B disjoint $\mathbb{Q}$-convex sets, the sets A^* and B^* are disjoint subsemigroups of X^* which are also closed under multiplication by positive rational numbers. Define the core of A^* by

$$\operatorname{cor} A^* := \{a^* \in A^* : \forall x^* \in X^* \, \exists n \in \mathbb{N} \; na^* + x^* \in A^*\}.$$

(C.f. [8], [10].) The set A being $\mathbb{Q}$-algebraically open, we have that $\operatorname{cor} A^* = A^*$. Indeed, if $a^* = (r, ra) \in A^*$ and $x^* = (s, sx) \in X^*$, then for large $n \in \mathbb{N}$, we get that

$$\frac{nr}{nr+s} a + \frac{s}{nr+s} x \in A.$$

Hence

$$n(r, ra) + (s, sx) = \left(nr + s, (nr+s) \left[\frac{nr}{nr+s} a + \frac{s}{nr+s} x \right] \right) \in A^*.$$

Now we are in the position to apply the Hahn–Banach type separation theorem for subsemigroups from [8]. Thus, there exists an additive function $\alpha^* : X^* \to \mathbb{R}$ such that

$$\alpha^*(a^*) < 0 \quad (a^* \in \operatorname{cor} A^* = A^*) \quad \text{and} \quad 0 \le \alpha^*(b^*) \quad (b^* \in B^*).$$

Define α and γ by

$$\alpha(x) := \alpha^*(0, x) \quad \text{and} \quad \gamma := -\alpha^*(1, 0).$$

Then, it follows from the separating property of α^* that

$$\alpha(a) - \gamma = \alpha^*(1, a) < 0 \quad (a \in A) \quad \text{and} \quad \alpha(b) - \gamma = \alpha^*(1, b) \ge 0 \quad (b \in B),$$

which is equivalent to the statement of the lemma. □

The main result of this paper is contained in the next theorem.

Theorem 2. *Let $f : X \to \mathbb{R}$ be a nonconstant function. Then f is a strictly midpoint-quasiaffine and $\mathbb{Q}$-radially upper semicontinuous function if and only if it can be represented in the form $f = g \circ \alpha$, where $\alpha : X \to \mathbb{R}$ is an additive function and $g : \mathbb{R} \to \mathbb{R}$ is an upper semicontinuous strictly increasing function. Furthermore, the representation $f = g \circ \alpha$ is unique in the following sense: If $f = g' \circ \alpha'$ with an additive α' and upper* *semicontinuous strictly increasing g', then there exists a positive constant $q > 0$ such that*

$$\alpha'(x) = q\alpha(x) \quad (x \in X) \quad \text{and} \quad g'(t) = g(t/q) \quad (t \in R).$$

Proof. The proof of the sufficiency is elementary and omitted. For the necessity, assume that f is a nonconstant strictly midpoint-quasiaffine and $\mathbb{Q}$-radially upper semicontinuous function. Then, by Theorem 1, it is also strictly $\mathbb{Q}$-quasiaffine. Denote by I the (nonempty) open interval $]\inf f, \sup f[$. It follows from the strict $\mathbb{Q}$-quasiaffinity that the range of f is contained in I. To see this, let $x \in X$ be arbitrary. We show that $f(x) > \inf f$, the proof of $f(x) < \sup f$ is analogous. On the contrary, assume that $f(x) = \inf f$. The function f is nonconstant, hence there exists $y \in X$ such that $f(x) \neq f(y)$, therefore $f(x) < f(y)$. By the midpoint-quasiaffinity, we have

$$\min(f(2x-y), f(y)) \leq f(x) \leq \max(f(2x-y), f(y)).$$

The right hand side inequality is strict because $f(x) < f(y)$. Hence $f(2x-y) < f(y)$. Thus, by the strict midpoint-quasiaffinity, $f(2x-y) < f(x)$, that is $\inf f < f(x)$.

