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The mean values of multiplicative functions IV

By GEDIMINAS STEPANAUSKAS (Vilnius)

Dedicated to Professors Imre Kátai and Zoltán Daróczy
on their 60th birthday

Abstract. The mean value theorem for the product of multiplicative functions with arguments from arithmetic progression when the variable of these progressions runs over primes is proved. This theorem is used for the investigation of the limit behaviour of a sum of additive functions.

1. Results

Let $g_l : \mathbb{N} \to \mathbb{C}$, $l = 1, \dots, s$ be multiplicative functions. Throughout the paper p and q denote primes; $c, c_1, \dots$ are positive constants; m, n, k are positive integers; $a_1, \dots, a_s$ are positive integers, also; and $b_1, \dots, b_s$ are integers.

In this paper we continue the investigations of publications [6], [7], [8], [9]. Let

$$\mathcal{G}(n) = \mathcal{G}(n; g_1, \dots, g_s) = g_1(a_1 n + b_1) \dots g_s(a_s n + b_s).$$

We consider the asymptotic behaviour of the sum

$$M_x(\mathcal{G}) = \frac{1}{\pi(x)} \sum_{p \le x} \mathcal{G}(p),$$

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as $x \to \infty$. This sum was earlier analyzed by I. KÁTAI [4].

Define the multiplicative functions g_{lr} and g^*_{lr}, $l = 1, \dots, s$, by

$$g_{lr}(p^m) = \begin{cases} g_l(p^m) & \text{if } p \le r, \\ 1 & \text{if } p > r, \end{cases} \qquad g^*_{lr}(p^m) = \begin{cases} 1 & \text{if } p \le r, \\ g_l(p^m) & \text{if } p > r; \end{cases}$$

and the multiplicative functions h_l, h_{lr}, $l = 1, \dots, s$, by means of the convolution $h_l = g_l * \mu$, $h_{lr} = g_{lr} * \mu$, where μ denotes the Möbius function.

Let us introduce some notations we shall use below. Let

$$(d_1, \dots, d_k),\ [d_1, \dots, d_k]$$

mean the greatest common divisor and the least common multiple of the integers $d_1, \dots, d_k$, respectively;

$$\varphi \quad \text{be the Euler function;} \quad a = \max(a_1, \dots, a_s), \quad b = \max(b_1, \dots, b_s);$$

$$\Delta_{kl} = a_k b_l - a_l b_k, \qquad 1 \le k < l \le s, \quad \Delta = \max_{1 \le k \le l \le s} |\Delta_{kl}|;$$

$$w_p = {\sum}' \frac{h_1(p^{m_1}) \dots h_s(p^{m_s})}{\varphi([p^{m_1}, \dots, p^{m_s}])}, \tag{1}$$

where the prime $'$ means that the summation is taken over all collections $(p^{m_1}, \dots, p^{m_s})$ with non-negative integer exponents m_l, $l = 1, \dots, s$, for which

$$(p^{m_l}, a_l) = 1, \quad (p^{m_l}, b_l) = 1, \qquad l = 1, \dots, s,$$

$$\text{and } (p^{m_k}, p^{m_l}) \mid \Delta_{kl}, \qquad 1 \le l < k \le s;$$

$$P(x) = \prod_{p \le x} w_p, \quad P(r, x) = \prod_{r < p \le x} w_p;$$

$$S(r, x) = \sum_{l=1}^{s} \sum_{r < p \le x} \frac{|g_l(p) - 1|^2}{p}.$$

We shall use the conditions:

$$\sum_{r < p \le x} \frac{\operatorname{Re}\big(g_1(p) + \dots + g_s(p)\big) - s}{p} \le C; \tag{A}$$

$$\big|g_l(n)\big| \le \psi_l(n) \ll (\log n)^{A_l}, \quad l = 1, \dots, s, \tag{B}$$

for $n \geq 2$, where A_l are non-negative constants and the functions ψ_l do not decrease,

$$\text{(C)}\qquad \begin{aligned}&\Psi(n) = \psi_1(a_1n+b_1)\dots\psi_s(a_sn+b_s), \quad A=\max(A_1,\dots,A_s);\\ &S(r,x)\leq\frac{1}{4},\ s\leq r,\ s(\log r)^{A-1}\leq r,\ \Delta\leq r,\ \Delta_{kl}\neq 0,\ a\leq r,\ b_l\leq r,\\ &a_l+b_l>0,\ (a_l,b_l)=1,\ b_l\neq 0,\ a_lx+b_l\leq x^{3/2},\\ &\qquad l=1,\dots,s,\ 1\leq k<l.\end{aligned}$$

We can formulate now our main result.

Theorem. *Let the multiplicative functions $g_1,\dots,g_s$ satisfy the conditions* (A), (B), *and* (C) *with some collection of required constants. Assume further that $B>0$, $\alpha\geq\alpha_0>0$, $1-1/s<\alpha<1$. Then for $2\leq r\leq\sqrt{\log x}$*

$$\text{(2)}\qquad \begin{aligned}&M_x(\mathcal{G})-P(x)\\ &\ll\left(\frac{1}{(\log x)^B}+\frac{(ax+b)^{s(1-\alpha)}}{x}\log x\right)\exp\left(\frac{csr^\alpha(\log r)^{A-1}}{(1-\alpha)^{A+1}}\right)\\ &+\Psi(x)s\left(\frac{(\log r)^{A-1}}{r}+\log\log(a|b|+2)\left(\big(S(r,x)\big)^{1/4}+\frac{\log\big(a+\frac{b}{x}\big)}{\log x}\right)\right),\end{aligned}$$

where the constant c and the one in the symbol $\ll$ may depend on B, on α_0, and on the constants from (A) *and* (B), *only.*

The conditions (C) are not essential. They could be weakened, but the proof of Theorem would be more difficult.

