*Publ. Math. Debrecen*
**52 / 3-4** (1998), 635–657

# Remarks on Walsh-Fourier multipliers

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*Dedicated to the 60th birthday of*
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**Abstract.** We consider special multiplier operators for one- and two-parameter Walsh-Paley system. These multipliers were defined and partly investigated in earlier papers SIMON [2], [3]. Thus e.g. their boundedness from the dyadic Hardy space $H^p$ into $L^p$ for some $p$ were shown. In the present work several consequences of our previous investigations will be proved. Furthermore, we make additional remarks and formulate some problems with respect to this area.

**1.** First of all the most important concepts, notations of the Walsh-Fourier analysis and auxiliary results will be formulated. (For details see the book SCHIPP–WADE–SIMON [1].) The one-dimensional Walsh(–Paley) system will be denoted by $w_n$ $(n = 0, 1, \dots)$. That is,

$$w_n := \prod_{k=0}^{\infty} r_k^{n_k},$$

where $r_k$ $(k = 0, 1, \dots)$ is the $k$-th Rademacher function and
$n = \sum_{k=0}^{\infty} n_k 2^k$ $(n_k = 0, 1; k = 0, 1, \dots)$ is the binary decomposition of $n$. If $f \in L^1[0, 1)$ let $\hat{f}$ be the sequence of the Walsh-Fourier coefficients $\hat{f}(k)$

*Mathematics Subject Classification*: Primary 42C10, 43A22; Secondary 47B38, 60G46.
*Key words and phrases*: martingales and Hardy spaces, $p$-atoms, Walsh functions, quasi-local operators, multipliers.
This research was supported by the Hungarian Scientific Research Funds (OTKA) T020497.

$(k = 0, 1, \dots)$ of $f$. We denote by $S_n f$ the $n$-th partial sum of the Walsh-Fourier series $\sum_{k=0}^{\infty} \hat{f}(k) w_k$, i.e. $S_n f := \sum_{k=0}^{n-1} \hat{f}(k) w_k$ $(n = 1, 2, \dots)$. The $n$-th $(C, 1)$-mean $\sigma_n f$ of $\sum_{k=0}^{\infty} \hat{f}(k) w_k$ is defined by $\sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f$ $(n = 1, 2, \dots)$. The functions $D_n := \sum_{k=0}^{n-1} w_k$, $K_n := \frac{1}{n} \sum_{k=1}^{n} D_k$ $(n = 1, 2, \dots)$ are the exact analogues of the well-known (trigonometric) kernel functions of Dirichlet's and Fejér's type, respectively. These functions have some good properties, useful also in the following investigations. First we mention a simple result with respect to Dirichlet kernels, which plays a central role in the Walsh-Fourier analysis:

$$D_{2^n}(x) = \begin{cases} 2^n & (0 \le x < 2^{-n}) \\ 0 & (2^{-n} \le x < 1) \end{cases} \qquad (n = 0, 1, \dots). \tag{1}$$

Furthermore, it is not hard to see that

$$\sum_{k=0}^{n-1} k w_k = n(D_n - K_n) \qquad (n = 1, 2, \dots). \tag{2}$$

Let $x \dot{+} y$ be the so-called dyadic sum of $x, y \in [0, 1)$ then the next relations hold for all $x \in [0, 1)$ and $s = 0, 1, \dots$:

$$0 \le K_{2^s}(x) = \frac{1}{2}\Big(2^{-s} D_{2^s}(x) + \sum_{l=0}^{s} 2^{l-s} D_{2^s}(x \dot{+} 2^{-l-1})\Big), \tag{3}$$

$$|K_l(x)| \le \sum_{t=0}^{s} 2^{t-s-1} \sum_{i=t}^{s} \big(D_{2^i}(x) + D_{2^i}(x \dot{+} 2^{-t-1})\big) \qquad (2^s \le l < 2^{s+1}). \tag{4}$$

Finally, we will use the identity

$$\sum_{k=2^s}^{\infty} \frac{w_k}{k} = \sum_{l=2^s+1}^{\infty} K_l \left(\frac{1}{l-1} - \frac{1}{l+1}\right) - \frac{K_{2^s}}{2^s+1} - \frac{D_{2^s}}{2^s} \qquad (s = 0, 1, \dots). \tag{5}$$

The Kronecker product $w_{n,m}$ $(n, m = 0, 1, \dots)$ of two Walsh systems is said to be the two-dimensional (or two-parameter) Walsh system. Thus $w_{n,m}(x, y) := w_n(x)w_m(y)$ $(x, y \in [0, 1))$. For the two-parameter Walsh-Fourier coefficients of a function $f \in L^1[0, 1)^2$ the same notations will be used as in the one-dimensional case. That is, let

$$\hat{f}(n, m) := \int_0^1 \int_0^1 f(x, y)w_{n,m}(x, y)\, dx\, dy \qquad (n, m = 0, 1, \dots)$$

and $\hat{f} := (\hat{f}(n, m); n, m = 0, 1, \dots)$. Furthermore, let

$$S_{n,m}f := \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \hat{f}(k, l)w_{k,l} \qquad (n, m = 1, 2, \dots)$$

be the $(n, m)$-th (rectangular) partial sum of the two-parameter Walsh-Fourier series $\sum_{k,l=0}^{\infty,\infty} \hat{f}(k, l)w_{k,l}$ of $f \in L^1[0, 1)^2$. It is easy to show that

$$S_{n,m}f(x, y) = \int_0^1 \int_0^1 f(t, u)D_n(x\dot{+}t)D_m(y\dot{+}u)\, dt\, du \qquad (x, y \in [0, 1)).$$

In the special case $n = 2^k, m = 2^l$ $(k, l = 0, 1, \dots)$ we have by (1)

$$S_{2^k,2^l}f(x, y) = 2^{k+l} \int_{I(x,y)} f \qquad (x, y \in [0, 1)),$$

where the dyadic rectangle $I(x, y)$ is defined by the Descartes product

$$I_{k,l}(x, y) := I_k(x) \times I_l(y).$$

Here $I_j(z)$ $(j = 0, 1, \dots; z \in [0, 1))$ stands for the (unique) dyadic interval

$$I_j(z) := [\nu 2^{-j}, (\nu + 1)2^{-j}) \qquad (\nu = 0, \dots, 2^j - 1)$$

containing $z$. If $k = l$ then $I_{k,k}(x, y)$ is a so-called dyadic square.

**2.** The Hardy spaces play important role in the Fourier analysis, especially in the theory of Walsh-Fourier series. The dyadic analogues of them can be defined as follows. (For details see the book of Weisz [4].) Let $n = 0, 1, \dots$ and denote $\mathcal{F}_n$ the $\sigma$-algebra generated by the dyadic intervals $[k2^{-n}, (k + 1)2^{-n})$ $(k = 0, 1, \dots, 2^n - 1)$. Obviously, the sequence

$\mathcal{F} := (\mathcal{F}_n, n = 0, 1, \dots)$ of $\sigma$-algebras is non-decreasing, i.e. $\mathcal{F}_n \subset \mathcal{F}_m$ if $n < m$. If $f$ is an integrable real function defined on $[0, 1)$ then $S_{2^n} f$ is the conditional expectation of $f$ relative to $\mathcal{F}_n$ $(n = 0, 1, \dots)$.

We are going to consider (dyadic) martingales with respect to $\mathcal{F}$. A sequence $f = (f_n, n = 0, 1, \dots)$ of integrable functions is said to be a martingale, if each $f_n$ is $\mathcal{F}_n$ measurable and $S_{2^n} f_m = f_n$ for all $n \leq m$ $(n, m = 0, 1, \dots)$. It is clear that for every $f \in L^1$ the sequence $(S_{2^n} f, n = 0, 1, \dots)$ is a martingale (called martingale obtained from $f$ and denoted likewise by $f$). Furthermore, the definition of $\hat{f}(k)$ $(k = 0, 1, \dots)$ can be extended to a martingale $f$ in a usual way. Consequently the Walsh-Fourier coefficients of $f \in L^1$ are the same as those of the martingale obtained from $f$.

