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Oscillation criteria for nonlinear differential equations with several deviating arguments

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Abstract. In this paper we reduce the problem of the oscillation of the solutions of nonlinear differential equations with several deviating arguments of the form $\frac{d}{dt}\frac{1}{a_{n-1}(t)}\frac{d}{dt}\cdots\frac{d}{dt}\frac{1}{a_1(t)}\frac{d}{dt}x(t)\pm f(t,x[g_1(t)],\dots,x[g_m(t)])=0$ to the problem of oscillation of a certain set of second order ordinary differential equations of the type

$$\left(\frac{1}{a_i(t)}\dot{y}(t)\right)^{\cdot}+Q_i(t)|y(t)|^{\lambda}\mathrm{sgn}y(t)=0,\ i=1,2,\dots,n-1 \text{ and } \lambda>0.$$

The obtained criterion extends the results by Lovelady, Kusano, Naito and Trench in such a way that they can be applied in cases of nonlinear differential equations with several deviating arguments.

1. Introduction

Consider the functional differential equation

$$(E;\ \delta)\qquad L_nx(t)+\delta f(t,x[g_1(t)],\dots,x[g_m(t)])=0,$$

where $\delta=\pm1$, $n\geq3$, $L_0x(t)=x(t)$, $L_kx(t)=\frac{1}{a_k(t)}(L_{k-1}x(t))^{\cdot}$, $k=1,2,\dots,n$, $\left(^{\cdot}=\frac{d}{dt}\right)$, $a_n(t)=1$, $a_i:[t_0,\infty)\to(0,\infty)$, i=$1,2,\dots,n-1$, $g_j:[t_0,\infty)\to\mathbf{R}$, $j=1,2,\dots,m$, $f:[t_0,\infty)\times\mathbf{R}^m\to\mathbf{R}$ are continuous and $\lim\limits_{t\to\infty}g_j(t)=\infty$, $j=1,2,\dots,m$.

We will assume that

$$(1)\qquad \int^{\infty}a_i(s)ds=\infty,\qquad i=1,2,\dots,n-1.$$

There exist continuous functions $q:[t_0,\infty)\to\mathbf{R}_+=[0,\infty)$ and $\sigma:[t_0,\infty)\to\mathbf{R}$, and nonnegative constants λ_i $(i=1,2,\dots,m)$ with

$\sum_{i=1}^{m} \lambda_i = \lambda > 0$ such that

$$f(t, x_1, \ldots, x_m)\mathrm{sgn} x_1 \geq q(t) \prod_{i=1}^{m} (|x_i|)^{\lambda_i} \text{ for } t \in [t_0, \infty) \tag{2}$$

and $x_1 x_i > 0 \quad (i = 1, 2, \ldots, m),$ and

$$\sigma(t) = \min\{t, g_1(t), \ldots, g_m(t)\}. \tag{3}$$

The domain of $L_n, D(L_n)$ is defined to be the set of all functions $x : [T_x, \infty) \to \mathbf{R}$ such that $L_j x(t),\ j = 0, 1, \ldots, n$ exist and are continuous on $[T_x, \infty),\ T_x \geq t_0$. A solution of equation $(E; \delta)$ is called oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory. Equation $(E; \delta)$ is said to be oscillatory if all of its solutions are oscillatory.

Equation $(E; \delta)$ is said to be almost oscillatory if:

(i) for $\delta = 1$ and n even, every solution of equation $(E; \delta)$ is oscillatory;

(ii) for $\delta = 1$ and n odd, every solution x of equation $(E; \delta)$ is either oscillatory or $|L_i x(t)| \to 0$ monotonically as $t \to \infty,\ i=0, 1, \ldots, n-1$;

(iii) for $\delta = -1$ and n even, every solution x of equation $(E; \delta)$ is oscillatory, $|L_i x(t)| \to 0$ monotonically as $t \to \infty,\ i = 0, 1, \ldots, n-1$ or else $|L_i x(t)| \to \infty$ monotonically as $t \to \infty,\ i = 0, 1, \ldots, n-1$;

(iv) for $\delta = -1$ and n odd, every solution x of equation $(E; \delta)$ is either oscillatory or $|L_i x(T)| \to \infty$ monotonically as $t \to \infty,\ i = 0, 1, \ldots, n-1$.

Next, equation $(E; \delta)$ is said to have "property A" if (i) and (ii) are satisfied and (iii) and (iv) are replaced by:

(iii)' for $\delta = -1$ and n even, every solution $x(t)$ of equation $(E; \delta)$ is either oscillatory or $|L_i x(t)| \to 0$ monotonically as $t \to \infty,\ i=0, 1, \ldots, n-1$;

(iv)' for $\delta = -1$ and n odd, every solution of equation $(E; \delta)$ is oscillatory, respectively.

Also, equation $(E; \delta)$ is said to have "property B" if (i) and (iv) are satisfied and (ii) and (iii) are replaced by:

(ii)' for $\delta = 1$ and n odd, every solution of equation $(E; \delta)$ is oscillatory.

(iii)' for $\delta = -1$ and n even, every solution $x(t)$ of equation $(E; \delta)$ is either oscillatory or $|L_i x(t)| \to \infty$ monotonically as $t \to \infty,\ i=0, 1, \ldots, n-1$, respectively.

The behavioral properties of the solutions of the equation $(E; \delta)$ and/or related equations have been discussed by numerous authors using various techniques and as recent contributions to this study we cite the papers of Chanturya [1], Grace and Lalli ([2] - [4]), Kitamura [5], Kusano and Naito [6], Lovelady [7] and [8], Philos [9] and [10], Trench ([11]–[13]), Willett [14] and Werbowski [15].

