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Characterizing polynomial functions by a mean value property

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Dedicated to Professors Zoltán Daróczy and Imre Kátai on their 60th birthday

Abstract. We generalize Flett's Mean Value Theorem to the case of functions defined in normed spaces. This is a motivation for considering functional equations related to the Flett mean value formula in a quite general setting. We solve them in the case where functions are defined in abelian groups and take values in a rational linear space.

1. Introduction

In [6] Sahoo and Riedel gave a generalization of Flett's Mean Value Theorem [2] as follows:

Theorem 1.1. *Let f be a real valued function differentiable in $[a,b]$, then there is a point $c \in (a,b)$ such that*

$$f(c) - f(a) = (c-a)f'(c) - \frac{1}{2}\frac{f'(b)-f'(a)}{b-a}(c-a)^2. \tag{1}$$

It is easy to see that if $f'(b) = f'(a)$, then this reduces to Flett's Mean Value Theorem.

In [4], following the approach of Aczél [1] and Haruki [3] who solved functional equations related to the Lagrange Mean Value Theorem, we

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replaced in (1) the derivative of f by an unknown function h and so we obtained the following functional equation:

$$f(c)-f(a)=(c-a)h(c)-\frac{1}{2}\frac{h(b)-h(a)}{b-a}(c-a)^2. \tag{2}$$

It turns out that specifying c to be equal to $\frac{a+3b}{4}$ implies that (2) characterizes cubic polynomials. More exactly, we obtained in [4] the following.

Theorem 1.2. *Let $t\in(0,1)$ be fixed. The pair (f,h) of real valued functions defined in $\mathcal{R}$ satisfies the equation*

$$\begin{aligned} f\left((1-t)a+tb\right)-f(a)=t(b-a)&h((1-t)a+tb)\\ &-\frac{1}{2}\frac{h(b)-h(a)}{b-a}(t(b-a))^2,\end{aligned} \tag{3}$$

for all $a,b\in\mathcal{R}$ if and only if

$$f(x)=\begin{cases} Ax^3+Bx^2+Cx+D & \text{if } t=\frac{3}{4}\\ Bx^2+Cx+D & \text{if } t\neq\frac{3}{4}\end{cases} \tag{4}$$

$$h(x)=\begin{cases} 3Ax^2+2Bx+C & \text{if } t=\frac{3}{4}\\ 2Bx+C & \text{if } t\neq\frac{3}{4}.\end{cases} \tag{5}$$

Let us note that solutions (4) and (5) are highly regular despite the fact that we do not assume any regularity *a priori*. An explanation to this effect might be given by the following argument. Let us first observe that the equation (3) may be written equivalently (with new variables $x=a$ and $y=b-a$) in the form

$$f\left(x+ty\right)-f(x)=\left[h(x+ty)-\frac{t}{2}\left(h(x+y)-h(x)\right)\right](ty). \tag{6}$$

Now the problem of solving (6) may be properly asked in the case of functions $f:G\to H$ and $h:G\to L(G,H)$, where G and H are real linear spaces and $L(G,H)$ denotes the space of all linear mappings from G into H. This setting is quite a natural extension of our original problem since it comes from Flett's equality where f' is replaced by h, and so in higher dimensional normed spaces one would expect h to be a differential, hence a linear mapping. Looking closer, one can even ask for solutions of (6) in the case where G is a semigroup where multiplication by t is defined (e.g. if $t = \frac{3}{4}$ and G is uniquely divisible by 2), H is a group with the same property (and uniquely divisible by 2), and h maps G into $\mathrm{Hom}(G, H)$. Taking $G = H = \mathcal{R}$, with the usual addition, we see that in fact *there is a regularity condition* implicitly assumed when solving (3). Namely, the form of (3) (or equivalently of (6)), with multiplication by ty on the righthand side tacitly imposes that we are looking for *linear*, and hence *continuous* homomorphisms h. We show below that dropping this kind of regularity we get generalized polynomial functions as solutions to (6), which turn out to be ordinary polynomials under rather slight regularity assumptions.

To strengthen our argument and dismiss the doubt that our generalization has not very much in common with Theorem 1.1 stated for functions defined in reals, let us extend this result to the multidimensional case. We have the following.