It is easy to apply our Theorem to the functions from a set G (for the definition of G see [8]). For example, let

$$\mathcal{A}_k=\big\{n\,\big|\,p^m\|n\Rightarrow m<k\big\}$$

denote a set of k-free positive integers.

Corollary 1. *For $x \ge 2$ and $s \ll (\log\log x)^{1/3}$*

$$\frac{1}{\pi(x)}\sum_{p\le x}\frac{\varphi(p+1)\dots\varphi(p+s)}{(p+1)\dots(p+s)} = \prod_p w_p + R(x),$$

$$\frac{1}{\pi(x)}\sum_{\substack{p\le x\\ p+1,\dots,p+s\in\mathcal{A}_k}} 1 = \prod_p v_p + Q(x).$$

The values of w_p and v_p are defined by (1) and can be evaluated as follows. Let $s=\xi p+\eta$, where ξ, η are integers for which $\xi \ge 0$, $0\le \eta < p$. Then

$$w_p = \begin{cases} 1+\dfrac{1}{p-1}\Bigg(\eta\bigg(\Big(1-\dfrac{1}{p}\Big)^{\xi+1}-1\bigg) \\ \quad +(p-\eta-1)\bigg(\Big(1-\dfrac{1}{p}\Big)^{\xi}-1\bigg)\Bigg) & \text{if } p\le s,\\ 1-\dfrac{s}{p(p-1)} & \text{if } p>s,\end{cases}$$

$$v_p = \begin{cases} 0 & \text{if } p^k\le s,\\ 1-\dfrac{s-\xi}{p^{k-1}(p-1)} & \text{if } p^k > s.\end{cases}$$

For the remainder terms we have

$$R(x) \ll \frac{s\log\log s}{(\log\log x)^{1/3}(\log\log\log x)^{3/2}},$$

$$Q(x) \ll \frac{s\log\log s}{(\log\log x)^{1/3}(\log\log\log x)^{2}}.$$

In case if s is fixed

$$R(x) \text{ and } Q(x) \ll_{\varkappa} \frac{1}{(\log\log x)^{\varkappa}},$$

where $\varkappa$, $0<\varkappa<s/(s-1)$, is arbitrary.

Let further the values s, a_l, b_l, $l=1,\dots,s$, not depend on x. Denote

$$\sum_{|f_l(p)|\le 1}\frac{f_l^2(p)}{p},\qquad l=1,\dots,s, \tag{3}$$

$$\sum_{|f_l(p)|>1}\frac{1}{p},\qquad l=1,\dots,s, \tag{4}$$

$$\sum_{\substack{|f_1(p)|\le 1\\ \cdots\cdots\cdots\\ |f_s(p)|\le 1}}\frac{f_1(p)+\dots+f_s(p)}{p}. \tag{5}$$

Let also be

$$\begin{gathered} a_l+b_l>0,\ (a_l,b_l)=1,\ b_l\ne 0,\ l=1,\dots,s;\\ \Delta_{jk}\ne 0,\quad 1\le j<k\le s. \end{gathered} \tag{6}$$

Corollary 2. *Let f_l, $l=1,\dots,s$, be real-valued additive functions, let the series* (3), (4), (5) *converge, and let the conditions* (6) *be fulfilled. Then the distribution functions*

$$\frac{1}{\pi(x)}\,\#\left\{p\mid p\le x,\ f_1(a_1p+b_1)+\dots+f_s(a_sp+b_s)\le z\right\}$$

converge weakly towards a limit distribution as $x\to\infty$, and the characteristic function of this limit distribution is equal to

$$\prod_p w_p,$$

where w_p is defined by (1) *with $g_l=e^{itf_l}$, $l=1,\dots,s$.*

Corollaries 1 and 2 clearly imply

Corollary 3. *Let s not depend on x. Then the distribution functions*

$$\frac{1}{\pi(x)}\,\#\left\{p\;\Big|\;p\le x,\ \frac{\varphi(p+1)\dots\varphi(p+s)}{(p+1)\dots(p+s)}\le e^z\right\}$$

converge weakly towards a limit distribution, as $x\to\infty$. The characteristic function of this limit distribution is

$$\prod_p w_p,$$

with

$$w_p = \begin{cases} 1 + \dfrac{1}{p-1}\Bigg(\eta\bigg(\Big(1-\dfrac{1}{p}\Big)^{it(\xi+1)} - 1\bigg) \\ \quad +(p-\eta-1)\bigg(\Big(1-\dfrac{1}{p}\Big)^{it\xi} - 1\bigg)\Bigg) & \text{if } p \le s, \\ 1 - \dfrac{s}{p-1}\bigg(\Big(1-\dfrac{1}{p}\Big)^{it} - 1\bigg) & \text{if } p > s, \end{cases}$$

where ξ and η are defined in Corollary 1.