LOVELADY [7] and [8] considered the retarded equation

$$(L;\ \delta) \qquad x^{(n)}(t) + \delta q(t) x[g(t)] = 0,$$

where $\delta = \pm 1$, $n > 3$, $g, q : [t_0, \infty) \to \mathbf{R}$ are continuous, $q(t) \geq 0$ and not identically zero for all large $t, g(t) \leq t$ and $g(t) \to \infty$ as $t \to \infty$, and established some interesting results. He has related oscillation of all solutions of equation $(L; \delta)$ to oscillation of certain associated second order equations.

LOVELADY's results for the equation $(L; 1)$ have been extended by Trench [11] to more general equations of the form

$$x^{(n)}(t) + f(t, x(t)) = 0,$$

where $n \geq 3$, $f : [t_0, \infty) \times \mathbf{R} \to \mathbf{R}$ is continuous,

$$\frac{f(t,x)}{x} \geq q(t) \geq 0 \text{ for } x \neq 0 \text{ and } t \geq t_0,$$

and q is defined as in equation $(L; \delta)$.

On the other hand, the result proved in [8] has been generalized by KUSANO and NAITO [6] to the general linear equation

$$L_n x(t) + q(t) x(t) = 0,$$

where L_n and q are defined as in equation $(E; \delta)$.

The interesting results of ([6]–[8] and [11]) have limited applications, since they are applicable only to linear or almost linear equations and fail to apply to other classes of nonlinear differential equations with general deviating arguments.

Therefore, the purpose of this paper is to extend the results of LOVEDAY, KUSANO and NAITO and TRENCH to equations of type $(E; \delta)$ with general deviating arguments.

The main results of this paper are presented in the form of four theorems. In Theorem 1, we give a sufficient condition for equation $(E; \delta)$ to be almost oscillatory, while Theorem 2 [respectively Theorem 3] concerns with a sufficient condition so that equation $(E; \delta)$ has property A [respectively property B]. Theorem 4 deals with the oscillatory behaviour of equation $(E; \delta)$ when $\lambda = 1$.

2. Main results

We begin by formulating preparatory results which are needed in proving our main results.

For functions $p_i : [t_0, \infty) \to \mathbf{R},\ i = 1, 2, \dots$, we define

$$I_0 = 1$$

$$I_i(t, s; p_i, \dots, p_1) = \int_s^t p_i(u) I_{i-1}(u, s; p_{i-1}, \dots, p_1)du,\ i = 1, 2, \dots .$$

It is easy to verify that for $i = 1, 2, \dots, n-1$

$$I_i(t, s; p_1, \dots, p_i) = (-1)^i I_i(s, t; p_i, \dots, p_1)$$

and

$$I_i(t, s; p_1, \dots, p_i) = \int_s^t p_i(u) I_{i-1}(t, u; p_1, \dots, p_{i-1})du.$$

The following two lemmas will be needed in the proofs of the main results.

Lemma 1. *If $x \in D(L_n)$, then for $t, s \in [t_0, \infty)$ and $0 \le i < k \le n$*

(i) $$L_i x(t) = \sum_{j=i}^{k-1} I_{j-i}(t, s; a_{i+1}, \dots, a_j) L_j x(s) + \int_s^t I_{k-i-1}(t, u; a_{i+1}, \dots, a_{k-1}) a_k(u) L_k x(u) du,$$

(ii) $$L_i x(t) = \sum_{j=i}^{k-1} (-1)^{j-i} I_{j-i}(s, t; a_j, \dots, a_{i+1}) L_j x(s) + (-1)^{k-i} \int_t^s I_{k-i-1}(u, t; a_{k-1}, \dots, a_{i+1}) a_k(u) L_k x(u) du.$$

This lemma is a generalization of Taylor's formula with remainder encountered in calculus. The proof is immediate.

Lemma 2. *Suppose condition (1) holds. If $x \in D(L_n)$ is of constant sign and not identically zero for all large t, then there exist a $t_x \ge t_0$ and an integer $\ell, 0 \le \ell \le n$ with $n + \ell$ even for $x(t)L_n x(t)$ nonnegative or $n + \ell$ odd for $x(t)L_n x(t)$ nonpositive and such that for every $t \ge t_x$*

$$\ell > 0 \text{ implies } x(t)L_k x(t) > 0 \qquad (k = 0, 1, \dots, \ell)$$

and

$$\ell \leq n-1 \text{ implies } (-1)^{\ell-k} x(t) L_k x(t) > 0 \quad (k = \ell, \ell+1, \dots, n).$$

This lemma generalizes a well–known lemma of KIGUARADZE and can be proved similarly.

It will be convenient to make use of the following notation in the remainder of this paper. For any $T \geq t_0$ and all $t \geq s \geq T$ we let

$$\alpha_i[t,s] = I_i(t,s;a_1,\dots,a_i), \quad i = 1,2,\dots,n-1;$$
$$\zeta_i[t,s] = I_i(t,s;a_i,\dots,a_1), \quad i = 1,2,\dots,n-1;$$
$$\beta_i[t,s] = I_{n-i-1}(t,s;a_{n-1},\dots,a_{i+1}), \quad i = 1,2,\dots,n-1;$$
$$R_i[t,T] = \int_T^t a_i(s)ds, \quad i = 1,2,\dots,n-1;$$
$$\gamma_{i,j}[\sigma(t),T] = \int_T^{\sigma(t)} \alpha_{i-2}[g_j(t),s]a_{i-1}(s)R_i[s,T]ds, \quad i = 2,3,\dots,n-1, \text{ and } j = 1,2,\dots,m;$$
$$\gamma_{1,j}[\sigma(t),T] = R_1[\sigma(t),T], \quad j = 1,2,\dots,m.$$