We say that a martingale $f = (f_n, n = 0, 1, \dots)$ is $L^p$-bounded for some $0 < p < \infty$ if

$$\|f\|_p := \sup_n \|f_n\|_p < \infty.$$

(The symbol $\|f_n\|_p$ denotes the usual $p$-norm or quasi-norm of $f_n$.) It is well-known that for $1 < p < \infty$ the assumption $\|f\|_p < \infty$ is equivalent to the existence of a real function from the space $L^p[0, 1)$, from which $f$ is obtained. For all $0 < p < \infty$ $L^p$ will denote the set of the $L^p$-bounded martingales. Hence, if $1 < p < \infty$ then $L^p$ and $L^p[0, 1)$ can be identified.

Hardy spaces can be defined in various manner. To this end let the maximal function and the quadratic variation of a martingale $f = (f_n, n = 0, 1, \dots)$ be denoted by

$$f^* := \sup_n |f_n| \qquad \text{and} \qquad Qf := \Big(\sum_{n=0}^{\infty} |f_n - f_{n-1}|^2\Big)^{1/2},$$

respectively, (where $f_{-1} := 0$). In particular, for $f \in L^1$ the maximal function can also be given by

$$f^*(x) = \sup_n 2^n \Big| \int_{I_n(x)} f \Big| = \sup_n |S_{2^n} f(x)| \qquad (x \in [0, 1)).$$

Furthermore, the quadratic variation of a martingale obtained from $f \in L^1[0, 1)$ is not another as

$$Qf := \Big(|\hat{f}(0)|^2 + \sum_{n=1}^{\infty} |S_{2^n} f - S_{2^{n-1}} f|^2\Big)^{1/2}.$$

We introduce the martingale Hardy spaces for $0 < p < \infty$ as follows: denote $H^p$ the space of martingales $f$ for which $\|f\|_{H^p} := \|f^*\|_p < \infty$. It is well-known that the following equivalences hold:

$$c_p\|f\|_{H^p} \le \|Qf\|_p \le C_p\|f\|_{H^p} \qquad (0 < p < \infty) \tag{6}$$

and $c_p\|f\|_{H^p} \le \|f\|_p \le C_p\|f\|_{H^p}$ $(1 < p < \infty)$, where $f \in H^p$. (Here and later $c_p, C_p$ will denote positive constants depending only on $p$ although not always the same in different occurences.)

The atomic decomposition is a useful characterization of some Hardy spaces. To demonstrate this we give first the concept of atoms: let $0 < p \le 1$, then a function $a \in L^\infty[0,1)$ is called a $p$-atom if either $a$ is identically equal to 1 or there exists a dyadic interval $I$ for which

$$\operatorname{supp} a \subset I, \quad \|a\|_\infty \le |I|^{-1/p} \quad \text{and} \quad \int_0^1 a = 0. \tag{7}$$

We shall say that $a$ is supported on $I$. Then a martingale $f = (f_n,\ n = 0,1,\dots)$ belongs to $H^p$ for $0 < p \le 1$ if and only if there exist a sequence $(a_k,\ k = 0,1,\dots)$ of $p$-atoms and a sequence $(\mu_k,\ k = 0,1,\dots)$ of real numbers such that $\sum_{k=0}^\infty |\mu_k|^p < \infty$ and

$$f = \sum_{k=0}^\infty \mu_k a_k. \tag{8}$$

Moreover, the following equivalence of norms holds:

$$c_p\|f\|_{H^p} \le \inf\Big(\sum_{k=0}^\infty |\mu_k|^p\Big)^{1/p} \le C_p\|f\|_{H^p} \qquad (f \in H^p),$$

where the infimum is taken over all decompositions of $f$ of the form (8).

In the two-dimensional case the above concepts will be defined as follows. Let $\mathcal{F}_{n,m}$ $(n,m = 0,1,\dots)$ be the $\sigma$-algebra generated by the dyadic rectangles $I_{n,m}(x,y)$ $(x,y \in [0,1))$. Hence,

$$\mathcal{F}_{n,m} := \sigma\big(\{[k2^{-n},(k+1)2^{-n}) \times [l2^{-m},(l+1)2^{-m}) : k = 0,\dots,2^n-1; \\ l = 0,\dots,2^m-1\}\big),$$

where $\sigma(\mathcal{S})$ denotes the $\sigma$-algebra generated by an arbitrary set system $\mathcal{S}$. Then the conditional expectation operator relative to $\mathcal{F}_{n,m}$ is not another as $S_{2^n,2^m}$. A sequence of integrable functions $f = (f_{n,m}; n, m = 0, 1, \dots)$ is said to be a martingale if

i) $f_{n,m}$ is $\mathcal{F}_{n,m}$ measurable for all $n, m = 0, 1, \dots$

and

ii) $S_{2^n,2^m} f_{k,l} = f_{n,m}$ for all $n, m, k, l = 0, 1, \dots$ such that $n \le k$ and $m \le l$.

For example, if $f \in L^1[0,1)^2$ then the sequence $(S_{2^n,2^m} f;\ n, m = 0, 1, \dots)$ is evidently a martingale (called martingale obtained from $f$). Of course, $f_1 := (f_{n,0},\ n = 0, 1, \dots)$ and $f_2 := (f_{0,m},\ m = 0, 1, \dots)$ are (one-dimensional) martingales with respect to the sequence of $\sigma$-algebras

$$\sigma\big(\{[j2^{-k}, (j+1)2^{-k}) : j = 0, \dots, 2^k - 1\}\big) \qquad (k = 0, 1, \dots).$$

The concept of the Walsh-Fourier coefficients can be extended to the martingales also in the two-parameter case. That is, $\hat{f}$ will be denote the sequence of the Walsh-Fourier coefficients of the function or martingale $f$.

Denote $\|g\|_p$ $(0 < p < \infty)$ the usual $L^p$-norm (or quasi-norm) of a measurable function $g$ defined on the unit square $[0,1)^2$. We say that a martingale $f = (f_{n,m}; n, m = 0, 1, \dots)$ is $L^p$-bounded if $\|f\|_p := \sup_{n,m} \|f_{n,m}\|_p < \infty$. The set of the $L^p$-bounded martingales will be denoted by $L^p$. If $p > 1$ then $L^p$ and $L^p[0,1)^2$ can be identified.

The maximal function $f^*$ and the quadratic variation $Qf$ of a martingale $f = (f_{n,m}; n, m = 0, 1, \dots)$ are defined by $f^* := \sup_{n,m} |f_{n,m}|$ and

$$Qf := \Big(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |f_{n,m} - f_{n-1,m} - f_{n,m-1} + f_{n-1,m-1}|^2\Big)^{1/2},$$

respectively, where $f_{-1,k} := f_{k,-1} := 0$ $(k = -1, 0, 1, \dots)$. It can be shown that for each $0 < p < \infty$ the norms (or quasi-norms) $\|f^*\|_p$ and $\|Qf\|_p$ are equivalent:

$$c_p \|f^*\|_p \le \|Qf\|_p \le C_p \|f^*\|_p.$$

We introduce the martingale Hardy spaces for $0 < p < \infty$ as follows: denote $H^p$ the space of martingales $f$ for which $\|f\|_{H^p} := \|f^*\|_p < \infty$. Hence, by the above mentioned equivalence $\|f^*\|_p \sim \|Qf\|_p$ we get $\|f\|_{H^p} \sim \|Qf\|_p$.