In the following corollary we give an example of a sum of additive arithmetical functions with shifted arguments which is uniformly distributed mod 1 on the set of primes. The additive functions may depend on x.

Let

$$f_1(a_1n+b_1) + \cdots + f_s(a_sn+b_s) = \mathcal{F}(n).$$

We say that $\mathcal{F}(n)$ is asymptotically uniformly distributed mod 1 on the set of primes if

$$\frac{1}{\pi(x)}\,\#\,\big\{p \mid p \le x,\ \text{fractional part of } \mathcal{F}(p) \in [\alpha,\beta)\big\} \to \beta - \alpha$$

as $x \to \infty$, for every α, β, $0 \le \alpha < \beta \le 1$.

Corollary 4. *Let real-valued additive arithmetical functions $f_1, \ldots, f_s$ be such, that the following conditions are satisfied:*

(a) *$f_l(p) \to 0$ as $p \to \infty$, for $l = 1, \ldots, s$;*

(b) *there exists r, $2 \le r \le \log\log x$, such that*

$$\sum_{r<p\le x} \frac{f_l^2(p)}{p} \to 0$$

as $x \to \infty$, for $l = 1, \ldots, s$;

(c) *at least one of the functions f_k satisfies*

$$\sum_{p\le x} \frac{f_k^2(p)}{p} \to \infty, \qquad x \to \infty.$$

Let the conditions (6) *also be fulfilled. Then the sum $\mathcal{F}(n)$ is asymptotically uniformly distributed* mod 1 *on the set of primes.*

For example, let

$$f_l(p) = \frac{1}{\log p}, \qquad l = 1, \ldots, s-1,$$

$$f_s(p) = \begin{cases} \dfrac{1}{\log p} & \text{if } p > \log\log x, \\ \dfrac{1}{(\log\log p)^{1/2}} & \text{if } p \le \log\log x. \end{cases}$$

Then the sum $\mathcal{F}(n)$ is asymptotically uniformly distributed mod 1 on the set of primes.

Finally we formulate a local limit law for the sum of additive integer-valued functions.

Let λ_k, $k \in \mathbb{Z}$, be defined by means of the equality

$$\sum_{k=-\infty}^{+\infty} \lambda_k e^{itk} = \prod_p w_p \quad \text{for } \forall t \in \mathbb{R},$$

where $w_p = w_p(t)$ are determined by (1) with $g_l = e^{itf_l}$. Let us put also

$$\varepsilon(x) = \left(\sum_{l=1}^{s} \sum_{\substack{p > \log\log x \\ f_l(p) \ne 0}} \frac{1}{p} \right)^{1/4}.$$

Corollary 5. *Let the integer-valued additive arithmetical functions* $f_1, \ldots, f_s$ *be such, that the series*

$$\sum_{f_l(p) \ne 0} \frac{1}{p}, \quad l = 1, \ldots, s,$$

converge and the conditions (6) *are fulfilled. Then*

$$\frac{1}{\pi(x)} \# \left\{ p \le x \,\middle|\, \mathcal{F}(p) = k \right\} = \lambda_k + O\left(\varepsilon(x) + \frac{1}{\log\log x} \right)$$

uniformly for all $k \in \mathbb{Z}$ *and* $x \ge 2$.

2. Auxiliary lemmas

We shall use some estimates and statements which we formulate as lemmas.

Lemma 1 (Turán–Kubilius). *Let $f(p)$ be complex numbers, $\xi \in \mathbb{N}$, $\eta \in \mathbb{Z}$, $\eta \neq 0$, $(\xi, \eta) = 1$, $\xi + \eta > 0$, $\xi x + \eta \leq x^K$, $0 < \beta < 1$. Then for $x \geq 2$*

$$\frac{1}{\pi(x)} \sum_{p \leq x} \Bigg| \sum_{\substack{q \mid \xi p + \eta \\ q \leq x^\beta}} f(q) - \sum_{\substack{q \leq x^\beta \\ q \nmid \xi}} \frac{f(q)}{q} \Bigg|^2 \ll \sum_{\substack{q \leq x^\beta \\ q \nmid \xi}} \frac{|f(q)|^2}{q},$$

where the constant in the symbol $\ll$ depends only on β and K.

For *the proof of Lemma 1*, it is enough to repeat the proof of Lemma 4.12 from [2] and to satisfy oursel that the conditions of Lemma 1 guarantee the uniformity of our estimate with respect to f, x, ξ, and η.

Consider further a system of congruences

$$(7) \qquad a_l n + b_l \equiv 0 \mod d_l, \quad (a_l, b_l) = 1, \; l = 1, \ldots, s.$$

The Chinese residue theorem includes as a special case the following

Lemma 2. *The system of congruences* (7) *has a solution if and only if $(a_l, d_l) = 1$, $l = 1, \ldots, s$, and*

$$(d_k, d_l) \,\big|\, (a_k b_l - a_l b_k), \quad 1 \leq k < l \leq s.$$

If the solution exists, it is exactly one residue class modulo $[d_1, \ldots, d_s]$.