In the following theorem, we give a sufficient condition for equation $(E;\delta)$ to be almost oscillatory. In fact, we relate oscillation of equation $(E;\delta)$ to oscillation of a certain set of second order nonlinear ordinary differential equations; namely

$$\text{(4; i)} \qquad \left(\frac{1}{a_i(t)}\dot{y}(t)\right)^{\cdot} + Q_i(t,T)|y(t)|^{\lambda}\operatorname{sgn} y(t) = 0, \ i = 1,2,\dots,n-1,$$

for T sufficiently large with $\sigma(t) > T$, where

$$Q_i(t,T) = a_{i+1}(t)\int^{\infty} \beta_{i+1}[u,t]q(u)\prod_{j=1}^{m}\left(\frac{\gamma_{i,j}[\sigma(u),T]}{R_i[u,T]}\right)^{\lambda_j} du, \qquad i = 1,2,\dots,n-2$$

$$Q_{n-1}(t,T) = q(t)\prod_{J=1}^{m}\left(\frac{\gamma_{n-1,j}[\sigma(t),T]}{R_{n-1}[t,T]}\right)^{\lambda_j}.$$

Theorem 1. *Suppose that conditions (1), (2) and (3) hold. Then equation $(E;\delta)$ is almost oscillatory if:*

(i) *for* $\delta = 1$ *and* n *even, the equations* $(4; i)$ $(i = 1, 3, \dots, n-1)$ *are oscillatory;*

(ii) *for* $\delta = 1$ *and* n *odd, the equations* $(4; i)$ $(i = 2, 4, \dots, n-1)$ *are oscillatory and for all large* T

$$\int^{\infty} \beta_0[s, T] q(s) ds = \infty; \tag{5}$$

(iii) *for* $\delta = -1$ *and* n *even, the equations* $(4; i)$ $(i = 2, 4, \dots, n-2)$ *are oscillatory, condition* (5) *and for all large* T

$$\int^{\infty} q(s) \prod_{i=1}^{m} (\alpha_{n-1}[g_i(s), T])^{\lambda_i} ds = \infty, \tag{6}$$

are satisfied;

(iv) *for* $\delta = -1$ *and* n *odd, the equations* $(4; i)$ $(i = 1, 3, \dots, n-2)$ *are oscillatory and condition* (6) *is satisfied.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation $(E; \delta)$. Without loss of generality, we assume that $x(t) \neq 0$ for all $t \geq t_0$. Furthermore, we suppose that $x(t) > 0$, $x[g_i(t)] > 0$ $(i = 1, 2, \dots, m)$ and $x[\sigma(t)] > 0$ for $t \geq t_0$, since the substitution $w = -x$ transforms equation $(E; \delta)$ into an equation of the same form subject to the assumptions of the theorem.

By Lemma 2, there exist a $t_1 \geq t_0$ and an integer $\ell \in \{0, 1, \dots, n\}$ with $n + \ell$ odd if $\delta = 1$ or $n + \ell$ even if $\delta = -1$ such that

$$\begin{cases} x(t) L_k x(t) > 0 & \text{for } t \geq t_1, \ (k = 1, 2, \dots, \ell), \\ (-1)^{\ell - k} x(t) L_k x(t) > 0 & \text{for } t \geq t_1, \ (k = \ell, \ell+1, \dots, n). \end{cases} \tag{7}$$

Suppose $\ell \in \{1, 2, \dots, n-2\}$. Then, from Lemma 1 (ii) we obtain

$$L_{\ell+1} x(t) = \sum_{j=\ell+1}^{n-1} (-1)^{j-\ell-1} I_{j-\ell-1}(t, s; a_j, \dots, a_{\ell+2}) L_j x(s) + (-1)^{n-\ell-1} \int_t^s I_{n-\ell-2}(u, t; a_{n-1}, \dots, a_{\ell+2}) L_n x(u) du$$

for $s \geq t \geq t_1$. Using (2), (7) and letting $s \to \infty$ we have

$$-L_{\ell+1} x(t) \geq \int_t^{\infty} \beta_{\ell+1}[u, t] q(u) \prod_{i=1}^{m} (x[g_i(u)])^{\lambda_i} du, \qquad t \geq t_1. \tag{8}$$

Again, from Lemma 1 (i) and $\ell \in \{2, 3, \dots, n-2\}$ we get

$$x(t) = \sum_{j=0}^{\ell-2} I_j(t, t_1; a_1, \dots, a_j) L_j x(t_1) +$$

$$+ \int_{t_1}^{t} I_{\ell-2}(t, u; a_1, \dots, a_{\ell-2}) a_{\ell-1}(u) L_{\ell-1} x(u) du$$

$$\geq \int_{t_1}^{t} \alpha_{\ell-2}[t, u] a_{\ell-1}(u) L_{\ell-1} x(u) du, \qquad t \geq t_1.$$

There exists a $t_2 \geq t_1$ so that $\sigma(t) \geq t_1$ for all $t \geq t_2$ and

$$x[g_j(t)] \geq \int_{t_1}^{\sigma(t)} \alpha_{\ell-2}[g_j(t), u] a_{\ell-1}(u) L_{\ell-1} x(u) du, \ j = 1, 2, \dots, m. \tag{9}$$