Now let $c^* \in I$ be an arbitrarily fixed element. Then the level sets A_{c^*} and $\overline{B}_{c^*}$ are nonempty disjoint $\mathbb{Q}$-convex sets. By Lemma 5, A_{c^*} is $\mathbb{Q}$-algebraically open. Therefore, we are in the position to apply the Hahn–Banach-type separation theorem of Lemma 6. Thus there exists an additive function $\alpha : X \to \mathbb{R}$ and a constant $\gamma_0 \in \mathbb{R}$ such that

$$\alpha(a) < \gamma_0 \quad (a \in A_{c^*}) \quad \text{and} \quad \gamma_0 \leq \alpha(b) \quad (b \in \overline{B}_{c^*}). \tag{9}$$

Our first aim is to show that α separates $\overline{A}_c$ and $\overline{B}_c$ for all $c \in I$, that is

$$\sup_{a \in \overline{A}_c} \alpha(a) = \inf_{b \in \overline{B}_c} \alpha(b). \tag{10}$$

First we prove the "$\leq$" inequality in (10). If this inequality is not satisfied, then there exist $a \in \overline{A}_c$ and $b \in \overline{B}_c$ such that

$$\alpha(a) > \alpha(b). \tag{11}$$

We distinguish two cases. *Case I:* $c < c^*$. Then $\overline{B}_{c^*} \subset B_c$. On the other hand, by (11), there exists $n \in \mathbb{N}$ such that

$$\alpha(a + n(a - b)) \geq \gamma_0.$$

Then $a + n(a - b)$ cannot be in A_{c^*}, hence $a + n(a - b) \in \overline{B}_{c^*} \subset B_c$. Thus, applying Lemma 4, we get

$$a = \frac{1}{n+1}(a + n(a - b)) + \frac{n}{n+1} b \in \frac{1}{n+1} B_c + \frac{n}{n+1} \overline{B}_c \subset B_c,$$

which contradicts $a \in \overline{A}_c$.

Case II: $c \geq c^*$. Then $A_{c^*} \subset A_c$. By (11), there exists $n \in \mathbb{N}$ such that

$$\alpha(b + n(b - a)) < \gamma_0.$$

Then $b + n(b - a)$ cannot be in $\overline{B}_{c^*}$, hence $b + n(b - a) \in A_{c^*} \subset A_c$. Thus, applying Lemma 4 again, we get

$$b = \frac{1}{n+1}(b + n(b - a)) + \frac{n}{n+1} a \in \frac{1}{n+1} A_c + \frac{n}{n+1} \overline{A}_c \subset A_c,$$

which contradicts $b \in \overline{B}_c$.

The contradictions obtained show that (10) is valid with "$\leq$" for all $c \in I$. To show that this inequality is actually an equality, observe that the sets $\overline{A}_c$ and $\overline{B}_c$ cover X, hence the sets $\{\alpha(a) : a \in \overline{A}_c\}$ and $\{\alpha(b) : b \in \overline{B}_c\}$ cover the range of α. The range of a nonzero additive function is everywhere dense in $\mathbb{R}$, thus the strict inequality "$<$" in (10) leads to an obvious contradiction.

Define now the function $\gamma : I \to \mathbb{R}$ by

$$\gamma(c) = \sup_{a \in \overline{A}_c} \alpha(a).$$

Then (10) can be rewritten as

$$\alpha(a) \leq \gamma(c) \quad \text{if} \quad f(a) \leq c \quad \text{and} \quad \alpha(b) \geq \gamma(c) \quad \text{if} \quad f(b) \geq c.$$

Therefore, taking $a = b = x$, $c = f(x)$ (and using that $f(x) \in I$), we get that

$$\alpha(x) = \gamma(f(x)) \quad \text{for all} \quad x \in X. \tag{12}$$

Our next aim is to show, that the function γ is continuous, increasing, unbounded from above and below, and the function g in the statement of the theorem can be obtained as its right inverse.