Theorem 1.3. *Let D be an open subset of a normed space G and let $a, b \in D$ be such that the segment $I := \{a + tb : t \in [0,1]\}$ is contained in D. Further, let $f : D \to \mathcal{R}$ be a function differentiable at every point of I. Then there is a point $c = a + t_0(b - a) \in I \setminus \{a, b\}$ such that*

$$f(c) - f(a) = \left[d_c f - \frac{t_0}{2}\left(d_b f - d_a f\right)\right](c - a)\,. \tag{7}$$

Proof. Put $h = b - a$ and define $F : [0,1] \to \mathcal{R}$ by

$$F(t) = f(a + th).$$

In view of our assumptions F is differentiable at every point of $[0,1]$. Moreover, as it is easy to calculate

$$F'(t) = d_{a+th} f(h) \tag{8}$$

for every $t \in [0,1]$. By Theorem 1.1 we infer that there is a $t_0 \in (0,1)$ such that

$$F(t_0) - F(0) = t_0 F'(t_0) - \frac{1}{2}\left(F'(1) - F'(0)\right) t_0^2$$

whence we immediately get (7) putting $c = a + t_0 h$ and using (8).

2. Functional equation

Before we proceed to solve the equation (6) let us prove a lemma which will be the main tool in the proof. The lemma is a "homomorphic" version of a result by L. Székelyhidi [7, Theorem 9.5], see also W. H. Wilson [8].

Lemma 2.1. *Let G and H be abelian groups, denote by $X = \mathrm{Hom}(G,H)$ the space of group homomorphisms from G to H and suppose that for fixed $n \in \mathbb{N}$ the functions $h, h_1, \dots, h_n : G \longrightarrow X$ satisfy*

$$\left[h(x) + \sum_{i=1}^{n} h_i(\alpha_i(x) + \beta_i(y))\right](y) = 0 \tag{9}$$

for every $x, y \in G$, where $\alpha_i, \beta_i \in \mathrm{Hom}(G,G)$ and $\alpha_i(G) \subset \beta_i(G)$ for $i = 1, \dots, n$ then $\Delta_y^{2n} h(x) = 0$, for every $x, y \in G$.

Proof. We proceed by induction. If $n = 1$ then (9) takes the form

$$[h(x) + h_1(\alpha_1(x) + \beta_1(y))](y) = 0, \quad \text{for all } x, y \in G. \tag{10}$$

Fixing $u \in G$ arbitrarily, letting v be such that $\alpha_1(u) + \beta_1(v) = 0$, and replacing x by $x + u$ and y by $y + v$ in (10), we get

$$\begin{aligned} &[h(x+u) + h_1(\alpha_1(x) + \beta_1(y))](y) \\ &= -[h(x+u) + h_1(\alpha_1(x) + \beta_1(y))](v). \end{aligned} \tag{11}$$

Subtracting (11) from (10) gives

$$[h(x) - h(x+u)](y) = [h(x+u) + h_1(\alpha_1(x) + \beta_1(y))](v). \tag{12}$$

Again replacing x by $x + u$ and also y by $y + v$ in (12), we get

$$[h(x+u) - h(x+2u)](y+v) = [h(x+2u) + h_1(\alpha_1(x) + \beta_1(y))](v),$$

whence

$$\begin{aligned} &[h(x+u) - h(x+2u)](y) \\ &= [2h(x+2u) - h(x+u) + h_1(\alpha_1(x) + \beta_1(y))](v). \end{aligned} \tag{13}$$

Subtracting (12) from (13) yields

$$\begin{aligned} &-\left[h(x+2u)-2h(x+u)+h(x))\right](y) \\ &\quad = 2\left[h(x+2u)-h(x+u)\right](v). \end{aligned} \tag{14}$$