From the definition of the operator L_n, we have

$$(L_{\ell-1} x(t))^{\cdot} = a_\ell(t) L_\ell x(t) \qquad \text{for } t \geq t_2. \tag{10}$$

Integrating (10) from t_2 to t and using the fact that $L_\ell x(t)$ is nonincreasing for $t \geq t_1$, we get

$$L_{\ell-1} x(t) = L_{\ell-1} x(t_2) + R_\ell[t, t_2] L_\ell x(t) - \int_{t_2}^{t} R_\ell[s, t_2] a_{\ell+1} L_{\ell+1} x(s) ds.$$

Since $L_{\ell+1} x(t) < 0$ for $t \geq t_1$ we obtain

$$L_{\ell-1} x(t) \geq R_\ell[t, t_2] L_\ell x(t) \text{ for } t \geq t_2. \tag{11}$$

From (11), we can easily see that the function

$$\frac{L_{\ell-1} x(t)}{R_\ell[t, t_2]} \text{ is nonincreasing for } t \geq t_3 \text{ for some } t_3 > t_2. \tag{12}$$

Thus, inequality (9) takes the form

$$\begin{aligned} x[g_j(t)] &\geq \frac{L_{\ell-1} x(t)}{R_\ell[t, t_2]} \int_{t_3}^{\sigma(t)} \alpha_{\ell-2}[g_j(t), u] a_{\ell-1}(u) R_\ell[u, t_2] du \\ &= \frac{\gamma_{\ell,j}[\sigma(t), t_2]}{R_\ell[t, t_2]} L_{\ell-1} x(t) \quad \text{for } t \geq t_3, \ j = 1, 2, \dots, m. \end{aligned} \tag{13}$$

Next, let $\ell = 1$. From (7), (12) and condition (3) we have

$$x[g_j(t)] \geq x[\sigma(t)] \geq \frac{R_1[\sigma(t), t_2]}{R_1[t, t_2]} x(t)$$
$$= \frac{\gamma_{1,j}[\sigma(t), t_2]}{R_1[t, t_2]} L_0 x(t) \text{ for } t \geq t_3,\ j = 1, 2, \dots, m.$$

That is inequality (13) holds for $\ell = 1$.

Combining (8) and (13) and using condition (3) we get

$$-L_{\ell+1}x(t) \geq \int_t^\infty \beta_{\ell+1}[u, t] q(u) \prod_{j=1}^m \left(\frac{\gamma_{\ell,j}[\sigma(u), t_2]}{R_\ell[u, t_2]}\right)^{\lambda_j} (L_{\ell-1}x(u))^\lambda du,$$

or

$$(14) \qquad -\left(\frac{1}{a_\ell(t)}(L_{\ell-1}x(t))^\cdot\right)^\cdot \geq Q_\ell(t, t_2)(L_{\ell-1}x(t))^\lambda \text{ for } t \geq t_3.$$

Set $w(t) = L_{\ell-1}x(t)$. Then inequality (14) takes the form

$$\left(\frac{1}{a_\ell(t)}\dot{w(t)}\right)^\cdot + Q_\ell(t, t_2)(w(t))^\lambda \leq 0 \text{ for } t \geq t_3.$$

Now, from Lemma 2 in [3] (also, see [1]), it follows that the equation

$$\left(\frac{1}{a_\ell(t)}\dot{w(t)}\right)^\cdot + Q_\ell(t, t_2)(w(t))^\lambda = 0,\ t \geq t_3$$

has a nonoscillatory solution. But this is impossible by the hypothesis.

Next, let $\ell = n-1$. From inequality (13), condition (2) and equation $(E;1)$, we have

$$-L_n x(t) = -\left(\frac{1}{a_{n-1}(t)}(L_{n-2}x(t))^\cdot\right)^\cdot$$
$$\geq q(t) \prod_{j=1}^m \left(\frac{\gamma_{n-1,j}[\sigma(t), t_2]}{R_{n-1}[t, t_2]}\right)^{\lambda_j} (L_{n-2}x(t))^\lambda$$
$$= Q_{n-1}(t, t_2)(L_{n-2}x(t))^\lambda \text{ for } t \geq t_3^* \text{ for some } t_3^* > t_2.$$

Let $v(t) = L_{n-2}x(t)$. Then the above inequality becomes

$$\left(\frac{1}{a_{n-1}(t)}\dot{v(t)}\right)^\cdot + Q_{n-1}(t, t_2)(v(t))^\lambda \leq 0 \text{ for } t \geq t_3^*.$$

Again, by Lemma 2 in [3], we see that the equation

$$\left(\frac{1}{a_{n-1}(t)}\dot{v(t)}\right)^{\cdot} + Q_{n-1}(t,t_2)(v(t))^{\lambda} = 0,\ t \geq t_3^*,$$

has a nonoscillatory solution, contradicting the hypothesis.

The proofs when $\ell = 0$ and $\ell = n$ are similar to the proofs of those cases in Theorems 1 and 3 in [2] and also in [9], and hence will be omitted. This completes the proof.

Remarks. **1.** Theorem 1 extends Lovelady's results in [7] and [8] and the work of Kusano and Naito [6] in such a way that they can be applied in cases of nonlinear differential equations and especially with general deviating arguments.

2. Theorem 1 extends and improves some of the known oscillation criteria appeared in the literature. In particular, Theorem 1 can be applied to cases in which Theorems 3.2, 3.3 and 5.1 in [5] and some of the results in [1]–[3] and [6]–[15] are not applicable. Such cases are described in the following examples.