The monotonicity property of γ is obvious from its definition. The range of the additive function α is dense, and A_c and $\overline{B}_c$ are complementary sets. Therefore,

$$\gamma(c) = \sup_{a\in\overline{A}_c} \alpha(a) \geq \sup_{a\in A_c} \alpha(a) \geq \inf_{a\in\overline{B}_c} \alpha(a).$$

Due to (10), the left and right hand sides are equal. Hence we have

$$\gamma(c) = \sup_{a\in A_c} \alpha(a). \tag{13}$$

To prove that γ is lower semicontinuous, fix an element $c_0 \in I$, $t \in \mathbb{R}$ and assume that $\gamma(c_0) > t$. Then, by (13), there exists $a_0 \in A_{c_0}$ such that $\alpha(a_0) > t$. If $a_0 \in A_{c_0}$, then $f(a_0) < c_0$. Thus, $f(a_0) < c$ if c is taken from a small neighbourhood U of c_0. Then $a_0 \in A_c$ and hence $\gamma(c) = \sup_{a\in A_c} \alpha(a) > t$ for $c \in U$.

An analogous argument and the relation

$$\gamma(c) = \inf_{a\in B_c} \alpha(a) \tag{14}$$

show that γ is also upper semicontinuous. Thus it must be continuous.

To see the unboundedness of γ from below, let $t \in \mathbb{R}$ be fixed. Then there exists $x \in X$ such that $\alpha(x) \leq t$. Let $c < f(x)$, $c \in I$. Then $x \in B_c$, and by (14), $\gamma(c) \leq \alpha(x) \leq t$. An analogous argument yields the unboundedness from above.

For $t \in \mathbb{R}$ define

$$g(t) := \sup\{c \in I : \gamma(c) \leq t\} = \sup G_t. \tag{15}$$

This function is real valued. Indeed, if $t \in \mathbb{R}$, then, by the unboundedness of γ from below, the set G_t behind the supremum sign is nonempty and thus $g(t) > -\infty$. On the other hand, there exists $c_0 \in I$ such that $\gamma(c_0)>t$. Then, for $c \in G_t$, we have $\gamma(c) < \gamma(c_0)$. Hence $c < c_0$, which means that c_0 is an upper bound for G_t. Thus $g(t) < +\infty$.

It follows from the continuity of γ that g is strictly increasing. Indeed, if $g(t_1) = g(t_2)$ for some $t_1 < t_2$, then the function γ does not take values in the interval $]t_1, t_2]$. This, together with the unboundedness and continuity of γ, yields an obvious contradiction.

Obviously, g can be expressed in the following form:

$$g(t) := \inf\{c \in I : \gamma(c) > t\} = \inf H_t. \tag{16}$$

Using this form, we can show that g is upper semicontinuous. For, let $g(t_0) < s_0$ for some t_0, s_0. Then, due to (16), there exists $c_0 \in I$ such that $c_0 < s_0$ and $\gamma(c_0) > t_0$. For t from a sufficiently small neighbourhood U of t_0, we have $\gamma(c_0) > t$, that is $c_0 \in H_t$. Thus $g(t) = \inf H_t < s_0$ for $t \in U$. Therefore, g is upper semicontinuous.

By (15), (16) and the continuity of γ, it is also easy to see that g is the right inverse of γ, that is, $\gamma(g(t)) = t$ for all $t \in \mathbb{R}$.

To complete the proof of the necessity, we show that $f(x) = g(\alpha(x))$. Applying g to both sides of (12), we have

$$g(\alpha(x)) = g(\gamma(f(x))) \qquad \text{for all} \quad x \in X.$$

Therefore, it suffices to prove that $g(\gamma(s)) = s$ if s is in the range of f. Clearly,

$$g(\gamma(s)) = \sup\{c \in I : \gamma(c) \le \gamma(s)\} \ge s$$

for all $s \in I$ and the inequality turns into an equality if and only if $\gamma(c) > \gamma(s)$ for all $c > s$. Therefore, it is enough to show that, for all $x \in X$,

$$\gamma(c) > \gamma(f(x)) = \alpha(x) \quad \text{if} \quad c > f(x).$$

Let $x \in X$, $c > f(x)$ and choose $u \in X$ such that $\alpha(u) > 0$. By the $\mathbb{Q}$-radial upper semicontinuity of f, there exists a rational number $r > 0$ such that $f(x + ru) < c$. Then we have $x + ru \in A_c$, and hence

$$\gamma(c) = \sup_{a \in A_c} \alpha(a) \ge \alpha(x + ru) > \alpha(x) = \gamma(f(x)).$$

Thus the proof of the necessity is complete.