The lefthand side of (14) does not depend on v, therefore it has to vanish (to see this put $v=0$ in the righthand side). Thus we get that for every $x,u\in G$

$$h(x+2u)-2h(x+u)+h(x)=\Delta_u^2 h(x)=0. \tag{15}$$

This establishes the base case for the induction. Now suppose that our assertion holds for some natural $n\geq 1$ and suppose that $h,h_1,\dots,h_{n+1}: G\longrightarrow X$ satisfy

$$\left[h(x)+\sum_{i=1}^{n+1} h_i(\alpha_i(x)+\beta_i(y))\right](y)=0, \tag{16}$$

for all $x,y\in G$, where $\alpha_i,\beta_i\in X$ and $\alpha_i(G)\subset\beta_i(G)$ for $i=1,\dots,n+1$. Let $u\in G$ be arbitrary and choose $v\in G$ such that $\alpha_{n+1}(u)+\beta_{n+1}(v)=0$. Substituting $x+u$ for x and $y+v$ for y in (16), we get

$$\begin{aligned} &\Bigg[h(x+u)+\sum_{i=1}^{n} h_i(\alpha_i(x)+\beta_i(y)+\alpha_i(u)+\beta_i(v)) \\ &\qquad +h_{n+1}(\alpha_{n+1}(x)+\beta_{n+1})(y)\Bigg](y+v)=0. \end{aligned} \tag{17}$$

Comparing (16) and (17) yields

$$\begin{aligned} &\left[\Delta_u h(x)+\sum_{i=1}^{n}\Delta_{\alpha_i(u)+\beta_i(v)} h_i(\alpha_i(x)+\beta_i(y))\right](y) \\ &= -\Bigg[h(x+u)+\sum_{i=1}^{n} h_i(\alpha_i(x)+\beta_i(y)+\alpha_i(u)+\beta_i(v)) \\ &\qquad +h_{n+1}(\alpha_{n+1}(x)+\beta_{n+1}(y))\Bigg](v). \end{aligned} \tag{18}$$

Again, substitute in (18) $x+u$ for x and $y+v$ for y to get

$$\begin{aligned}\text{(19)}\qquad &\Bigg[\Delta_u h(x+u) + \sum_{i=1}^{n} \Delta_{\alpha_i(u)+\beta_i(v)} h_i(\alpha_i(x) \\ &\qquad +\beta_i(y)+\alpha_i(u)+\beta_i(v))\Bigg](y+v) \\ &= -\Bigg[h(x+u) + \sum_{i=1}^{n} h_i(\alpha_i(x)+\beta_i(y)+\alpha_i(u)+\beta_i(v)) \\ &\qquad +h_{n+1}(\alpha_{n+1}(x)+\beta_{n+1}(y))\Bigg](v).\end{aligned}$$

Subtracting (18) from (19) we get

$$\text{(20)}\qquad \left[\Delta_u^2 h(x) + \sum_{i=1}^{n} \Delta^2_{\alpha_i(u)+\beta_i(v)} h_i(\alpha_i(x)+\beta_i(y))\right](y) = H(x,u,v)(v),$$

where $H(x,u,v) \in X$ is a suitable homomorphism. The righthand side of (20) does not depend on y and repeating the argument in the base case, we get

$$\text{(21)}\qquad \left[\Delta_u^2 h(x) + \sum_{i=1}^{n} \Delta^2_{\alpha_i(u)+\beta_i(v)} h_i(\alpha_i(x)+\beta_i(y))\right](y) = 0$$

for all $x, y, u \in G$. Using the induction hypothesis we conclude that

$$\text{(22)}\qquad 0 = \Delta_u^{2n}\Delta_u^2 h(x) = \Delta_u^{2(n+1)} h(x),$$

which finishes the proof.

We will also need the following technical lemma.

Lemma 2.2. *Let K be an abelian group, and assume that H is a rational linear space. Let $N \in \mathbb{N}$ be fixed and let $1 \le k_1 < k_2 < \cdots < k_N$ be positive integers. If functions $B_i : K \to H$ satisfy*

$$B_i(2z) = 2^{k_i} B_i(z)$$

for every $i \in \{1, \ldots, N\}$ *then*

$$\sum_{i=1}^{N} B_i(z) = 0$$

if and only if $B_i = 0$ *for every* $i \in \{1, \ldots, N\}$.

Proof. Induction.

In the sequel we admit the following hypothesis.