Example 1. Consider the fourth order equation

$$(E_1;\ \delta) \qquad \left(\frac{1}{t}\left(\frac{1}{t}\left(\frac{1}{t}\dot{x(t)}\right)^{\cdot}\right)^{\cdot}\right)^{\cdot} + \delta c t^{c_1}|x[t^{c_2}]|^{\lambda}\operatorname{sgn} x[t^{c_2}] = 0,\ t > 0,$$

where $\delta = \pm 1,\ \lambda,\ c$ and c_2 are positive constants and c_1 is any constant. Here, we take

$$a_i(t) = t,\ i = 1,2,3,\ q(t) = ct^{c_1},\ g(t) = t^{c_2} \text{ and } f(x) = |x|^{\lambda}\operatorname{sgn} x.$$

It is easy to check that all conditions of Theorem 1 are satisfied for equation (E_1, δ) in the following special cases:

(i) $\quad \lambda = \frac{5}{3},\ c_1 = -\frac{14}{3},\ c_2 = \frac{1}{2}$ and $c > 0$ (i.e. superlinear),

(ii) $\quad \lambda = 1,\ c_1 = -7,\ c_2 = 1$ and $c > 0$ (i.e. linear), and

(iii) $\quad \lambda = \frac{1}{3},\ c_1 = -\frac{13}{3},\ c_2 = 3$ and $c > 0$ (i.e. sublinear),

and hence equation $(E_1; \delta)$ is almost oscillatory. One can easily check that the results in [1]–[3], [6]–[15] and Theorems 3.2, 3.3 and 5.1 in [5] are not applicable to equation $(E_1; \delta)$ for the above cases.

In what follows we let

$$\mathcal{D} = \{t \in [t_0, \infty) \mid g_i(t) \leq t \quad (i = 1,2,\dots,m)\}$$
$$\mathcal{A} = \{t \in [t_0, \infty) \mid g_i(t) \geq t \quad (i = 1,2,\dots,m)\}.$$

In the case when the deviating arguments $g_i(t)$ $(i = 1, 2, \dots, m)$ are of mixed type, we assume that there exist nondecreasing continuous functions

$$\tau, \xi : [t_0, \infty) \to \mathbf{R}$$

such that

$$\begin{aligned} &\tau(t) \le t \le \xi(t) && \text{for } t \ge t_0, \\ &g_i(t) \le \tau(t) && \text{for } t \in \mathcal{D}, \quad i = 1, 2, \dots, m, \\ &\xi(t) \le g_i(t) && \text{for } t \in \mathcal{A}, \quad i = 1, 2, \dots, m. \end{aligned}$$

Also, we let

$$\mathcal{D}(t) = \mathcal{D} \cap [\tau(t), t] \text{ and } \mathcal{A}(t) = \mathcal{A} \cap [t, \xi(t)].$$

In the following result we present a sufficient condition so that equation $(E; \delta)$ has property A.

Theorem 2. *Let conditions* (1) – (3) *hold. Then equation* $(E; \delta)$ *has property* A *if:*

(i) *for* $\delta = 1$ *and* n *even, the equations* $(4; i)$ $(i = 1, 3, \dots, n-1)$ *are oscillatory;*

(ii) *for* $\delta = 1$ *and* n *odd, the equations* $(4; i)$ $(i = 2, 4, \dots, n-1)$ *are oscillatory and condition (5) holds;*

(iii) *for* $\delta = -1$, $\lambda > 1$ *and* n *even, the equations* $(4; i)$ $(i = 2, 4, \dots, n-2)$ *are oscillatory, condition (5) and*

$$\int\limits_{\mathcal{A}} a_j[\xi(t)]\dot{\xi}(t) \int\limits_{\mathcal{A}(t)} I_{n-j-1}(\xi(t), s; a_{j+1}, \dots, a_{n-1}) q(s) \times \prod_{i=1}^{m} (\alpha_{j-1}[g_i(s), \xi(t)])^{\lambda_i} ds dt = \infty \tag{15}$$

for some $j = 1, 2, \dots, n-1$ *are satisfied.*

(iv) *for* $\delta = -1$, $\lambda > 1$ *the equations* $(4; i)$, $(i = 1, 3, \dots, n-2)$ *are oscillatory and condition* (15) *is satisfied.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation $(E; \delta)$, say $x(t) > 0$ for $t \ge t_0$. In view of Lemma 2, $x(t)$ satisfies inequalities (7) for some $\ell \in \{0, 1, \dots, n\}$. Proceeding as in the proof of Theorem 1, we see that the case $\ell \in \{0, 1, \dots, n-1\}$ is impossible. Next, we let $\ell = n$. This is the case when $\delta = -1$. Then from (7) we have

$$L_j x(t) > 0 \text{ for } t \ge t_1 \text{ and } j = 0, 1, \dots, n. \tag{16}$$

Applying Lemma 1 (i), for $j = 0, 1, \dots, n-1$ we obtain

$$L_j x(u) = \sum_{i=j}^{n-1} I_{i-j}(u, t; a_{j+1}, \dots, a_i) L_i x(t) + \int_t^u I_{n-j-1}(u, s; a_{j+1}, \dots, a_{n-1}) L_n x(s) ds.$$

Using (16) we get

$$L_j x[\xi(t)] \geq \int_t^{\xi(t)} I_{n-j-1}(\xi(t), s; a_{j+1}, \dots, a_{n-1}) q(s) \prod_{i=1}^{m} (x[g_i(s)])^{\lambda_i} ds. \tag{17}$$

Next, we show the fact that

$$x[g_i(s)] \geq \alpha_{j-1}[g_i(s), \xi(t)] L_{j-1} x[\xi(t)] \tag{18}$$

for $t < s < \xi(t)$ and for any $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n-1$. If $j = 1$, (18) follows from the fact that $x(t)$ and $\xi(t)$ are nondecreasing functions for $t \geq t_1$.