In the last part of the proof, we prove the uniqueness of the representation as stated in the theorem.

Assume that $f = g' \circ \alpha'$, where α' is an additive function and g' is an upper semicontinuous strictly increasing function. First observe that $g \circ \alpha = g' \circ \alpha'$ implies

$$\{x \in X : \alpha(x) \le 0\} = \{x \in X : \alpha'(x) \le 0\}. \tag{17}$$

Indeed, g and g' are strictly increasing, hence

$$\begin{aligned}\{x : \alpha'(x) \le 0\} &= \{x : g(\alpha(x)) \le g(0)\} = \{x : f(x) \le f(0)\}\\ &= \{x : g'(\alpha'(x)) \le g'(0)\} = \{x : \alpha'(x)) \le 0\}.\end{aligned}$$

Let $x_0 \in X$ be fixed such that $\alpha(x_0) > 0$. Then, by (17), $\alpha'(x_0) > 0$ holds, too. We show that

$$\alpha'(x) = \frac{\alpha'(x_0)}{\alpha(x_0)}\alpha(x) = q\alpha(x) \quad \text{for} \quad x \in X. \tag{18}$$

Let $x \in X$ and choose two rational sequences (r_n) and (s_n) such that (r_n) is monotone increasing, (s_n) is monotone decreasing and

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} s_n = \frac{\alpha(x)}{\alpha(x_0)}.$$

Then, we have $r_n \le \alpha(x)/\alpha(x_0) < s_n$ and hence

$$\alpha(r_n x_0 - x) \le 0 < \alpha(s_n x_0 - x) \quad \text{for} \quad n \in \mathbb{N}.$$

Using (17), these inequalities are equivalent to

$$\alpha'(r_n x_0 - x) \le 0 < \alpha'(s_n x_0 - x) \quad \text{for} \quad n \in \mathbb{N}.$$

Hence $r_n \le \alpha'(x)/\alpha'(x_0) < s_n$. Taking the limit $n \to \infty$, we obtain

$$\frac{\alpha'(x)}{\alpha'(x_0)} = \frac{\alpha(x)}{\alpha(x_0)},$$

which proves (18).

Define now $g^* : \mathbb{R} \to \mathbb{R}$ by $g^*(t) = g(t/q)$ $(t \in \mathbb{R})$. Then

$$g^*(\alpha'(x)) = g(\alpha'(x)/q) = g(\alpha(x)) = f(x) = g'(\alpha'(x))$$

for all $x \in X$. Therefore the two functions g^* and g' coincide on a dense subset of $\mathbb{R}$. Being upper semicontinuous and increasing, they must coincide everywhere, i.e. $g'(t) = g(t/q)$ for $t \in \mathbb{R}$.

Thus the proof of the theorem is complete. $\square$

The following result is the lower semicontinuous counterpart of the above theorem. It can be proved in a completely analogous way.

Theorem 3. *Let $f : X \to \mathbb{R}$ be a nonconstant function. Then f is a strictly midpoint-quasiaffine and $\mathbb{Q}$-radially lower semicontinuous function if and only if it can be represented in the form $f = g \circ \alpha$, where $\alpha : X \to \mathbb{R}$ is an additive function and $g : \mathbb{R} \to \mathbb{R}$ is a lower semicontinuous strictly increasing function. Furthermore, the representation $f = g \circ \alpha$ is unique in the sense of Theorem 2.*

If the function f is both $\mathbb{Q}$-radially upper and lower semicontinuous then, necessarily, the function g has stronger properties.