Introduce the following $\sigma$-algebras:

$$\mathcal{F}_{n,\infty} := \sigma\Big(\bigcup_{k=0}^{\infty} \mathcal{F}_{n,k}\Big),\ \mathcal{F}_{\infty,m} := \sigma\Big(\bigcup_{k=0}^{\infty} \mathcal{F}_{k,m}\Big) \qquad (n,m=0,1,\dots).$$

We assume that $f_{n,\infty}$, $f_{\infty,m}$ are defined as measurable functions with respect to $\mathcal{F}_{n,\infty}$ and $\mathcal{F}_{\infty,m}$ and $S_{2^k,2^l} f_{n,\infty}=f_{k,l}$, $S_{2^j,2^l} f_{\infty,m}=f_{j,l}$ $(k,l,n,m=0,1,\dots;\ k\le n; j\le m)$, respectively.

The diagonal maximal function of a martingale $f=(f_{n,m};\ n,m=0,1,\dots)$ is defined by $f^{\diamond} := \sup_n |f_{n,n}|$. Analoguosly, the so-called hybrid maximal functions of $f$ are given by $f_1^{\natural} := \sup_n |f_{n,\infty}|$, $f_2^{\natural} := \sup_m |f_{\infty,m}|$. The functions $f_1^{\natural}$, $f_2^{\natural}$ play the same role, further we concern with $f^{\natural} := f_1^{\natural}$ only.

It is easy to see that in case $f\in L^1[0,1)$ the above maximal functions can also be computed by

$$f^*(x,y) = \sup_{m,n} \frac{1}{|I_{n,m}(x,y)|}\Big|\int_{I_{n,m}(x,y)} f\Big|,$$

$$f^{\diamond}(x,y) = \sup_{n} \frac{1}{|I_{n,n}(x,y)|}\Big|\int_{I_{n,n}(x,y)} f\Big|,$$

$$f^{\natural}(x,y) = \sup_{n} \frac{1}{|I_{n}(x)|}\Big|\int_{I_{n}(x)} f(t,y)\,dt\Big|.$$

(Here $|I_{n,m}(x,y)|$ and $|I_n(x)|$ stand for the two- and one-dimensional Lebesgue measure of the sets in question, respectively.) The corresponding quadratic variations of a martingale $f=(f_{n,m};n,m=0,1,\dots)$ are introduced as follows:

$$Q^{\diamond} f := \Big(\sum_{n=0}^{\infty} |f_{n,n}-f_{n-1,n-1}|^2\Big)^{1/2},$$

$$Q^{\natural} f := \Big(\sum_{n=0}^{\infty} |f_{n,\infty}-f_{n-1,\infty}|^2\Big)^{1/2}.$$

The following equivalences are well-known: $\|Q^\diamond f\|_p \sim \|f^\diamond\|_p$, $\|Q^\natural f\|_p \sim \|f^\natural\|_p$ and $\|f^*\|_q \sim \|f^\diamond\|_q \sim \|f^\natural\|_q \sim \|f\|_q$ $(0 < p < \infty,\ 1 < q < \infty)$.

Define the spaces $H^p_\diamond$ and $H^p_\natural$ of Hardy type as the set of martingales $f$ such that

$$\|f\|_{H^p_\diamond} := \|Q^\diamond f\|_p < \infty \text{ and } |f\|_{H^p_\natural} := \|Q^\natural f\|_p < \infty \quad (0 < p < \infty),$$

respectively.

Unfortunatly, the atomic characterization of Hardy spaces is much more complicated in the two-dimensional case than in the one-dimensional. Indeed, in the two-dimensional case the support of an atom in $H^p$ is not a dyadic rectangle but an open set (see Weisz [5]). However, a finer atomic decomposition can be given, that is, the atoms can be decomposed into elementary rectangle particles. This makes possible in some investigations to examine only atoms supported on dyadic rectangles. To their definition let $0 < p \leq 1$. A function $a \in L^2[0,1)^2$ is called a rectangle $H^p$-atom if either $a$ is identically equal to 1 or there exists a dyadic rectangle $I$ such that

$$\tag{9} \begin{gathered} \operatorname{supp} a \subset I, \quad \|a\|_2 \leq |I|^{1/2-1/p}, \\ \int_0^1 a(x,t)\,dt = \int_0^1 a(u,y)\,du = 0 \quad (x,y \in [0,1)). \end{gathered}$$

We shall say that $a$ is supported on $I$.

The atomic characterization of $H^p_\diamond$ $(0 < p \leq 1)$ is similar to the one-dimensional case. Namely, a bounded measurable function $a$ is an $H^p_\diamond$-atom if $a \equiv 1$ or there exists a dyadic square $I$ such that

$$\tag{10} \operatorname{supp} a \subset I, \quad \|a\|_\infty \leq |I|^{-1/p}, \quad \int_0^1\int_0^1 a = 0.$$

We shall say also in this case that $a$ is supported on $I$. Then a martingale $f = (f_{n,m}; n,m = 0,1,\dots)$ is in $H^p_\diamond$ if and only if there exist a sequence $(a_k, k = 0,1,\dots)$ of $H^p_\diamond$-atoms and a sequence $(\lambda_k, k = 0,1,\dots)$ of real numbers such that $\sum_{k=0}^\infty |\lambda_k|^p < \infty$ and

$$\tag{11} \sum_{k=0}^\infty \lambda_k S_{2^n,2^n} a_k = f_{n,n} \qquad (n = 0,1,\dots).$$

Moreover, $c_p \inf \big(\sum_{k=0}^{\infty} |\lambda_k|^p\big)^{1/p} \le \|f\|_{H_\diamond^p} \le C_p\big(\sum_{k=0}^{\infty} |\lambda_k|^p\big)^{1/p}$, where the infimum is taken over all decompositions of $f$ of the form (11).

Later we need the concept of some Hardy-Lorentz spaces. To their definition we denote by $\tilde{g}$ the non-increasing rearrangement of a measurable function $g$:

$$\tilde{g}(t) := \inf\{\alpha : |\{|g| > \alpha\}| \le t\} \qquad (t > 0),$$

where $|\{|g| > \alpha\}| := |\{x : |g(x)| > \alpha\}|$ is the distribution function of $g$. For $0 < p < \infty$, $0 < q < \infty$ the Lorentz space $L^{p,q}$ is defined as follows:

$$\|g\|_{p,q} := \Big(\int_0^\infty \tilde{g}(t)^q t^{q/p} \frac{dt}{t}\Big)^{1/q}.$$

If $0 < p \le \infty, q = \infty$ then let $\|g\|_{p,\infty} := \sup_{t>0} t^{1/p}\tilde{g}(t)$. The space $L^{p,q}$ is given by $L^{p,q} := \{g : \|g\|_{p,q} < \infty\}$.

The weak $L^p$ space $L_*^p$ $(0 < p < \infty)$ consists of all measurable functions $g$ for which

$$\|g\|_{L_*^p} := \sup_{\alpha>0} \alpha|\{|g| > \alpha\}|^{1/p} < \infty,$$

while we set $\|g\|_{L_*^\infty} := \|g\|_\infty$. It follows immediately that $L^{p,p} = L^p$, $L^{p,\infty} = L_*^p$ $(0 < p \le \infty)$.

Now, we define the martingale Hardy-Lorentz spaces $H^{p,q}$, $H_\diamond^{p,q}, H_\natural^{p,q}$ $(0 < p, q \le \infty)$ as the set of all martingales $f$ such that

$$\|f\|_{H^{p,q}} := \|Qf\|_{p,q} < \infty, \quad \|f\|_{H_\diamond^{p,q}} := \|Q^\diamond f\|_{p,q} < \infty,$$
$$\|f\|_{H_\natural^{p,q}} := \|Q^\natural f\|_{p,q} < \infty,$$

respectively. We remark that in the case $p = q$ the equalities $H^{p,p} = H^p$, $H_\diamond^{p,p} = H_\diamond^p$, $H_\natural^{p,p} = H_\natural^p$ hold. Moreover, for $1 < p < \infty, 0 < q \le \infty$ we get $H^{p,q} \sim H_\diamond^{p,q} \sim H_\natural^{p,q} \sim L^{p,q}$.