Lemma 3 ([9, Lemma 3]). *Assume $\gamma \geq 0$. Then uniformly in u, $0 \leq u < 1$,*

$$\sum_{n=1}^{\infty} u^n n^\gamma \ll \frac{u}{(1-u)^{\gamma+1}}.$$

Lemma 4 ([9, Lemma 4]). *Assume $c_1 \leq \gamma \leq 1$. Then uniformly in γ and $u \geq 2$*

$$\sum_{p \leq u} \frac{1}{p^{1-\gamma}} \ll \frac{u^\gamma}{\log u}.$$

Lemma 5. *Assume* $\gamma > 0$, $\beta \in \mathbb{R}$. *Then uniformly for all* $u \geq 2$

$$\sum_{p>u} \frac{(\log p)^{\beta}}{p^{1+\gamma}} \ll \frac{(\log u)^{\beta}}{u^{\gamma} \log u}.$$

Proof. The assertion of the lemma follows from the following inequality. Let $\delta > 0$. Then

$$\begin{aligned}\sum_{p>u} \frac{1}{p^{1+\delta}} &= \sum_{k=1}^{\infty} \sum_{2^{k-1}u<p\leq 2^k u} \frac{1}{p^{1+\delta}} \\ &\ll \sum_{k=1}^{\infty} \frac{1}{(2^{k-1}u)^{\delta} \log(2^{k-1}u)} \sum_{2^{k-1}u<p\leq 2^k u} \frac{\log p}{p} \\ &\ll \frac{1}{u^{\delta} \log u} \sum_{k=1}^{\infty} \frac{1}{2^{(k-1)\delta}} (\log 2 + O(1)) \ll \frac{1}{u^{\delta} \log u}.\end{aligned}$$

Lemma 6 (Brun–Titchmarsh, [5, Ch. 5, Theorem 2.1]). *Let* γ *be a real number,* $0 < \gamma < 1$. *Then the inequality*

$$\pi(x, d, v) \ll \frac{x}{\varphi(d) \log x}$$

holds uniformly for all $x \geq 2$ *and integers* v *and* d *with* $1 \leq d \leq x^{\gamma}$.

Lemma 7 ([3, Theorem 3.7]). *The inequality*

$$\pi(x, d, v) \ll \frac{x}{\varphi(d) \log \frac{x}{d}}$$

holds uniformly for all $x \geq 2$ *and integers* d *and* v *with* $1 \leq d < x$, $(v, d) = 1$.

Lemma 8 ([5, Ch. 4, Theorem 7.5]). *There is a positive constant* c_2, *such that uniformly for all* $x \geq 2$

$$\pi(x) - \operatorname{li} x \ll x e^{-c_2 \sqrt{\log x}}.$$

Lemma 9 (Bombieri [1]). *Let K be a positive real number. Then there exists a further real number L, such that uniformly for all $x \geq 2$*

$$\sum_{d \leq x^{1/2}(\log x)^{-L}} \max_{(d,v)=1} \left| \pi(x,d,v) - \frac{\operatorname{li} x}{\varphi(d)} \right| \ll \frac{x}{(\log x)^K}.$$

Lemma 10 ([5, Ch. 2, Theorem 4.2]). *Let k_1, k_2 be positive integers, l_1, l_2 be integers, $(k_j, l_j) = 1$ for $j = 1, 2$, $k_1 l_2 - k_2 l_1 \neq 0$. Then*

$$\#\big\{n \mid n \leq x,\ k_j n + l_j \text{ are primes for } j = 1,2\big\} \ll \frac{x}{\log^2 x} \prod_{p \mid k_1 k_2 (k_1 l_2 - k_2 l_1)} \left(1 - \frac{1}{p}\right)^{-1}$$

uniformly in $x \geq 2$, k_j, l_j, $j = 1, 2$.

Lemma 11. *Let k and d be positive integers, $(k,d) = 1$, v be an integer, $v \neq 0$. Then*

$$\#\big\{(p,q) \mid p \leq x, kq - dp = v\big\} \ll \frac{x}{\varphi(k) \log^2 \frac{x}{k}} \prod_{p \mid dv} \left(1 - \frac{1}{p}\right)^{-1}$$

uniformly in $x \geq 2$, k, d, and v.

Proof. It follows from the condition $(k,d) = 1$ that the equation

$$km - dn = v \tag{8}$$

has solutions in positive integers m and n. All these solutions have the form

$$m = m_0 + dt, \quad n = n_0 + kt, \qquad t = t_0, t_0 + 1, \ldots,$$

where m_0, n_0 is the integer solution of the equation (8),

$$km_0 - dn_0 = v, \qquad 0 \leq n_0 < k,$$

and t_0 is the least positive integer for which $m_0 + dt_0 > 0$. The number of such pairs (m,n), where both components are primes (p,q), $p \leq x$, can be estimated by Lemma 10. Thus

$$\begin{aligned}
&\#\big\{(p,q) \mid p \le x,\, kq - ap = v\big\}\\
&= \#\left\{t \,\middle|\, t \le \frac{x-n_0}{k},\, m_0 + dt \text{ and } n_0 + kt \text{ are primes}\right\}\\
&\ll \frac{x}{\varphi(k)\log^2 \frac{x}{k}} \prod_{p\mid dv}\left(1-\frac{1}{p}\right)^{-1}
\end{aligned}$$

and Lemma 11 is proved.