Let $j \geq 2$. From Lemma 1 (i), we have

$$x[g_i(s)] = \sum_{p=0}^{j-2} I_p(g_i(s), \xi(t); a_1, \dots, a_p) L_p x[\xi(t)] + \int_{\xi(t)}^{g_i(s)} I_{j-2}(g_i(s), u; a_1, \dots, a_{j-2}) a_{j-1}(u) L_{j-1} x(u) du$$

for $s \in \mathcal{A}(t)$, $t \geq t_1$. Using (16) and noting that $L_{j-1}x$ is increasing on $[t_1, \infty)$, we easily get (18) from the above equation. Combining (17) with (18), we obtain

$$\frac{L_j x[\xi(t)]}{(L_{j-1} x[\xi(t)])^{\lambda}} \geq \int_t^{\xi(t)} I_{n-j-1}(\xi(t), s; a_{j+1}, \dots, a_{n-1}) q(s) \prod_{i=1}^{m} (\alpha_{j-1}[g_i(s), \xi(t)] L_{j-1} x[\xi(t)])^{\lambda_i} ds,$$

or

$$\frac{(L_{j-1}x[\xi(t)])^{\cdot}\dot{\hat{\xi}}(t)}{(L_{j-1}x[\xi(t)])^{\lambda}} \geq a_j[\xi(t)]\dot{\hat{\xi}}(t)\times$$

$$\int_t^{\xi(t)} I_{n-j-1}(\xi(t),s;a_{j+1},\dots,a_{n-1})q(s)\prod_{i=1}^{m}(\alpha_{j-1}[g_i(s),\xi(t)])^{\lambda_i}ds.$$

Integrating the above inequality on $\mathcal{A}$, $t \geq t_1$, we get

$$\int_{\mathcal{A}} a_j[\xi(t)]\dot{\hat{\xi}}(t)\times$$

$$\int_{\mathcal{A}(t)} I_{n-j-1}(\xi(t),s;a_{j+1},\dots,a_{n-1})q(s)\prod_{i=1}^{m}(\alpha_{j-1}[g_i(s),\xi(t)])^{\lambda_i}dsdt$$

$$\leq \int_{t_1}^{\infty}\frac{(L_{j-1}x[\xi(t)])^{\cdot}\dot{\xi}(t)}{(L_{j-1}x[\xi(t)])^{\lambda}}dt \leq \int_{L_{j-1}x[\xi(t_1)]}^{\infty}\frac{d\eta}{\eta^{\lambda}} < \infty,$$

which contradicts (15). This completes the proof.

In the following theorem we give a sufficient condition so that equation $(E;\delta)$ has property B.

Theorem 3. *Suppose that conditions* (1)–(3) *hold. A sufficient condition for equation* $(E;\delta)$ *to have property* B *is that:*

(i) *when* $\delta = 1$ *and* n *is even, the equations* $(4;i)$ $(i = 1,3,\dots,n-1)$ *are oscillatory;*

(ii) *when* $\delta = 1$, $\lambda < 1$ *and* n *is odd, the equations* $(4;i)$ $(i = 2,4,\dots,n-1)$ *are oscillatory and*

(19)
$$\int_{\mathcal{D}} a_{j+1}[\tau(t)]\dot{\tau}(t)\int_{\mathcal{D}(t)}\beta_{j+1}[s,\tau(t)]q(s)\prod_{i=1}^{m}(\zeta_j[\tau(t),g_i(s)])^{\lambda_i}dsdt = \infty$$

for some $j = 0,1,\dots,n-2$, *holds;*

(iii) *when* $\delta = -1$, $\lambda > 1$ *and* n *is even, the equations* $(4;i)$ $(i = 2,4,\dots,n-2)$ *are oscillatory, condition (6) and (19) hold;*

(iv) *when* $\delta = -1$ *and* n *is odd, the equations* $(4;i)$, $(i = 1,3,\dots,n-2)$ *are oscillatory and condition* (6) *holds.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation $(E;\delta)$, say $x(t) > 0$ for $t \geq t_0$. In view of Lemma 2, $x(t)$ satisfies inequalities (7) for some $\ell \in \{0, 1, \ldots, n\}$. Proceeding as in the proof of Theorem 1, we see that the case $\ell \in \{1, 2, \ldots, n\}$ is impossible. Next, we let $\ell = 0$. This is the case when $\delta = 1$ and n is odd, or $\delta = -1$ and n is even. Thus from (9) we have

$$(-1)^k L_k x(t) > 0 \text{ for } t \geq t_1, \qquad k = 0, 1, \ldots, n-1. \tag{20}$$

We first show that

$$x[g_i(s)] \geq \zeta_j[\tau(t), g_i(s)]|L_j x[\tau(t)]| \tag{21}$$

for $s \in \mathcal{D}(t),\ t \geq t_2 \geq t_1$ and any $j = 0, 1, \ldots, n-2$. If $j = 0$, this follows from the fact that $\tau(t)$ is nondecreasing and $x(t)$ is decreasing. Let $j \geq 1$. From Lemma 1 (ii) we obtain

$$\begin{aligned} x[g_i(s)] = &\sum_{p=0}^{j-1} (-1)^p I_p(\tau(t), g_i(s); a_p, \ldots, a_1) L_p x[\tau(t)] \\ &+ (-1)^j \int\limits_{g_i(s)}^{\tau(t)} I_{j-1}(u, g_i(s); a_{j-1}, \ldots, a_1) a_j(u) L_j x(u) du, \end{aligned}$$

which, in view of the decreasing nature of $|L_j x(t)|$, readily implies (21).