**3.** The classical theorem of Riesz-Thorin or the Marcinkiewicz theorem on the interpolation between some function spaces are well-known in the Fourier analysis (see e.g. Zygmund [6]). The analoguous statements with respect to the Hardy or Hardy-Lorentz spaces can be formulated as follows, first of all in the one-dimensional case. To this end let $T$ be a sublinear oparator which is bounded from $H^{p_k}$ into $L^{p_k}$ for some $0 < p_0 < p_1 \le \infty$. Then the following theorem holds.

**Theorem A** (Weisz [4]). *If $p_0 < p < p_1, 0 < q \le \infty$, then $T : H^{p,q} \to L^{p,q}$ is bounded.*

In the case $p_0 \le 1 \le p_1$ we can write in Theorem A $p = 1$ and $q = \infty$, which leads to the estimation $\|Tf\|_{1,*} \le C\|Qf\|_{1,*}$. Since $Q$ is of weak type (1,1) we get $\|Tf\|_{1,*} \le C\|f\|_{1,*}$, i.e. $T$ is of weak type (1,1).

In the two-dimensional case the analoguous theorem on interpolation reads as follows.

**Theorem B** (Weisz [4]). *Let $X \in \{H, H_\diamond, H_\natural\}$ and $T$ be a sublinear operator which is bounded from $X^{p_k}$ into $L^{p_k}$ for some $0 < p_0 < p_1 \le \infty$. Then $T : X^{p,q} \to L^{p,q}$ is bounded for all $p_0 < p < p_1, 0 < q \le \infty$.*

If $X := H_\diamond$ and $p_0 \le 1 \le p_1$, then the weak type (1,1) of $T$ follows similarly to the one-dimensional case, since $Q^\diamond$ is of weak type (1,1). However, if $X := H$ (and $p_0 \le 1 \le p_1$), we get only the estimation

$$\|Tf\|_{1,*} \le C\|Qf\|_{1,*} \le C\|f^\natural\|_1 \le C\|f\|_{H^1_\natural},$$

since $Q$ is of weak type $(H^1_\natural, L^1)$. That is, $T$ has weak type $(H^1_\natural, L^1)$. We remark that in the two-dimensional case $Q$ is not of weak type (1,1).

To the application of the previous theorems we need to show the $(H^p, L^p)$-boundedness of the operator $T$ in question, that is, the inequality $\|Tf\|_p \le C_p\|f\|_{H^p}$ for all $f \in H^p$. In the one-dimensional case and for $0 < p \le 1$ – taking into consideration the atomic characterization (8) of $H^p$ – it is enough to prove that $\|Ta\|_p \le C_p$ holds for all $p$-atoms $a \in H^p$. If $T$ is bounded from $L^s$ into $L^s$ for some $1 \le s \le \infty$, then the above suffiecient condition $\|Ta\|_p \le C_p$ can be weakened as follows (Weisz [5], Simon [2]):

$$\int_{[0,1)\setminus I} |Ta|^p \le C_p. \tag{12}$$

Here $a$ is an arbitrary $p$-atom supported on the dyadic interval $I$ (see the definition (7) of atoms). If (12) is true, then $T$ is called $p$-quasi-local (Weisz [5]). Hence, $p$-quasi-locality together with $(L^s, L^s)$-boundedness of a sublinear operator $T$ implies that $T : H^p \to L^p$ is bounded. Applying Theorem A we get

**Theorem C** (Weisz [4]). *Let $0 < p \le 1, 1 \le s \le \infty$ and assume that the sublinear operator $T$ is $p$-quasi-local and $(L^s, L^s)$-bounded. Then $T : H^{u,v} \to L^{u,v}$ is bounded for all $p < u < s$ and $0 < v \le \infty$. Especially, $T$ is of weak type* (1,1).

Since $\|a\|_p \le 1$ holds for all $p$-atoms $a$, thus the assumption $\int_{[0,1)\setminus I} |Ta|^p \le C_p \|a\|_p^p$ implies the $p$-quasi-locality (12) of $T$. This motivates the concept of the strong $p$-quasi-locality. Namely, a sublinear operator $T$ is called strong $p$-quasi-local if

$$\int_{[0,1)\setminus I} |Tf|^p \le C_p \|f\|_p^p \tag{13}$$

is true for all $f$ such that $\operatorname{supp} f \subset I$ (for some dyadic interval $I$) and $\int_0^1 f = 0$. (For $p = 1$ see Schipp–Wade–Simon [1].) It is not hard to see that the weak type (1,1) of $T$ can be deduced directly from its strong 1-quasi-locality, assumed the $(L^s, L^s)$-boundedness of $T$ for some $1 \le s \le \infty$. This was proved for $s = \infty$ in Schipp–Wade–Simon [1]. For the sake of the completeness we give the proof here for other exponents $1 \le s < \infty$. It can be assumed evidently that $p > 1$. Let $f \in L^1, y > \|f\|_1$ and $f = g + h$ be the corresponding Calderon-Zygmund decomposition (see for example Schipp–Wade–Simon [1]). That is, $g \in L^\infty[0,1)$, $\|g\|_\infty \le Cy$, $h = \sum_{k=0}^\infty h_k$, the function $h_k$ is supported on a dyadic interval $I_k$, the intervals $I_k$ $(k = 0, 1, \dots)$ are pairwise disjoint, $\int_0^1 h_k = 0$ $(k = 0, 1, \dots)$ and $|\Omega| := |\bigcup_{k=0}^\infty I_k| \le C\|f\|_1/y, \|h\|_1 \le y|\Omega|$. Then

$$|\{|Tf| > y\}| \le |\{|Tg| > y/2\}| + |\{|Th| > y/2\}| =: A + B,$$

where by the strong 1-quasi-locality we get

$$\begin{aligned} B &\le |\Omega| + |\{x \in [0,1) \setminus \Omega : |Th(x)| > y/2\}| \le |\Omega| + \frac{2}{y} \int_{[0,1)\setminus\Omega} |Th| \\ &\le |\Omega| + \frac{2}{y} \sum_{k=0}^\infty \int_{[0,1)\setminus I_k} |Th_k| \le |\Omega| + \frac{C}{y} \sum_{k=0}^\infty \|h_k\|_1 \end{aligned}$$

$$= |\Omega| + \frac{C}{y}\|h\|_1 \le C|\Omega| \le C\frac{\|f\|_1}{y}.$$

On the other hand

$$A \le \left(\frac{2}{y}\right)^s \int_0^1 |Tg|^s \le \frac{C_s}{y^s}\|g\|_s^s$$
$$\le \frac{C_s}{y^s}\|g\|_\infty^{s-1}\|g\|_1 \le \frac{C_s}{y^s}y^{s-1}(\|f\|_1 + \|h\|_1) \le \frac{C_s}{y}\|f\|_1.$$

As we mentioned already the atomic characterization of the two-dimensional $H^p$ spaces is complicated. Although the elements of $H^p$ cannot be decomposed into rectangle $H^p$-atoms, in the investigation of the $H^p$-quasi-local operators it is enough to take these atoms. It will be assumed that the operator $T$ is sublinear. Then $T$ is called $H^p$-quasi-local (see Weisz [5]) if there exists $\delta > 0$ such that for every rectangle $H^p$-atom $a$ supported on the dyadic rectangle $I$ and for all $r = 0, 1, \ldots$ one has

$$\int_{[0,1)^2 \setminus I^r} |Ta|^p \le C_p 2^{-\delta r}. \tag{14}$$

Here $I^r$ is the dyadic rectangle defined as follows: $I^r := I_1^r \times I_2^r$, where $I = I_1 \times I_2$ for some dyadic intervals $I_1, I_2$ and $I_j^r$ is the (unique) dyadic interval for which $I_j \subset I_j^r$ and $|I_j^r| = 2^r|I_j|$ $(j = 1, 2)$. If $0 < p \le 1$ and $T$ is of type $(L^2, L^2)$, then a theorem of Weisz [5] implies the boundedness of $T : H^p \to L^p$. Now, applying Theorem B we get the two-dimensional variant of Theorem C.