3. Proofs

PROOF of the Theorem. Let l belong to the set $\{1,\dots,s\}$. The following estimates will be used later. We obtain from (B), Lemma 3 and Lemma 4, that

$$\begin{aligned}
\sum_{d>z}\frac{|h_{lr}(d)|}{\varphi(d)} &\le \frac{1}{z^\alpha}\sum_{d=1}^{\infty}\frac{|h_{lr}(d)|d^\alpha}{\varphi(d)} = \frac{1}{z^\alpha}\prod_{p\le r}\left(1+\sum_{m=1}^{\infty}\frac{\big|h_{lr}(p^m)\big|p^{m\alpha}}{\varphi(p^m)}\right)\\
&\le \frac{1}{z^\alpha}\prod_{p\le r}\left(1+\frac{c_3(\log p)^{A_l}}{p^{1-\alpha}}\left(1-\frac{1}{p^{1-\alpha}}\right)^{-A_l-1}\right)\\
&\le \frac{1}{z^\alpha}\prod_{p\le r}\left(1+\frac{c_4(\log p)^{A_l}}{(1-\alpha)^{A_l+1}p^{1-\alpha}}\right) \qquad (9)\\
&\le \frac{1}{z^\alpha}\exp\left(\frac{c_5}{(1-\alpha)^{A_l+1}}\sum_{p\le r}\frac{(\log p)^{A_l}}{p^{1-\alpha}}\right)\\
&\le \frac{1}{z^\alpha}\exp\left(\frac{c_6 r^\alpha(\log r)^{A_l-1}}{(1-\alpha)^{A_l+1}}\right),
\end{aligned}$$

where the constant c_6 does not depend on α. Analogously

$$\sum_{d=1}^{\infty}\frac{|h_{lr}(d)|}{\varphi(d)} \le \exp\big(c_7(\log r)^{A_l}\log\log r\big) \qquad (10)$$

and

$$\sum_{d=1}^{\infty} \frac{|h_{lr}(d)|}{(\varphi(d))^{1-\alpha}} \le \exp\left(\frac{c_8 r^{\alpha}(\log r)^{A_l-1}}{(1-\alpha)^{A_l+1}}\right). \tag{11}$$

Split the left-hand side of (2) in the following way:

$$\begin{aligned} M_x(\mathcal{G}) - P(x) = P(r,x)\big(M_x(\mathcal{G}_r) - P(r)\big) \\ + \frac{1}{\pi(x)} \sum_{p \le x} \mathcal{G}_r(p)\big(\mathcal{G}_r^*(p) - P(r,x)\big), \end{aligned}$$

where

$$\mathcal{G}_r(n) = \mathcal{G}(n; g_{1r}, \dots, g_{sr}) \quad \text{and} \quad \mathcal{G}_r^*(n) = \mathcal{G}(n; g_{1r}^*, \dots, g_{sr}^*).$$

Let $\omega(n)$ mean the number of distinct prime divisors of the number n and

$$M_{x0}(\mathcal{G}_r) = \frac{1}{\pi(x)} \sum_{\substack{p \le x \\ (p,b_l)=1, \forall l}} \mathcal{G}_r(p).$$

The value of $M_x(\mathcal{G}_r)$ can be written in the form

$$\begin{aligned} M_x(\mathcal{G}_r) &= M_{x0}(\mathcal{G}_r) + O\bigg(\frac{\Psi(x)}{\pi(x)}\big(\omega(|b_1|) + \dots + \omega(|b_s|)\big)\bigg) \\ &= M_{x0}(\mathcal{G}_r) + O\bigg(\frac{\Psi(x)}{\pi(x)}\big(|b_1| + \dots + |b_s|\big)\bigg) = M_{x0}(\mathcal{G}_r) + O\Big(\frac{1}{\sqrt{x}}\Big). \end{aligned}$$

Put

$$\begin{aligned} R_1 &= \big|M_{x0}(\mathcal{G}_r) - P(r)\big|, \\ R_2 &= \frac{1}{\pi(x)}\bigg|\sum_{p \le x} \mathcal{G}_r(p)\big(\mathcal{G}_r^*(p) - P(r,x)\big)\bigg|. \end{aligned}$$

Then evidently

$$M_x(\mathcal{G}) - P(x) \ll \big|P(r,x)\big|\Big(R_1 + \frac{1}{\sqrt{x}}\Big) + R_2. \tag{12}$$