(A) $(G, +)$ is an abelian group, $(H, +)$ is a rational linear space and $t \in \mathbb{Q} \cap (0, 1)$ is such that the mappings $G \ni x \to tx \in G$, $G \ni x \to (2-t)x \in G$, $G \ni x \to (1-t)x \in G$, are automorphisms.

Remark 2.3. Let us note that if $t = \frac{p}{q}$, $p, q \in \mathbb{N}$, $p < q$, $(p, q) = 1$, then assumptions on G appearing in the above hypothesis (A) are equivalent to say that G is uniquely divisible by $p, q, 2q - p$ and $q - p$. For instance, if $t = \frac{3}{4}$ then (A) is satisfied by any abelian group G, uniquely divisible by 3, 4 and 5, or simply by 5!. Of course, (A) is satisfied if we assume that G is a rational linear space, but admitting only the latter case would certainly limit the results that follow.

Now, let us prove the main result for equation (6).

Theorem 2.4. *Let the hypothesis (A) be satisfied and denote by* X *the set* $\mathrm{Hom}(G, H)$ *of all homomorphisms mapping* G *into* H. *The functions* $f : G \to H$ *and* $h : G \to X$ *are solutions of the functional equation* (6) *if and only if*

$$f(x) = \begin{cases} A(x,x)(x) + B(x)(x) + C(x) + D & \text{if } t = \frac{3}{4}, \\ B(x,x) + C(x) + D & \text{if } t \neq \frac{3}{4} \end{cases} \tag{23}$$

$$h(x) = \begin{cases} 3A(x,x) + 2B(x) + C & \text{if } t = \frac{3}{4}, \\ 2B(x) + C & \text{if } t \neq \frac{3}{4} \end{cases} \tag{24}$$

where $A : G \times G \to X$ *is a biadditive symmetric function such that the function*

$$G \times G \times G \ni (x, y, z) \to A(x, y)(z) \in H \tag{25}$$

is symmetric, $B : G \to X$ *is additive and such that the function*

$$G \times G \ni (x, y) \to B(x)(y) \in H \tag{26}$$

is symmetric, $C \in X$ and $D \in H$ are constants.

Proof. Since multiplication by t is an automorphism in G and in H (and hence also in X) we get that any additive function from G into H or into X is t-homogeneous. Using this fact and symmetry of mappings (25) and (26) it is easy to check that the functions f, h given by (23) and (24) do satisfy the functional equation (6).

To show that these are the only solutions, substitute in (6) $x-ty$ for x to get

$$f(x) - f(x-ty) = \left[h(x) - \frac{t}{2}\left(h(x+(1-t)y) - h(x-ty)\right)\right](ty).$$

Now, replace y by $-y$ in the above to obtain

$$f(x) - f(x+ty) = -\left[h(x) - \frac{t}{2}\left(h(x-(1-t)y) - h(x+ty)\right)\right](ty). \tag{27}$$

Adding (6) to (27) and multiplying both sides by 2 we infer that

$$\begin{aligned}\big[(2-t)h(x+ty) - (2-t)h(x) - th(x+y)\\ +th(x-(1-t)y)\big](ty) = 0.\end{aligned} \tag{E}$$

Now, substituting in the above $x+(1-t)y$ for x and dividing both sides by t we get

$$\left[h(x) - \frac{2-t}{t}h(x+(1-t)y) + \frac{2-t}{t}h(x+y) - h(x+(2-t)y)\right](y) = 0.$$

By Lemma 2.1 we see that h is a polynomial function of degree at most 5, i.e. h is of the form

$$h(x) = A_o + \sum_{i=1}^{5} A_i^*(x) \tag{28}$$

where $A_o \in X$ is a constant, $A_1^* : G \to X$ is additive, and A_i^*, $i = 2,3,4,5$ are diagonalizations of i-additive symmetric functions $A_i : G^i \to X$.