**Theorem D** (Weisz [4]). *Let $0 < p \le 1$ and assume that the sublinear operator $T$ is $H^p$-quasi-local and bounded from $L^2$ into $L^2$. Then $T : H^{u,v} \to L^{u,v}$ is bounded for all $p < u < 2, 0 < v \le \infty$. In particular, $T$ is of weak type $(H^1_\natural, L^1)$.*

The definition of the quasi-locality of sublinear operators defined on $H^p_\diamond$ $(0 < p \le 1)$ is similar to the one-dimensional case. Namely, a sublinear operator $T$ is called $H^p_\diamond$-quasi-local if

$$\int_{[0,1)^2 \setminus I} |Ta|^p \le C_p$$

for all $H^p_\diamond$-atoms $a$ supported on the dyadic square $I$. Then it is clear that the atomic characterization (11) of $H^p_\diamond$ implies the boundedness of $T : H^p_\diamond \to L^p$. Moreover, the assumption $\int_{[0,1)\setminus I} |Ta|^p \le C_p$ can be modified as follows: there exists $r = 0, 1, \dots$ such that

$$\int_{[0,1)^2\setminus I^r} |Ta|^p \le C_p$$

holds for every $H^p_\diamond$-atom $a$ supported on the dyadic square $I$.

The $H^p_\diamond$-quasi-locality leads by Theorem B to

**Theorem E** (Weisz [4]). *Let $0 < p \le 1, 1 \le s \le \infty$ and assume the sublinear operator $T$ is $H^p_\diamond$-quasi-local and $(L^s, L^s)$-bounded. Then $T$ is bounded from $H^{u,v}_\diamond$ into $L^{u,v}$ for all $p < u < s, 0 < v \le \infty$. Moreover, $T$ is of weak type* (1,1).

The strong $p$-quasi-locality of $T$ can be defined in analoguous way as in the one-dimensional case. Namely, $T$ is called such an operator if $\int_{[0,1)^2\setminus I} |Tf|^p \le C_p \|f\|_p^p$ is true for all $f$ supported on the dydadic square $I$ such that $\int_0^1 \int_0^1 f = 0$. The decomposition lemma of Calderon-Zygmund can also be applied in the two-dimensional case. Thus the strong 1-quasi-locality of $T$ implies its weak type (1,1).

**4.** In earlier papers (see Simon [2], [3]) we investigated some multiplier operators defined on Hardy spaces $H^p$ $(0 < p \le 1)$. That is, in the one-dimensional case let $\lambda = (\lambda_n, n = 0, 1, \dots)$ be a bounded sequence of real numbers and consider the operator $T_\lambda$ by the rule $\widehat{T_\lambda f} = \lambda \hat{f}$. By well-known Parseval's equality $T_\lambda$ is defined at least on $L^2$. Moreover, $T_\lambda : L^2 \to L^2$ is a bounded linear operator. If $\Lambda_j^{(\lambda)} := \sum_{k=2^j}^{2^{j+1}-1} \lambda_k w_k$ $(j = 0, 1, \dots)$ and $0 < p \le 1$, then the assumption

$$\sup_N \sum_{j=N}^{\infty} 2^N \int_{2^{-N}}^1 \Big( \int_0^{2^{-N}} |\Lambda_j^{(\lambda)}(x \dot{+} t)| \, dt \Big)^p dx < \infty \tag{15}$$

implies the $p$-quasi-locality of $T_\lambda$ (Simon [2]). Therefore Theorem C can be applied with $p_1 := 2$ and the boundedness of $T_\lambda : H^{u,v} \to L^{u,v}$ $(p < u < 2, 0 < q \le \infty)$ follows. Especially, we get the boundedness of $T_\lambda : L^s \to L^s$

for all $1 < s \le 2$ and consequently by means of duality for $2 < s < \infty$, too. Hence, we can write in Theorem C in place of $s$ an arbitrary number $1 < s < \infty$. This proves

**Theorem 1.** *Let $\lambda$ be a bounded sequence of real numbers, $0 < p \le 1$ and assume that (15) holds. Then $T_\lambda : H^{u,v} \to L^{u,v}$ is bounded for all $p < u < \infty$, $0 < v \le \infty$. Especially, $T_\lambda$ is of weak type (1,1).*

For $p = 1$ the assumption (15) has a very simple form (Simon [2]):

$$\sup_N \sum_{j=N}^{\infty} \int_{2^{-N}}^{1} |\Lambda_j^{(\lambda)}| < \infty. \tag{16}$$

On the other hand, if (16) is true, then $T_\lambda$ is strong 1-quasi-local. Indeed, let $f$ be a function supported on the dyadic interval $I$ such that $\int_0^1 f = 0$. Without loss of generality we may assume that $I = [0, 2^{-N})$ for some $N = 0, 1, \ldots$. Then $\hat{f}(k) = 0$ $(k = 0, 1, \ldots, 2^N - 1)$, therefore $T_\lambda f = \sum_{j=N}^{\infty} \sum_{k=2^j}^{2^{j+1}-1} \lambda_k \hat{f}(k) w_k$ and

$$\begin{aligned}
\int_{[0,1)\setminus I} |T_\lambda f| &= \int_{2^{-N}}^{1} \Big| \sum_{j=N}^{\infty} \sum_{k=2^j}^{2^{j+1}-1} \lambda_k \hat{f}(k) w_k \Big| \\
&\le \sum_{j=N}^{\infty} \int_{2^N}^{1} \Big| \int_0^{2^N} f(t) \Lambda_j^{(\lambda)}(x \dot{+} t)\, dt \Big|\, dx \\
&\le \sum_{j=N}^{\infty} \int_0^{2^{-N}} |f(t)| \int_{2^{-N}}^{1} \big| \Lambda_j^{(\lambda)}(x \dot{+} t) \big|\, dx\, dt \\
&= \sum_{j=N}^{\infty} \int_0^{2^{-N}} |f(t)|\, dt \int_{2^{-N}}^{1} \big| \Lambda_j^{(\lambda)}(x) \big|\, dx \\
&= \|f\|_1 \sum_{j=N}^{\infty} \int_{2^{-N}}^{1} \big| \Lambda_j^{(\lambda)} \big|.
\end{aligned}$$

We do not know whether $T_\lambda$ is strong $p$-quasi-local under the assumption (15) for $0 < p < 1$.

Now, let $\lambda_0 := 1, \lambda_{2^n+k} := (2^n + k)2^{-n}$ $(k = 0, 1, \ldots, 2^n - 1)$. It was shown in Simon [2] that (15) is fulfilled, i.e. $T_\lambda$ is $p$-quasi-local for all

$0 < p \leq 1$. Moreover, the same statement holds for $T_{1/\lambda}$ if $1/2 < p \leq 1$. Hence, by Theorem 1 we get

**Corollary 1.** *If $0 < p < \infty$, $0 < q \leq \infty$ and the sequence $\lambda$ is defined as above, then $T_\lambda$ is bounded from $H^{p,q}$ into $L^{p,q}$. Moreover, if $1/2 < p < \infty$, $0 < q \leq \infty$, then $T_{1/\lambda} : H^{p,q} \to L^{p,q}$ is also bounded. In particular, $T_\lambda$, $T_{1/\lambda}$ are of weak type (1,1).*

The assumption (16) is not enough to the $(L^1, L^1)$-boundedness of $T_\lambda$. Namely, the next theorem is true.