From the definition of h_{lr} and then from Lemma 2, we obtain that

$$M_{x0}(\mathcal{G}_r) = \frac{1}{\pi(x)} \sum_{\substack{p\le x \\ (p,b_l)=1,\forall l}} \sum_{d_1|a_1p+b_1} h_{1r}(d_1)\cdots \sum_{d_s|a_sp+b_s} h_{sr}(d_l)$$

$$= \frac{1}{\pi(x)} \sum_{d_1\le a_1x+b_1} h_{1r}(d_1)\cdots \sum_{d_s\le a_sx+b_s} h_{sr}(d_s) \sum_{\substack{p\le x \\ (p,b_l)=1,\forall l \\ d_l|a_lp+b_l,\forall l}} 1$$

$$= \frac{1}{\pi(x)} \sideset{}{''}\sum_{d_l\le a_lx+b_l,\forall l} h_{1r}(d_1)\dots h_{sr}(d_s)\pi\big(x,[d_1,\dots,d_s],v\big),$$

where the double prime $''$ means that the summation in taken over all vectors $(d_1,\dots,d_s)\in\mathbb{N}^s$ for which $(d_l,a_l)=1$, $(d_l,b_l)=1$, $\forall l$, and $(d_k,d_l)|\Delta_{kl}$, $1\le k<l\le s$, and where v is the only integer for which

$$a_lv+b_l\equiv 0 \mod d_l, \quad \forall l,$$

and $0\le v\le [d_1,\dots,d_s]-1$. It is clear also, that $(v,[d_1,\dots,d_s])=1$.

Since

$$P(r) = \sideset{}{''}\sum \frac{h_{1r}(d_1)\dots h_{sr}(d_s)}{\varphi([d_1,\dots,d_s])},$$

we can write

$$R_1 \le R_{11}+R_{12}+R_{13}+R_{14},$$

where

$$R_{11} = \frac{1}{\pi(x)} \sideset{}{''}\sum_{d_l\le z,\forall l} \big|h_{1r}(d_1)\dots h_{sr}(d_s)\big|$$

$$\times \left|\pi(x,[d_1,\dots,d_s],v)\frac{\operatorname{li} x}{\varphi([d_1,\dots,d_s])}\right|,$$

$$R_{12} = \frac{1}{\pi(x)} \sideset{}{''}\sum_{d_l\le z,\forall l} \frac{\big|h_{1r}(d_1)\dots h_{sr}(d_s)\big|}{\varphi([d_1,\dots,d_s])}|\operatorname{li} x-\pi(x)|,$$

$$R_{13} = \sum_{l=1}^{s} \sideset{}{''}\sum_{d_l>z} \frac{\big|h_{1r}(d_1)\dots h_{sr}(d_s)\big|}{\varphi([d_1,\dots,d_s])},$$

and

$$R_{14} = \frac{1}{\pi(x)} \sum_{k=1}^{s} \sum_{\substack{d_l \le a_l x + b_l, \forall l \\ d_k > z}}'' \left| h_{1r}(d_1) \dots h_{sr}(d_s) \right| \pi(x, [d_1, \dots, d_s], v)$$

with some z, $z^s \le x^{1/3}$, which we shall choose later.

By Lemma 6

$$\begin{aligned} R_{11} \ll \frac{1}{\pi(x)} \max_{\substack{d \le z^s \\ (d,v)=1}} \left| \pi(x,d,v) - \frac{\operatorname{li} x}{\varphi(d)} \right|^{\alpha} \left(\frac{x}{\log x} \right)^{1-\alpha} \\ \times \sum_{d_l \le z, \forall l}'' \frac{\left| h_{1r}(d_1) \dots h_{sr}(d_s) \right|}{\big(\varphi([d_1, \dots, d_s])\big)^{1-\alpha}}, \end{aligned}$$

and by Lemma 9 it is

$$\begin{aligned} &\ll \frac{1}{(\log x)^{(K-1)\alpha}} \sum_{d_l \le z, \forall l}'' \Bigg(\prod_{1 \le j < k \le s} (d_j, d_k) \Bigg)^{1-\alpha} \frac{\left| h_{1r}(d_1) \dots h_{sr}(d_s) \right|}{\big(\varphi(d_1, \dots, d_s)\big)^{1-\alpha}} \\ &\ll \frac{1}{(\log x)^{(K-1)\alpha}} \Bigg(\prod_{1 \le j < k \le s} |\Delta_{jk}| \Bigg)^{1-\alpha} \sum_{d_1=1}^{\infty} \frac{\left| h_{1r}(d_1) \right|}{(\varphi(d_1))^{1-\alpha}} \dots \sum_{d_s=1}^{\infty} \frac{\left| h_{sr}(d_s) \right|}{(\varphi(d_s))^{1-\alpha}}. \end{aligned}$$

Keeping in mind that

$$\prod_{1 \le j < k \le s} |\Delta_{jk}| \ll \Delta^{s(s-1)/2}$$

and using also the inequality (11), we obtain that

$$\begin{aligned} R_{11} &\ll \frac{1}{(\log x)^{(K-1)\alpha}} \Delta^{s(s-1)/2} \exp\left(\frac{c_9 s r^{\alpha} (\log r)^{A-1}}{(1-\alpha)^{A+1}} \right) \\ &\ll \frac{1}{(\log x)^{(K-1)\alpha}} \exp\left(\frac{c_{10} s r^{\alpha} (\log r)^{A-1}}{(1-\alpha)^{A+1}} \right). \end{aligned}$$