Theorem 4. *Let $f : X \to \mathbb{R}$ be a nonconstant function. Then f is a strictly midpoint-quasiaffine and $\mathbb{Q}$-radially continuous function if and only if it can be represented in the form $f = g \circ \alpha$, where $\alpha : X \to \mathbb{R}$ is an additive function and $g : \mathbb{R} \to \mathbb{R}$ is an upper semicontinuous strictly increasing function which is continuous on the range of the additive function α. Furthermore, the representation $f = g \circ \alpha$ is unique in the sense of Theorem 2.*

Proof. If f has the representation $f = g \circ \alpha$ then, by Theorem 2, it is strictly midpoint-quasiaffine and $\mathbb{Q}$-radially upper semicontinuous. It is immediate, that due to the continuity of g on the range of α, it is also $\mathbb{Q}$-radially lower semicontinuous.

Conversely, if f is strictly midpoint-quasiaffine and $\mathbb{Q}$-radially continuous then, by Theorem 2, it can be represented in the form $f = g \circ \alpha$, where α is additive, and g is strictly increasing and upper semicontinuous. To prove the continuity of g on the range of α, let $t = \alpha(x)$ be an arbitrary element, where $x \in X$. Choose $y \in X$ such that $\alpha(y) < \alpha(x)$ and denote $x_n := (1/n)y + (1 - 1/n)x$. Then, by the $\mathbb{Q}$-radial continuity of f, $f(x_n)$ tends to $f(x)$, that is $g(\alpha(x_n)) \to g(t)$ as $n \to \infty$. By the choice of y, the sequence $t_n = \alpha(x_n)$ is strictly monotone increasing and g is monotone, hence

$$g(t) = \lim_{n\to\infty} g(t_n) = \lim_{s\to t-0} g(s) = \liminf_{s\to t} g(s).$$

Thus g is lower semicontinuous at t.

The statement concerning the uniqueness is a consequence of Theorem 2. □

The following result shows that strictly midpoint-quasiaffine functions are either regular or very irregular. This result gives an insight into the irregularity results of Császár [2], [3] and Marcus [6].

Corollary 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonconstant function. If f is a strictly midpoint-quasiaffine and $\mathbb{Q}$-radially continuous function, then either f is monotone, or the restriction of f to any measurable set of positive Lebesgue measure, is not measurable.*

Proof. By Theorem 2, $f = g \circ \alpha$, where α is an additive function and g is strictly monotone. Assume that f is not monotone, then α cannot be of the form $\alpha(x) = cx$. Hence α is noncontinuous additive function. Taking the inverse γ of g, we have $\alpha = \gamma \circ f$. Therefore, if f is measurable on a set of positive Lebesgue measure, then α is also measurable and, therefore, it is continuous. The contradiction shows that f cannot be regular.

4. Quasi-additive functions

In some recent papers Tabor [11], [12] has introduced the notion of quasi-additive function. If X and Y are normed spaces, then a function $f : X \to Y$ is called quasi-additive if there exists $0 \le \varepsilon < 1$ such that

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon \min\{\|f(x+y)\|, \|f(x) + f(y)\|\} \quad \text{for} \quad x, y \in X. \tag{19}$$

The main results of the papers [11], [12], [1] show that quasi-additive functions have regularity properties very similar to that of additive functions.

Our next result gives an explanation for this fact by showing that real-valued quasi-additive functions on X can always be decomposed into the form $g \circ \alpha$, where g is a continuous quasi-additive function and α is an additive function.

Theorem 5. *A function $f : X \to \mathbb{R}$ is quasi-additive if and only if there exist an additive function $\alpha : X \to \mathbb{R}$ and a continuous, strictly increasing quasi-additive function $g : \mathbb{R} \to \mathbb{R}$ such that $f = g \circ \alpha$.*

Proof. It is easy to see the sufficiency of the condition. It remains to prove its necessity.