**Theorem 2.** *Let $\lambda$ be the sequence as in Corollary 1. Then $T_\lambda$ is not bounded from $L^1$ into $L^1$.*

Proof. For $n = 1, 2, \ldots$ let

$$f_n := D_{2^n} - 1 = \sum_{k=0}^{n-1} \sum_{j=0}^{2^k-1} w_{2^k+j}.$$

Then by (2)

$$\begin{aligned} T_\lambda f_n &= \sum_{k=0}^{n-1} \sum_{j=0}^{2^k-1} \frac{2^k+j}{2^k} w_{2^k+j} = \sum_{k=0}^{n-1} \frac{r_k}{2^k} \Big( 2^k D_{2^k} + \sum_{j=0}^{2^k-1} j w_j \Big) \\ &= \sum_{k=0}^{n-1} r_k (2D_{2^k} - K_{2^k}) = \sum_{k=0}^{n-1} \big( 2(D_{2^{k+1}} - D_{2^k}) - r_k K_{2^k} \big) \\ &= 2(D_{2^n} - 1) - \sum_{k=0}^{n-1} r_k K_{2^k} = 2f_n - \sum_{k=0}^{n-1} r_k K_{2^k}. \end{aligned}$$

Further we examine the sum $\sum_{k=0}^{n-1} r_k K_{2^k}$. Let $x \in [0, 1)$, then by (3)

$$\begin{aligned} &\sum_{k=0}^{n-1} r_k(x) K_{2^k}(x) = \frac{1}{2} \sum_{k=0}^{n-1} r_k(x) \Big( 2^{-k} D_{2^k}(x) + \sum_{l=0}^{k} 2^{l-k} D_{2^k}(x \dotplus 2^{-l-1}) \Big) \\ &= \frac{1}{2} \bigg( 2r_0(x) + \sum_{k=1}^{n-1} r_k(x) \Big( 2^{-k} D_{2^k}(x) + D_{2^k}(x) + \sum_{l=0}^{k} 2^{l-k} D_{2^k}(x \dotplus 2^{-l-1}) \Big) \bigg) \end{aligned}$$

$$= r_0(x) + \frac{1}{2}\sum_{k=1}^{n-1} 2^{-k} r_k(x) D_{2^k}(x) + \frac{1}{2}\sum_{k=1}^{n-1}\big(D_{2^{k+1}}(x) - D_{2^k}(x)\big)$$

$$+\frac{1}{2}\sum_{k=1}^{n-1} r_k(x)\sum_{l=0}^{k-1} 2^{l-k} D_{2^k}(x\dot{+}2^{-l-1}).$$

Since

$$\Big\| r_0 + \frac{1}{2}\sum_{k=1}^{n-1} 2^{-k} r_k D_{2^k} + \frac{1}{2}\sum_{k=1}^{n-1}(D_{2^{k+1}} - D_{2^k})\Big\|_1$$

$$= \Big\| r_0 + \frac{1}{2}\sum_{k=1}^{n-1} 2^{-k} r_k (D_{2^k} + \frac{1}{2}(D_{2^n} - 1)\Big\|_1 \le 1 + \frac{1}{2}\sum_{k=1}^{n-1} 2^{-k} + 1 < 3,$$

we need to investigate only the function

$$F_n(x) := \sum_{k=1}^{n-1} r_k(x)\sum_{j=0}^{k-1} 2^{j-k} D_{2^k}(x\dot{+}2^{-l-1}).$$

To this end let us write $F_n(x)$ in the following form:

$$F_n(x) = \sum_{j=0}^{n-1} 2^j \sum_{k=j+1}^{n-1} r_k(x) D_{2^k}(x\dot{+}2^{-l-1}) = \sum_{j=0}^{n-1} 2^j d_{jn}(x),$$

where $d_{jn}(x) := \sum_{k=j+1}^{n-1} r_k(x) 2^{-k} D_{2^k}(x\dot{+}2^{-l-1})$. Let $j = 0, \ldots, n-2$ and $k = j+1, \ldots, n-1$. The basic property (1) of Dirichlet's kernels implies

$$r_k(x) 2^{-k} D_{2^k}(x\dot{+}2^{-l-1}) = \begin{cases} 1 & (2^{-j-1} \le x < 2^{-j-1} + 2^{-k-1}) \\ -1 & (2^{-j-1} + 2^{-k-1} \le x < 2^{-j-1} + 2^{-k}) \\ 0 & (\text{otherwise}). \end{cases}$$

The intervals $I_{jk} := [2^{-j-1} + 2^{-k-1}, 2^{-j-1} + 2^{-k})$ $(j = 0, \ldots, n-2;$ $k = j+1, \ldots, n-1)$ are pairwise disjoint and

$$d_{jn}(x) = -1 \qquad (x \in I_{jj+1}).$$

Therefore

$$\|F_n\|_1 \ge \sum_{j=0}^{n-1} \int_{I_{jj+1}} |F_n| \ge \sum_{j=0}^{n-2} \int_{I_{jj+1}} 2^j |d_{jn}| = \sum_{j=0}^{n-2} 2^j 2^{-j-2} = \frac{n-1}{4}.$$

Since $\|T_\lambda f_n\|_1 \geq \frac{1}{2}\|F_n\|_1 - 3 \geq \frac{n-1}{8} - 3$ and $\|f_n\|_1 < 2$, the operator $T_\lambda$ cannot be $(L^1, L^1)$-bounded. This proves our theorem.

It remains open whether $T_{1/\lambda}$ is bounded from $L^1$ into $L^1$.

In the two-dimensional case we have investigated only special multipliers. Namely, let the sequence of real numbers $\lambda = (\lambda_{k,l}; k, l = 0, 1, \dots)$ be given as follows:

$$\lambda_{0,0} := 1, \quad \lambda_{i,0} := \frac{i}{2^n}, \quad \lambda_{0,j} := \frac{j}{2^m}, \quad \lambda_{i,j} := \frac{ij}{2^{n+m}}$$

$(n, m = 0, 1, \dots;\ i = 2^n, \dots, 2^{n+1} - 1;\ j = 2^m, \dots, 2^{m+1} - 1)$ and define $T_\lambda$ as $\widehat{T_\lambda f} = \lambda \hat{f}$. It is clear that $T_\lambda$ is the two-dimensional variant of the multiplier operator from Corollary 1. In Simon [3] we proved that $T_\lambda$ is $H^p$-quasi-local for all $0 < p \leq 1$. Moreover, if $1/2 < p \leq 1$, then also $T_{1/\lambda}$ is $H^p$-quasi-local. From this it follows by Theorem D that $T_\lambda$ is bounded from $H^{u,v}$ into $L^{u,v}$ for all $0 < u < 2$, $0 < v \leq \infty$. Furthermore, if $1/2 < p \leq 1$, $0 < v \leq \infty$, then the same statements holds also for $T_{1/\lambda}$. In particular, the operators $T_\lambda$, $T_{1/\lambda}$ are bounded from $L^s$ into $L^s$ $(1 < s \leq 2)$ and by duality for all $1 < s < \infty$. Hence, Theorem D leads to

**Theorem 3.** *Let $\lambda$ be defined as above and $0 < p < \infty, 1/2 < u < \infty$, $0 < q \leq \infty$. Then the operators $T_\lambda : H^{p,q} \to L^{p,q}$ and $T_{1/\lambda} : H^{u,q} \to L^{u,q}$ are bounded, respectively. Especially, $T_\lambda$, $T_{1/\lambda}$ are weak type $(H^1_\natural, L^1)$.*

We do not know whether $T_\lambda, T_{1/\lambda}$ are of weak type (1,1). However, the following theorem is true.