Setting $x = 0$ in (6), we get (cf. (A))

$$f(x) = \left[h(x) - \frac{t}{2}\left[h\left(\frac{x}{t}\right) - h(0)\right]\right](x) + D \tag{29}$$

or, taking into account t-homogeneity of additive mappings

$$f(x) = D + A_o(x) + \sum_{i=1}^{5} \left(1 - \frac{1}{2}t^{1-i}\right) A_i^*(x)(x). \tag{30}$$

where $D \in H$ is a constant. Substituting (28) and (30) into (6) we see that the obtained equality imposes no conditions on D and A_o. Further, denote for $i \in \{1, \ldots, 5\}$

$$\begin{aligned} B_i(x,y) = &\left(1 - \frac{1}{2t^{1-i}}\right)[A_i^*(x+ty)(x+ty) - A_i^*(x)(x)] \\ &- \left[A_i^*(x+ty) - \frac{t}{2}\left(A_i^*(x+y) - A_i^*(x)\right)\right](ty). \end{aligned}$$

Then we have for every $i \in \{1, \ldots, 5\}$

$$B_i(2x, 2y) = 2^{i+1} B_i(x,y)$$

and from (6) we derive

$$\sum_{i=1}^{5} B_i(x,y) = 0.$$

From Lemma 2.2 (with $K = G \times G$ and $z = (x,y)$) it follows that $B_i = 0$ for $i \in \{1, \ldots, 5\}$. In other words, we get the following equations

$$\begin{aligned} &\left(1 - \frac{1}{2t^{1-i}}\right)[A_i^*(x+ty)(x+ty) - A_i^*(x)(x)] \\ &= \left[A_i^*(x+ty) - \frac{t}{2}\left(A_i^*(x+y) - A_i^*(x)\right)\right](ty). \end{aligned} \tag{31}$$

for every $i \in \{1, \ldots, 5\}$. Elementary though tedious calculations, which we omit here, show that under our assumptions on t the mappings A_3, A_4 and A_5 have to vanish. For $i = 1$ the equation (31) reduces to

$$\frac{t}{2}\left(A_1^*(x)(y) + A_1^*(y)(x) + tA_1(y)(y)\right) = \frac{t}{2}\left(2A_1^*(x)(y) + tA_1(y)(y)\right)$$

whence it follows that

$$A_1^*(x)(y) = A_1^*(y)(x)$$

for every $x, y \in G$. Defining $B : G \to H$ by $B = \frac{1}{2}A_1^*$ we see that B satisfies the assertion.

For $i = 2$ we get from (31) the equation

$$\begin{aligned}&\left(1-\frac{1}{2t}\right)\left[t\left(A_2(x,x)(y)+2A_2(x,y)(x)\right)\right.\\&\left.+t^2\left(2A_2(x,y)(y)+A_2(y,y)(x)\right)+t^3A_2(y,y)(y)\right]\\&=tA_2(x,x)(y)+t^2A_2(x,y)(y)+\left(t^3-\frac{t^2}{2}\right)A_2(y,y)(y).\end{aligned}$$

Fixing y and using Lemma 2.2 we see that the above equation is equivalent to the following system.

$$2\,(2t-1)\,A_2(x,y)(x) = A_2(x,x)(y) \tag{32}$$

$$2\,(1-t)\,A_2(x,y)(y) = (2t-1)\,A_2(y,y)(x) \tag{33}$$

for every $x, y \in G$. Interchanging x and y in (33), comparing with (32) and using divisibility by rationals in H, we see that either

$$A_2(x,y)(x) = 0 \tag{34}$$

for every $x, y \in G$, or

$$(2t-1)^2 = t-1.$$

The latter equality holds only if $t = 0$ or $t = \frac{3}{4}$. In other cases (34) holds which in view of (32) and well known fact that a symmetric and biadditive function is uniquely determined by its values on the diagonal, implies $A_2 = 0$. We assumed that $t \in \mathbb{Q} \cap (0,1)$. It remains therefore to consider the case $t = \frac{3}{4}$. Then (32) becomes

$$A_2(x,x)(y) = A_2(x,y)(x).$$

Substitute $x + z$ for x in the above equality to get

$$2A(x,z)(y) = A_2(x,y)(z) + A_2(z,y)(x). \tag{35}$$

Interchanging y and z in (35) we obtain

$$2A(x,y)(z) = A_2(x,z)(y) + A_2(z,y)(x). \tag{36}$$

After substracting (36) from (35) we infer that

$$A_2(x,y)(z) = A_2(x,z)(y)$$

which in view of symmetry of A_2 easily implies symmetry of the mapping $G^3 \ni (x,y,z) \to A_2(x,y)(z) \in H$. To finish the proof it is enough to define $A : G \times G \to X$ by $A = \frac{1}{3}A_2$.