Assume that $f : X \to \mathbb{R}$ is quasi-additive. We may assume that f is non identically zero. Then it is nonconstant (since $f(0) = 0$). It follows from (19) that, for all $x, y \in X$,

$$f(y) - \varepsilon|f(y)| \le f(x+y) - f(x) \le f(y) + \varepsilon|f(y)|. \tag{20}$$

Since $0 \leq \varepsilon < 1$ hence the sign of both sides coincide with that of $f(y)$, that is $\operatorname{sign} f(y) = \operatorname{sign}(f(x+y) - f(x))$. Hence

$$\operatorname{sign}\left(f\left(\frac{x+y}{2}\right) - f(x)\right) = \operatorname{sign} f\left(\frac{y-x}{2}\right) = \operatorname{sign}\left(f(y) - f\left(\frac{x+y}{2}\right)\right).$$

Therefore, f is strictly midpoint-quasiaffine. (Cf. Baran [1, Lemma 1].)

It follows from (20) that

$$|f(x+y) - f(x)| \leq (1+\varepsilon)|f(y)| \quad (x, y \in X).$$

On the other hand, by [12, Lemma 3], we also have

$$\left|f\left(\frac{z}{2^n}\right)\right| \leq \left(\frac{1+\varepsilon}{2}\right)^n |f(z)| \quad (z \in X,\, n \in \mathbb{N}).$$

Thus, combining these two inequalities,

$$\left|f\left(x + \frac{z}{2^n}\right) - f(x)\right| \leq 2\left(\frac{1+\varepsilon}{2}\right)^{n+1} |f(z)| \quad (x, z \in X,\, n \in \mathbb{N}).$$

It follows from this inequality that f satisfies the following radial continuity property

$$\lim_{n\to\infty} f\left(\left(1 - \frac{1}{2^n}\right)x + \frac{1}{2^n}y\right) = f(x) \quad (x, y \in X).$$

This, together with the $\mathbb{Q}$-quasiaffinity of f, means that f is $\mathbb{Q}$-radially continuous. Therefore, we can apply Theorem 4 to obtain that $f = g \circ \alpha$, where g is an upper semicontinuous strictly monotone function, α is an additive function and g is continuous on the range of α.

Substituting this form of f into (19), we obtain that

$$\|g(s+t) - g(s) - g(t)\| \leq \varepsilon \min\{\|g(s+t)\|, \|g(s) + g(t)\|\} \tag{21}$$

for all s, t form the range of α. The range of α is dense in $\mathbb{R}$, hence, for all $s, t \in \mathbb{R}$, we can find decreasing sequences $s_n \to s$ and $t_n \to t$ in the range of α. Substituting s_n, t_n into (21), taking the limit $n \to \infty$ and using the upper semicontinuity (which is equivalent to the right continuity) of g, we obtain that (21) is also valid for all $s, t \in \mathbb{R}$. That is, g is quasi-additive on $\mathbb{R}$.

The function g is continuous at zero (since zero is in the range of α). Therefore, by [12, Theorem 1] it is continuous everywhere. Thus the proof of the theorem is complete. $\square$

It follows from this result, exactly in the same way as Corollary 1 in Section 3, that noncontinuous quasi-additive functions have irregularity properties as noncontinuous additive functions (see the results in [1, 11, 12, 13]).

5. Jensen-convex functions

Let X be a linear space and D be a convex subset of X. A function $f : D \to \mathbb{R}$ is said to be a *Jensen-convex function* (or a *midpoint-convex function*) if it satisfies the Jensen functional inequality

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} \quad (x, y \in D).$$

It is obvious that convex functions are Jensen-convex functions, moreover, if g is a convex function and α is an additive function, then $f = g \circ \alpha$ is Jensen-convex. However, the converse of this statement is not valid without any further assumptions. (Cf. [5, Example V.3.2, p. 127].) In our next result we show that this is the case if $D = X$ and, in addition, f is *strictly midpoint-quasiconcave*, that is

$$\min(f(x), f(y)) \le f\left(\frac{x+y}{2}\right)$$

for all $x, y \in X$ and if $f(x) \ne f(y)$ then

$$\min(f(x), f(y)) < f\left(\frac{x+y}{2}\right).$$

Theorem 6. *Let $f : X \to \mathbb{R}$. Then f is a strictly midpoint-quasiconcave and Jensen-convex function if and only if there exist a strictly increasing continuous convex function $g : R \to \mathbb{R}$ and an additive function $\alpha : X \to \mathbb{R}$ such that $f = g \circ \alpha$.*

Proof. The sufficiency of the condition is obvious. To prove the necessity, assume that f is a strictly midpoint-quasiconcave and Jensen-convex function. The Jensen-convexity of f implies that f is also strictly midpoint-quasiconvex, since

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} \le \max(f(x), f(y)) \quad (x, y \in X)$$

and there is strict inequality at the second place if $f(x) \ne f(y)$.