**Theorem 4.** *Let $\lambda$ be the sequence as in the previous theorem. Then $T_\lambda$ is not strong 1-quasi-local.*

Proof. Let $n = 0, 1, \dots$ and

$$g_n(x, y) := r_n(x) D_{2^n}(x) D_{2^n}(y) = \sum_{k=2^n}^{2^{n+1}-1} w_k(x) \sum_{j=0}^{2^n-1} w_j(y)$$

$$(x, y \in [0, 1)).$$

Then $\operatorname{supp} g_n \subset [0, 2^{-n}) \times [0, 2^{-n})$, $\|g_n\|_1 = 1$ and $\int_0^1 \int_0^1 g_n(x, y)\, dx\, dy = 0$. On the other hand by (1), (2)

$$T_\lambda g_n(x,y) = 2^{-n}\sum_{k=2^n}^{2^{n+1}-1} kw_k(x)\Big(\sum_{l=0}^{n-1} 2^{-l}\sum_{j=2^l}^{2^{l+1}-1} jw_j(y)+1\Big)$$

$$= \sum_{k=0}^{2^n-1}\frac{2^n+k}{2^n} r_n(x)w_k(x)\Big(\sum_{l=0}^{n-1}\sum_{j=o}^{2^l-1}\frac{2^l+j}{2^l} r_l(y)w_j(y)+1\Big)$$

$$= r_n(x)\Big(D_{2^n}(x)+2^{-n}\sum_{k=0}^{2^n-1} kw_k(x)\Big)$$

$$\times\Big(\sum_{l=0}^{n-1} r_l(y)(D_{2^l}(y)+2^{-l}\sum_{j=0}^{2^l-1} jw_j(y))+1\Big)$$

$$= r_n(x)\Big(2D_{2^n}(x)-K_{2^n}(x)\Big)\Big(\sum_{l=0}^{n-1} r_l(y)(2D_{2^l}(y)-K_{2^l}(y))+1\Big)$$

$$= r_n(x)\Big(2D_{2^n}(x)-K_{2^n}(x)\Big)\Big(2D_{2^n}(y)-1-\sum_{l=0}^{n-1} r_l(y)K_{2^l}(y)\Big).$$

If $2^{-n}\le x<1, 2^{-n}\le y<1$, then

$$T_\lambda g_n(x,y) = r_n(x)K_{2^n}(x)\Big(1+\sum_{l=0}^{n-1} r_l(y)K_{2^l}(y)\Big).$$

Therefore by the proof of Theorem 2 it follows that (see also (13))

$$\int_{[0,1)^2\setminus[0,2^{-n})^2}|T_\lambda g_n| \ge \int_{2^{-n}}^1\int_{2^{-n}}^1 |T_\lambda g_n(x,y)|\,dx\,dy = \int_{2^{-n}}^1 K_{2^n}$$

$$\times\int_{2^{-n}}^1 |1+\sum_{l=0}^{n-1} r_lK_{2^l}| \ge \int_{2^{-n}}^1 K_{2^n}\Big(\int_{2^{-n}}^1 |\sum_{l=0}^{n-1} r_lK_{2^l}|-1\Big)\ge Cn\int_{2^{-n}}^1 K_{2^n},$$

where $C>0$ is a constant and $n$ is large enough. Furthermore by (3) we get

$$\int_{2^{-n}}^{1} K_{2^n} = 2^{-n-1}\sum_{l=0}^{n-1} 2^l \int_{2^{-n}}^{1} D_{2^n}(x\dot{+}2^{-l-1})\,dx$$

$$= 2^{-n-1}\sum_{l=0}^{n-1} 2^l\Big(\int_0^1 D_{2^n}(x\dot{+}2^{-l-1})\,dx - \int_0^{2^{-n}} D_{2^n}(x\dot{+}2^{-l-1})\,dx\Big)$$

$$= 2^{-n-1}\sum_{l=0}^{n-1} 2^l = \frac{2^n-1}{2^{n+1}},$$

thus $\int_{[0,1)^2\setminus[0,2^{-n})^2} |T_\lambda g_n| \geq Cn$, if $n$ is large enough. This completes the proof.

Further we investigate the so-called restricted multiplier operators $T_\lambda^{(\alpha)}$, $T_{1/\lambda}^{(\alpha)}$. To this end let $\alpha$ be a non-negative real number and

$$T_\lambda^{(\alpha)} f := \sum_{n,m}^{(\alpha)} \sum_{i=2^n}^{2^{n+1}-1} \sum_{j=2^m}^{2^{m+1}-1} \lambda_{i,j}\hat{f}(i,j)w_{i,j} =: \sum_{n,m}^{(\alpha)} \Delta_{n,m}(f),$$

where $\sum_{n,m}^{(\alpha)}$ means the summation with respect to the indices $n, m = 0, 1, \ldots$ such that $|n-m| \leq \alpha$. The following theorem is true.

**Theorem 5.** *The operators* $T_\lambda^{(\alpha)}$, $T_{1/\lambda}^{(\alpha)}$ *are* $H_\diamond^p$*-quasi-local for all* $0 < p \leq 1$.

Proof. We follow the method of Weisz [5] and Simon [3]. Only an outline will be given because the proof is similar to one of the corresponding result proved in Simon [3]. Let $T = T_\lambda^{(\alpha)}$ or $T = T_{1/\lambda}^{(\alpha)}$ and $a$ be an $H_\diamond^p$-atom. It can be evidently assumed that $a$ is supported on the dyadic square $I := [0, 2^{-N}) \times [0, 2^{-N})$, i.e. (see the definiton (10) of $H_\diamond^p$-atoms), $\|a\|_\infty \leq 2^{2N/p}$ and $\int_0^1\int_0^1 a = \int_0^{2^{-N}}\int_0^{2^{-N}} a = 0$. Moreover, $\hat{a}(i,j) = 0$ if $i < 2^N$ and $j < 2^N$, i.e. $\Delta_{n,m}(a) = 0$ if $n < N$ and $m < N$. Therefore we can suppose that $n \geq N$ or $m \geq N$. Let $r = 0, 1, \ldots$ be a natural number such that $r - 1 < \alpha \leq r$. If $n = N, N+1, \ldots$; $m = 0, 1, \ldots$ and $|n-m| \leq \alpha$, then $m \geq n - \alpha \geq N - r$. Similarly, if $m = N, N+1, \ldots$;

$n = 0, 1, \dots$ and $|n - m| \le \alpha$, then $n \ge N - r$. This means that

$$Ta = \sum_{\substack{n,m=N-r;\\ |n-m|\le\alpha}}^{\infty} \Delta_{n,m}(a).$$

To the $H^p_\diamond$-quasi-locality of $T$ we need to integrate $|Ta|^p$ on the set $[0,1)^2 \setminus I^r$. Let

$$A_1 := [2^{-N+r}, 1) \times [0, 2^{-N}), \qquad A_2 := [2^{-N}, 1) \times [2^{-N+r}, 1),$$
$$A_3 := [0, 2^{-N}) \times [2^{-N+r}, 1), \qquad A_4 := [2^{-N+r}, 1) \times [2^{-N}, 1).$$

We will show that

$$\int_{A_i} |Ta|^p \le C_p \qquad (i = 1, 2, 3, 4). \tag{17}$$

It is clear that the proof for $i = 3$ and for $i = 4$ is the same as for $i = 1$ and for $i = 2$, respectively. To the proof it is enough to modify the proof of the analoguous estimations given in SIMON [3]. Therefore we give details only for $T^{(\alpha)}_\lambda$ and for $i = 1$.