Again, we apply Lemma 1 (ii):

$$\begin{aligned} |L_{j+1} x[\tau(t)]| &= (-1)^{j+1} L_{j+1} x[\tau(t)] \\ &= \sum_{i=j+1}^{n-1} (-1)^i I_{i-j-1}(t, \tau(t); a_i, \ldots, a_{j+1}) L_i x(t) \\ &\qquad + (-1)^n \int\limits_{\tau(t)}^{t} I_{n-j-2}(s, \tau(t); a_{n-1}, \ldots, a_j) L_n(s) ds. \end{aligned}$$

Thus,

$$|L_{j+1} x[\tau(t)]| \geq \int\limits_{\tau(t)}^{t} \beta_{j+1}[s, \tau(t)] q(s) \prod_{i=1}^{m} (x[g_i(s)])^{\lambda_i} ds. \tag{22}$$

Combining (21) with (22) we have

$$|L_{j+1} x[\tau(t)]| \geq (|L_j x[\tau(t)]|)^{\lambda} \int\limits_{D(t)} \beta_{j+1}[s, \tau(t)] q(s) \prod_{i=1}^{m} (\zeta_j[\tau(t), g_i(s)])^{\lambda_i} ds$$

or

$$a_{j+1}[\tau(t)]\dot{\tau(t)} \int_{\mathcal{D}(t)} \beta_{j+1}[s,\tau(t)]q(s) \prod_{i=1}^{m} (\zeta_j[\tau(t), g_i(s)])^{\lambda_i} ds \le \frac{|(L_j x[\tau(t)])^{\cdot}|\dot{\tau(t)}}{(|L_j x[\tau(t)]|)^{\lambda}}.$$

Integrating the above in $\mathcal{D} \cap [t_2, \infty)$ we have

$$\int_{\mathcal{D}\cap[t_2,\infty)} a_{j+1}[\tau(t)]\dot{\tau(t)} \int_{\mathcal{D}(t)} \beta_{j+1}[s,\tau(t)]q(s) \prod_{i=1}^{m} (\zeta_j[\tau(t), g_i(s)])^{\lambda_i} ds dt \le \int_0^{|L_j x[\tau(t_2)]|} \frac{dw}{w^{\lambda}} < \infty,$$

which contradicts (19). This completes the proof.

The following theorem is concerned with the oscillatory behavior of equation $(E;\delta)$ when f satisfies condition (2) with $\lambda = 1$.

Theorem 4. *Let conditions* (1), (3) *and* (2) *with* $\lambda = 1$ *hold. Equation* (E,δ) *is oscillatory if:*

(i) *for* $\delta = 1$ *and* n *even, the equations* $(4;i)$ $(i = 1, 3, \dots, n-1)$ *are oscillatory;*

(ii) *for* $\delta = 1$ *and* n *odd, the equations* $(4;i)$ $(i = 2, 4, \dots, n-1)$ *are oscillatory, and*

$$\limsup_{t\to\infty} \int_{\mathcal{D}(t)} \beta_j[s,\tau(t)]q(s) \prod_{i=1}^{m} (\zeta_j[\tau(t), g_i(s)])^{\lambda_i} ds > 1 \tag{23}$$

for some $j = 0, 1, \dots, n-1$, *is satisfied;*

(iii) *for* $\delta = -1$ *and* n *even, the equations* $(4;i)$ $(i = 2, 4, \dots, n-2)$ *are oscillatory, condition (23) and*

$$\limsup_{t\to\infty} \int_{\mathcal{A}(t)} I_{n-j-1}[\xi(t), s; a_{j+1}, \dots, a_{n-1})q(s) \times \prod_{i=1}^{m} (\alpha_j[g_i(s), \xi(t)])^{\lambda_i} ds > 1 \tag{24}$$

for some $j = 0, 1, \dots, n-1$, *are satisfied;*

(iv) *for* $\delta = -1$ *and* n *odd, the equations* $(4;i)$, $(i = 1, 3, \dots, n-2)$ *are oscillatory and condition* (24) *is satisfied.*

PROOF. Let $x(t)$ be a nonoscillatory solution of equation $(E;\delta)$. Assume $x(t) > 0$ for $t \geq t_0$. In view of Lemma 2, $x(t)$ satisfies the inequalities (7) for some $\ell \in \{0, 1, \dots, n\}$. Proceeding as in the proof of Theorem 1, we see that the case $\ell \in \{1, 2, \dots, n-1\}$ is impossible. Now, we consider the following two cases:

Case 1: Let $\ell = 0$. This is the case when $\delta = 1$ and n is odd or $\delta = -1$ and n is even. Then from (7) we get [20]). As in the proof of Theorem 3, we obtain the inequalities (21) and (22). From (22) with $j+1$ replaced by j

$$|L_j x[\tau(t)]| \geq \int_{\tau(t)}^{t} \beta_j[s, \tau(t)] q(s) \prod_{i=1}^{m} (|x[g_i(s)]|)^{\lambda_i} ds. \tag{25}$$

Combining (21) with (25) yields

$$|L_j x[\tau(t)]| \geq |L_j x[\tau(t)]| \int_{\mathcal{D}(t)} \beta_j[s, \tau(t)] q(s) \prod_{i=1}^{m} (\zeta_j[\tau(t), g_i(s)])^{\lambda_i} ds,$$

which contradicts (23).