On the other hand, the Jensen-convexity also yields that

$$f\left(rx + (1-r)y\right) \le rf(x) + (1-r)f(y)$$

for all $x, y \in X$ and rational number $r \in [0,1]$. Taking the limsup with respect $r \to 0$, we obtain

$$\limsup_{\substack{r \to 0^+ \\ r \in \mathbb{Q}}} f\left(rx + (1-r)y\right) \le f(y),$$

that is, f is $\mathbb{Q}$-radially upper semicontinuous.

Now, we can apply Theorem 2. Therefore, there exist a strictly increasing upper semicontinuous g and an additive α such that $f = g \circ \alpha$. Substituting this representation of f into the Jensen-convexity inequality, we obtain that

$$g\left(\frac{s+t}{2}\right) \le \frac{g(s)+g(t)}{2}$$

for all s, t in the range of α. Using the same argument as in the proof Theorem 5, we obtain that this inequality is valid for all $s, t \in \mathbb{R}$. Therefore, g is an upper semicontinuous Jensen-convex function. Then, by the Berstein–Doetsch theorem [5, Theorem VI.4.2, p. 145], it follows that g is continuous, which completes the proof. $\square$

In our next result we consider functions that satisfy a stronger inequality than that of the midpoint-quasiaffinity. Namely, we replace the means min and max by quasiarithmetic means.

Let $I \subset \mathbb{R}$ be an open interval and $\phi, \psi : I \to \mathbb{R}$ are continuous strictly increasing functions. We consider functions $f : X \to I$ satisfying

$$\phi^{-1}\left(\frac{\phi(f(x))+\phi(f(y))}{2}\right) \le f\left(\frac{x+y}{2}\right) \le \psi^{-1}\left(\frac{\psi(f(x))+\psi(f(y))}{2}\right) \tag{23}$$

$$(x, y \in X).$$

Theorem 7. *Let ϕ and ψ as above. A function $f : X \to I$ satisfies (23) if and only if there exists an additive function $\alpha : X \to \mathbb{R}$ and a strictly increasing and continuous function $g : \mathbb{R} \to I$ such that $f = g \circ \alpha$, furthermore, $\phi \circ g$ is concave and $\psi \circ g$ is convex.*

Proof. The sufficiency of the stated conditions is immediate. It remains to prove the necessity. It follows from (23) that f is strictly midpoint-quasiaffine. Furthermore, (23) also implies that $\phi \circ f$ is Jensen-concave and $\psi \circ f$ is Jensen-convex. Hence, for all rational $r \in [0, 1]$, for all $x, y \in X$,

$$\begin{aligned}\phi^{-1}\left(r\phi(f(x)) + (1-r)\phi(f(y))\right) &\leq f\left(rx + (1-r)y\right)\\ &\leq \psi^{-1}\left(r\psi(f(x)) + (1-r)\psi(f(y))\right)\end{aligned}$$

Therefore, f is $\mathbb{Q}$-radially continuous. The proof now can be completed exactly in the same way as that of Theorem 6. □

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KAZIMIERZ NIKODEM
DEPARTMENT OF MATHEMATICS
TECHNICAL UNIVERSITY OF ŁÓDŹ
BRANCH IN BIELSKO–BIAŁA
UL. WILLOWA 2
PL 43–309 BIELSKO–BIAŁA
POLAND

E-mail: knik@merc.pb.bielsko.pl

ZSOLT PÁLES
INSTITUTE OF MATHEMATICS AND INFORMATICS
LAJOS KOSSUTH UNIVERSITY
H–4010 DEBRECEN, P.O. BOX 12
HUNGARY

E-mail: pales@math.klte.hu

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