3. A remark on semigroups

The main tool in proving Theorem 2.3 was Lemma 2.1 which we proved in the case of mappings h defined in abelian groups. Similarly, Székelyhidi's Theorem 9.5 from [7] has also been proved in the case where the domain is an abelian group. We will show that in a particular case which is important because of the origin of our problem, it is enough to assume that the domain is a semigroup. Namely, if $t = \frac{3}{4}$, as in the original problem stemming from Flett's Theorem, and $H = \mathbb{C}$ then equation (E) becomes

$$5h\left(x + \frac{3}{4}y\right) - 5h\left(x\right) - 3h\left(x+y\right) + 3h\left(x - \frac{1}{4}y\right) = 0,$$

or, after introducing new variables $u = x - \frac{1}{4}y$ and $v = \frac{1}{4}$ and rearranging the terms

$$3h\left(u+5v\right) - 5h\left(u+4v\right) + 5h\left(u+v\right) - 3h\left(u\right) = 0. \tag{37}$$

Without using Székelyhidi's result we can prove the following.

Theorem 3.1. *Let $(G,+)$ be an abelian semigroup. A function $h : G \to \mathbb{C}$ is a solution of* (37) *if and only if*

$$\Delta_v^3 h(u) = 0.$$

Proof. The "if" is easy to check. Assume now that h solves (37). Let k and n be positive integers and substitute in (37) first kv instead of v and then $u + nv$ instead of u. We get

$$\begin{aligned} &3h\left(u+(n+5k)v\right) - 5h\left(u+(n+4k)v\right) \\ &+5h\left(u+(n+k)v\right) - 3h\left(u+nv\right) = 0, \end{aligned} \tag{38}$$

for every k, $n \in \mathbb{N}$ and $u, v \in G$. Fix u and v and define a function $\varphi : \mathbb{N} \cup \{0\} \to \mathbb{C}$ by

$$\varphi(r) = h(u + rv).$$

Let us note that φ depends on u and v. Because of (37), φ satisfies the following difference equation

$$3\varphi(n + 5k) - 5\varphi(n + 4k) + 5\varphi(n + k) - 3\varphi(n) = 0, \tag{39}$$

for every $k, n \in \mathbb{N}$. Let $k = 1$. Then from (37) we derive

$$3\varphi(n + 5) - 5\varphi(n + 4) + 5\varphi(n + 1) - 3\varphi(n) = 0. \tag{40}$$

The characteristic equation for (40) has 1 as a triple root and the remaining roots are $\lambda = -\frac{2}{3} + i\frac{\sqrt{5}}{3}$ and λ^{-1}. Thus φ has the following form

$$\varphi(r) = a + br + cr^2 + d\lambda^r + e\lambda^{-r}, \tag{41}$$

where a, b, c, d and e are some complex constants, depending on u and v. Substituting (41) into (39) we easily get

$$d\lambda^{2n} z_k + e w_k = 0 \tag{42}$$

where $z_k = 3\left(\lambda^k\right)^5 - 5\left(\lambda^k\right)^4 + 5\left(\lambda^k\right) - 3$ and $w_k = 3\left(\lambda^{-k}\right)^5 - 5\left(\lambda^{-k}\right)^4 + 5\left(\lambda^{-k}\right) - 3$. We can choose a $k \in \mathbb{N}$ so that $z_k \neq 0 \neq w_k$. Now, letting n be equal 1 and 2, we easily derive from (42) that $d = e = 0$. Thus, finally, φ has to be of the form

$$\varphi(r) = a + br + cr^2$$

whence we get

$$h(u + rv) = a(u, v) + b(u, v)r + c(u, v)r^2$$

for every $u, v \in G$ and $r \in \mathbb{N} \cup \{0\}$, where a, b, c are some functions mapping $G \times G$ into $\mathbb{C}$. It can be immediately checked that h satisfies

$$h(u + 3v) - 3h(u + 2v) + 3h(u + v) - h(u) = 0$$

which ends the proof.

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