Using the same ideas as in the estimation of R_{11}, from Lemma 8 and the inequality (10) we deduce that

$$R_{12} \ll \exp(-c_{11} \sqrt{\log x}) \Delta^{s(s-1)/2} \exp\big(c_{12} s (\log r)^A \log\log r \big)$$

and from the inequalities (10) and (9) that

$$\begin{aligned} R_{13} &\ll \Delta^{s(s-1)/2} \exp\big(c_{13} s (\log r)^A \log\log r\big) \sum_{l=1}^{s} \sum_{d_l > z} \frac{|h_{lr}(d_l)|}{\varphi(d_l)} \\ &\ll \Delta^{s(s-1)/2} \exp\big(c_{13} s (\log r)^A \log\log r\big) s \frac{1}{z^\alpha} \exp\left(\frac{c_{14} r^\alpha (\log r)^{A-1}}{(1-\alpha)^{A+1}}\right) \\ &\ll \frac{1}{z^\alpha} \exp\left(\frac{c_{15} s r^\alpha (\log r)^{A-1}}{(1-\alpha)^{A+1}}\right). \end{aligned}$$

Analogously

$$\begin{aligned} R_{14} &\ll \frac{1}{\pi(x)} \sum_{k=1}^{s} \sideset{}{''}\sum_{\substack{d_l \le a_l x + b_l, \forall l \\ d_k > z}} \big|h_{1r}(d_1) \dots h_{sr}(d_s)\big| \left(\frac{x}{[d_1, \dots, d_s]} + 1\right) \\ &\ll \left(\frac{1}{z^\alpha} + \frac{(ax+b)^{s(1-\alpha)}}{x}\right) \log x \exp\left(\frac{c_{16} s r^\alpha (\log r)^{A-1}}{(1-\alpha)^{A+1}}\right). \end{aligned}$$

Let us choose $z = (\log x)^{c_{17}}$ with sufficiently large constant c_{17}. Then the estimates of R_{11}, R_{12}, R_{13}, and R_{14} imply that

$$R_1 \ll \left(\frac{1}{(\log x)^B} + \frac{(ax+b)^{s(1-\alpha)}}{x} \log x\right) \exp\left(\frac{c s r^\alpha (\log r)^{A-1}}{(1-\alpha)^{A+1}}\right). \tag{13}$$

If $p > r$, then

$$w_p - 1 = \sum_{m=1}^{\infty} \frac{h_1(p^m) + \dots + h_s(p^m)}{\varphi(p^m)}$$

and it follows from (B), (C) and Lemma 3, that

$$\begin{aligned} |w_p - 1| &\le \frac{p}{p-1} \left(\frac{\big(|h_1(p)| + \dots + |h_s(p)|\big)^2}{p}\right)^{1/2} \frac{1}{p^{1/2}} + \frac{c_{18} s (\log p)^A}{p^2} \\ &\le \frac{p}{p-1} \left(\frac{s S(r,x)}{p}\right)^{1/2} + \frac{c_{18} s (\log p)^A}{p^2} \le \frac{p}{p-1} \frac{1}{2} + \frac{c_{18} (\log p)^A}{p}. \end{aligned}$$

Thereby it is clear that

$$|w_p - 1| \le \frac{3}{4} \tag{14}$$

if $p > p_0$, where p_0 is large enough and depends only on the constants from (B).

Without loss of generality we may assume that $r \geq p_0$. We estimate the value of $P(r,x)$ using (14), Lemma 3, Lemma 5 and the conditions (A), (B) and (C). Thus

$$\begin{aligned} P(r,x) &= \exp\left\{ \sum_{r<p\leq x} \log\left(1 + \sum_{m=1}^{\infty} \frac{h_1(p^m) + \cdots + h_s(p^m)}{(\varphi(p))^m}\right)\right\} \\ &= \exp\left\{ \sum_{r<p\leq x} \left(\frac{h_1(p) + \cdots + h_s(p)}{p} + O\biggl(\frac{s(\log p)^A}{p^2}\biggr)\right.\right. \\ &\qquad \left.\left. + O\biggl(\frac{s}{p}\,\frac{|h_1(p)|^2 + \cdots + |h_s(p)|^2}{p}\biggr)\right)\right\} \qquad (15) \\ &= \exp\left\{ \sum_{r<p\leq x} \frac{h_1(p) + \cdots + h_s(p)}{p} \right. \\ &\qquad \left. + O\biggl(\frac{s(\log r)^{A-1}}{r} + S(r,x)\biggr)\right\} \ll 1. \end{aligned}$$

Put

$$\begin{aligned} P_{r1} &= \Bigl\{p \mid p \leq x,\ \exists q > r \text{ and } \exists l,\ \text{that } q \mid a_l p + b_l \text{ and } |h_l(p)| > \frac{1}{2}\Bigr\}, \\ P_{r2} &= \bigl\{p \mid p \leq x,\ \exists q > r \text{ and } \exists l,\ \text{that } q^2 \mid a_l p + b_l\bigr\} \setminus P_{r1}, \\ P_{r3} &= \{p \mid p \leq x\} \setminus (P_{r1} \cup P_{r2}). \end{aligned}$$

Split further R_2 into three sums R_{21}, R_{22}, R_{23} over p from P_{r1}, P_{r2}, P_{r3}, respectively.