Case 2. Let $\ell = n$. This is the case when $\delta = -1$. Then from (7) we get (16). As in the proof of Theorem 2, we have the inequalities (17) and (18). From (18) with $j-1$ replaced by j

$$x[g_i(s)] \geq \alpha_j[g_i(s), \xi(t)] L_j x[\xi(t)]. \tag{26}$$

Combining (17) with (26) yields

$$L_j x[\xi(t)] \geq$$
$$L_j x[\xi(t)] \int_{\mathcal{A}(t)} I_{n-j-1}(\xi(t), s; a_{j+1}, \dots, a_{n-1}) q(s) \prod_{i=1}^{m} (\alpha_j[g_i(s), \xi(t)])^{\lambda_i} ds,$$

which contradicts (24). This completes the proof.

The following example is illustrative.

Example 2. Consider the equation

$$(E_2;\ -1) \qquad L_n x(t) = ct^{1-n}(x[t + \sin t])^{1/3}(x[t + 2\sin t])^{2/3}, \qquad t > 0,$$

where $n \geq 3$

$$L_0 x(t) = x(t), \qquad L_k x(t) = \frac{1}{t}(L_{k-1}x(t))^{\cdot}, \quad k = 1, 2, \dots, n,$$

$a_n \equiv 1$, and c is a positive constant.

Here we let

$a_i(t) = t,\ i = 1, 2, \ldots, n-1,\ g_1(t) = t + \sin t,\ g_2(t) = t + 2\sin t,\ q(t) = ct^{1-n},\ \lambda_1 = \frac{1}{3}$ and $\lambda_2 = \frac{2}{3}$. Thus $\sigma(t) = t - 2$ and

$$\beta_0[t, s] = \alpha_{n-1}[t, s] = \frac{1}{2^{n-1}(n-1)!}(t^2 - s^2)^{n-1}, \quad t \geq s \geq t_0.$$

Now,

$$\mathcal{D} = \bigcup_{k=0}^{\infty}((2k+1)\pi, (2k+2)\pi) \text{ and } \mathcal{A} = \bigcup_{k=0}^{\infty}(2k\pi, (2k+1)\pi).$$

Define

$$\tau(t) = \begin{cases} t + \sin t & \text{for } t \in \mathcal{D} \\ t & \text{for } t \notin \mathcal{D} \end{cases} \text{ and } \xi(t) = \begin{cases} t + \sin t & \text{for } t \in \mathcal{A} \\ t & \text{for } t \notin \mathcal{A}. \end{cases}$$

If we choose $t_k = (2k+1)\pi + \frac{\pi}{2},\ (k = 1, 2, \ldots)$, then

$$\mathcal{D}(t) = \mathcal{D} \cap [\tau(t_k), t_k] = [\tau(t_k), t_k],$$

and

$$\begin{aligned} \int\limits_{\mathcal{D}(t)} \beta_0[s, \tau(t_k)]q(s)ds &= \frac{c}{2^{n-1}(n-1)!} \int\limits_{\tau(t_k)}^{t_k} (s^2 - \tau^2(t_k))^{n-1} \frac{1}{s^{n-1}} ds \\ &\geq \frac{c}{2^{n-1}(n-1)!} \int\limits_{\tau(t_k)}^{t_k} (s - \tau(t_k))^{n-1} ds = \frac{c}{2^{n-1}n!}. \end{aligned}$$

Thus, condition (23) is satisfied for $c > 2^{n-1}n!$. If we choose $t_k = 2k\pi + \frac{\pi}{2}$ $(k = 1, 2, \ldots)$, then we can prove by a similar argument as above that the condition (24) is also satisfied for $c > 2^{n-1}n!$ Therefore, all conditions of Theorem 4 (iii) and (iv) are satisfied and hence equation $(E_2; -1)$ is oscillatory.

Remarks. **1.** The results of this paper are applicable to a larger class of nonlinear differential equations which includes superlinear, sublinear and linear equations. We also mention that we do not stipulate that the functions $g_i(t)$ $(i = 1, 2, \ldots, m)$ in equation $(E; \delta)$ be either retarded, advanced or of mixed type. Hence our theorems may hold for ordinary, retarded, advanced and mixed type equations.

2. When condition (1) is satisfied, the disconjugate operator L_n,

$$L_n = \frac{d}{dt}\frac{1}{a_{n-1}(t)}\cdots\frac{1}{a_1(t)}\frac{d}{dt} \tag{$\star$}$$

is said to be in the canonical form (see Trench [13]) and if (1) is violated, L_n is no longer in canonical form.

According to Trench [13] any operator of the form ($\star$) can be uniquely represented in a canonical form with a different set of $\tilde{a}_i$'s obtained from the original a_i's. More precisely, we can find $\tilde{a}_i$, $i = 1, 2, \ldots, n-1$ such that

$$L_n = \frac{d}{dt}\frac{1}{\tilde{a}_{n-1}(t)}\cdots\frac{1}{\tilde{a}_1(t)}\frac{d}{dt},$$

so that

$$\int^{\infty} \tilde{a}_i(s)ds = \infty, \qquad i = 1, 2, \ldots, n-1,$$

and the $\tilde{a}_i(t)$, $i = 1, 2, \ldots, n-1$ are determined up to positive multiplicative constants with product 1. Since actual computations of $\tilde{a}_i(t)$ are tedious and hard to obtain, we only mention that it is possible to obtain an analogue of our main result without the explicit computation of $\tilde{a}_i(t)$, $i = 1, 2, \ldots, n-1$. This can be done by using the concept of a principal system for an operator of the form ($\star$) introduced by Trench [13]. Here we omit the details.

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