By our previous remarks it follows that

$$\int_{A_1} |Ta|^p = \int_{2^{-N+r}}^{1} \int_0^{2^{-N}} |Ta|^p$$

$$\le \int_{2^{-N+r}}^{1} \int_0^{2^{-N}} \sum_{i=N-r}^{\infty} \Big| \sum_{k=2^i}^{2^{i+1}-1} \frac{k}{2^i} \sum_{j=N-r, |i-j|\le\alpha}^{\infty} \sum_{l=2^j}^{2^{j+1}-1} \frac{l}{2^j} \hat{a}(k,l) w_{k,l} \Big|^p$$

$$= \sum_{i=N-r}^{\infty} \int_{2^{-N+r}}^{1} \int_0^{2^{-N}} \Big| \int_0^{2^{-N}} \int_0^{2^{-N}} a(s,t) \sum_{k=2^i}^{2^{i+1}-1} \frac{k}{2^i} w_k(x \dot{+} s)$$

$$\times \sum_{\substack{j=N-r,\\ |i-j|\le\alpha}}^{\infty} \sum_{l=2^j}^{2^{j+1}-1} \frac{l}{2^j} w_l(y+t) ds\, dt \Big|^p dx\, dy.$$

Using Hölder's inequality we conclude that

$$\int_{A_1}|Ta|^p\le 2^{-N(1-p)}\sum_{i=N-r}^{\infty}\int_{2^{-N+r}}^{1}\Big(\int_0^{2^{-N}}\Big|\int_0^{2^{-N}}\int_0^{2^{-N}}a(s,t)\sum_{k=2^i}^{2^{i+1}-1}\frac{k}{2^i}w_k(x\dot{+}s)$$

$$\times\sum_{\substack{j=N-r,\\|i-j|\le\alpha}}^{\infty}\sum_{l=2^j}^{2^{j+1}-1}\frac{l}{2^j}w_l(y\dot{+}t)ds\,dt\Big|dy\Big)^p dx$$

$$\le 2^{-N(1-p)}\sum_{i=N-r}^{\infty}\int_{2^{-N+r}}^{1}\Big(\int_0^{2^{-N}}\int_0^{2^{-N}}\Big|\int_0^{2^{-M}}a(s,t)$$

$$\times\sum_{\substack{j=N-r,\\|i-j|\le\alpha}}^{\infty}\sum_{l=2^j}^{2^{j+1}-1}\frac{l}{2^j}w_l(y\dot{+}t)dt\Big|dy\times\Big|\sum_{k=2^i}^{2^{i+1}-1}\frac{k}{2^i}w_k(x\dot{+}s)\Big|ds\Big)^p dx.$$

From this it follows by Cauchy's inequality that

$$\int_{A_1}|Ta|^p\le 2^{-N(1-p)}$$

$$\times\sum_{i=N-r}^{\infty}\int_{2^{-N+r}}^{1}\Big(\int_0^{2^{-N}}2^{-M/2}\Big[\int_0^1\Big|\int_0^{2^{-N}}a(s,t)$$

$$\times\sum_{\substack{j=N-r,\\|i-j|\le\alpha}}^{\infty}\sum_{l=2^j}^{2^{j+1}-1}\frac{l}{2^j}w_l(y\dot{+}t)dt\Big|^2dy\Big]^{1/2}\times\Big|\sum_{k=2^i}^{2^{i+1}-1}\frac{k}{2^i}w_k(x\dot{+}s)\Big|ds\Big)^p dx$$

$$\le 2^{-N(1-p/2)}\sum_{i=N-r}^{\infty}\int_{2^{-N+r}}^{1}\Big(\int_0^{2^{-N}}\Big[\int_0^1|a(s,t)|^2dt\Big]dy\Big]^{1/2}\Big|\sum_{k=2^i}^{2^{i+1}-1}\frac{k}{2^i}w_k(x\dot{+}s)\Big|ds\Big)^p dx.$$

Now, applying the formulas (1), (2) and (3) we obtain

$$\int_{A_1}|Ta|^p\le 2^{-N(1-p/2)}$$

$$\times\sum_{i=N-r}^{\infty}\int_{2^{-N+r}}^{1}\Big(\int_0^{2^{-N}}\Big[\int_0^1|a(s,t)|^2dt\Big]^{1/2}\sum_{l=0}^{i}2^{l-i-1}D_{2^i}(x\dot{+}s\dot{+}2^{-l-1})ds\Big)^p dx$$

$$\le 2^{-N(1-p/2)} \sum_{i=N-r}^{\infty} 2^{-(i+1)p}$$

$$\times \int_{2^{-N+r}}^{1} \Big( \sum_{l=0}^{N-r-1} 2^{l} \int_0^{2^{-N}} \Big[\int_0^1 |a(s,t)|^2 dt\Big]^{1/2} D_{2^i}(x \dot{+} s \dot{+} 2^{-l-1}) ds\Big)^p dx$$

$$\le 2^{-N(1-p/2)} \sum_{i=N-r}^{\infty} 2^{-(i+1)p}$$

$$\times \Big( \sum_{l=0}^{N-r-1} 2^{pl} \int_{2^{-N+r}}^{1} \Big(\int_0^{2^{-N}} \Big[\int_0^1 |a(s,t)|^2 dt\Big]^{1/2} D_{2^i}(x \dot{+} s \dot{+} 2^{-l-1}) ds\Big)^p dx$$

$$= 2^{-N(1-p/2)} \sum_{i=N-r}^{\infty} 2^{-(i+1)p}$$

$$\times \Big( \sum_{l=0}^{N-r-1} 2^{pl} \int_{2^{-l-1}}^{2^{-l-1}+2^{-N}} \Big(\int_0^{2^{-N}} \Big[\int_0^1 |a(s,t)|^2 dt\Big]^{1/2} D_{2^i}(x \dot{+} s \dot{+} 2^{-l-1}) ds\Big)^p dx$$

$$\le 2^{-N(1-p/2)} \sum_{i=N-r}^{\infty} 2^{-(i+1)p}$$

$$\times \Big( \sum_{l=0}^{N-r-1} 2^{pl} \int_{2^{-l-1}}^{2^{-l-1}+2^{-N}} \Big(\Big[\int_0^{2^{-N}} \int_0^1 |a(s,t)|^2 dt\, ds\Big]^{1/2}$$

$$\times \Big[\int_0^{2^{-N}} D_{2^i}^2(x \dot{+} s \dot{+} 2^{-l-1}) ds\Big]^{1/2}\Big)^p dx$$

$$= 2^{-N(1-p/2)} \sum_{i=N-r}^{\infty} 2^{-(i+1)p} \|a\|_2^p \sum_{l=0}^{N-r-1} 2^{pl-N} 2^{ip/2}$$

$$\le 2^{-N(1-p/2)} 2^{-Np+2N)} 2^{-N} 2^{-p} \sum_{i=N-r}^{\infty} 2^{-ip/2} 2^{p(N-r)}$$

$$\le C_p 2^{Np/2} 2^{-(N-r)p/2} \le C_{p,r}.$$

Applying Theorem E and the duality argument we get

**Corollary 2.** *Let $\lambda$ be defined as above, $\alpha \ge 0$, $0 < p < \infty$, $0 < q \le \infty$. Then the operators $T_\lambda^{(\alpha)}$, $T_{1/\lambda}^{(\alpha)}$ are bounded from $H_\diamond^{p,q}$ into $L^{p,q}$. Especially, $T_\lambda^{(\alpha)}$, $T_{1/\lambda}^{(\alpha)}$ are of weak type (1,1).*

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*(Received October 22, 1997; revised February 9, 1998)*