It follows from (B) and (15) that

$$R_{21} \ll \frac{\Psi(x)}{\pi(x)} \sum_{p\in P_{r1}} 1 \ll \frac{\Psi(x)}{\pi(x)} \sum_{l=1}^{s} \sum_{\substack{r<q\leq a_l x + b_l \\ |h_l(q)|>1/2}} \sum_{\substack{p\leq x \\ a_l p + b_l \equiv 0 \mod q}} 1. \qquad (16)$$

Let

$$y = \exp\Bigl(1 - \bigl(S(r,x)\bigr)^{1/2}\Bigr) \log x.$$

Split the right-hand side of (16) into two sums R_{211} and R_{212} including terms for which $r < q \le y$ and $y < q \le a_l x + b_l$, respectively. Then

$$R_{211} \ll \frac{\Psi(x)}{\pi(x)} \sum_{l=1}^{s} \sum_{\substack{r<q\le y \\ |h_l(q)|>1/2}} \max_{1\le v<q} \pi(x,q,v).$$

By Lemma 7

$$\begin{aligned} R_{211} &\ll \frac{\Psi(x)}{\pi(x)} \sum_{l=1}^{s} \sum_{\substack{r<q\le y \\ |h_l(q)|>1/2}} \frac{x}{q \log \frac{x}{q}} \\ &\ll \frac{\Psi(x)}{\pi(x)} \frac{x}{\log \frac{x}{y}} \sum_{l=1}^{s} \sum_{r<q\le y} \frac{|h_l(q)|^2}{q} \ll \Psi(x)\big(S(r,x)\big)^{1/2}. \end{aligned}$$

Changing the order of summation in the expression R_{212}, we have

$$\begin{aligned} R_{212} &\ll \frac{\Psi(x)}{\pi(x)} \sum_{l=1}^{s} \sum_{y<q\le a_l x+b_l} \sum_{1\le k\le \frac{a_l x+b_l}{q}} \sum_{\substack{p\le x \\ a_l p+b_l=kq}} 1 \\ &= \frac{\Psi(x)}{\pi(x)} \sum_{l=1}^{s} \sum_{1\le k\le \frac{a_l x+b_l}{y}} \sum_{y<q\le \frac{a_l x+b_l}{k}} \sum_{\substack{p\le x \\ a_l p+b_l=kq}} 1. \end{aligned} \tag{17}$$

Observing now that the inner sum is empty when $(a_l, k) \ne 1$ and using Lemma 11 and a few well-known estimates, we get that

$$\begin{aligned} R_{212} &\ll \frac{\Psi(x)}{\pi(x)} \sum_{l=1}^{s} \sum_{1\le k\le \frac{a_l x+b_l}{y}} \frac{x}{\varphi(k) \log^2 \frac{x}{k}} \frac{a_l |b_l|}{\varphi(a_l|b_l|)} \\ &\ll \frac{\Psi(x)}{\pi(x)} s \log\log(a|b|+2) \frac{x}{\log^2 \frac{xy}{ax+b}} \sum_{1\le k\le \frac{ax+b}{y}} \frac{1}{\varphi(k)} \\ &\ll \Psi(x) s \log\log(a|b|+2) \log x \frac{x}{\log^2 \frac{xy}{ax+b}} \log \frac{ax+b}{y}. \end{aligned} \tag{18}$$

Since

$$\log\frac{xy}{ax+b}=\log y-\log\Big(a+\frac{b}{x}\Big)\geq c_{19}\log y\geq\frac{c_{19}}{2}\log x$$

with sufficiently small positive constant c_{19}, we obtain

$$R_{212}\ll\Psi(x)s\log\log(a|b|+2)\left(\frac{\log\left(a+\frac{b}{x}\right)}{\log x}+\big(S(r,x)\big)^{1/2}\right).$$

We have now that

$$R_{21}\ll\Psi(x)s\log\log(a|b|+2)\left(\frac{\log\left(a+\frac{b}{x}\right)}{\log x}+\big(S(r,x)\big)^{1/2}\right). \tag{19}$$

The value of R_{22} does not exceed

$$\begin{aligned}&\frac{\Psi(x)}{\pi(x)}\sum_{l=1}^{s}\sum_{r<q\leq(a_lx+b_l)^{1/2}}\sum_{\substack{p\leq x\\ a_lp+b_l\equiv0 \mod q^2}}1\\ &\leq\frac{\Psi(x)}{\pi(x)}\sum_{l=1}^{s}\sum_{r<q\leq(a_lx+b_l)^{1/2}}\max_{1\leq v<q^2}\pi(x,q^2,v).\end{aligned}$$

For the estimation of $\pi(x,q^2,v)$ in the range $r<q\leq x^{1/4}$, we use Lemma 7. In the range $x^{1/4}<q\leq(a_lx+b_l)^{1/2}$, we estimate $\pi(x,q^2,v)$ trivially. Thus

$$R_{22}\ll\frac{\Psi(x)}{\pi(x)}\sum_{l=1}^{s}\Bigg(\sum_{r<q\leq x^{1/4}}\frac{x}{q^2\log\frac{x}{q^2}}+\sum_{x^{1/4}<q\leq x^{1/2}}\frac{x}{q^2}+\sum_{x^{1/2}<q\leq(a_lx+b_l)^{1/2}}1\Bigg).$$

Now Lemma 5 implies that

$$R_{22}\ll\Psi(x)s\bigg((r\log r)^{-1}+x^{-1/4}+\frac{(ax+b)^{1/2}}{x}\bigg). \tag{20}$$

For the estimation of R_{23} we use the inequality

$$e^u-e^v\ll|u-v|\left(|e^u|+|e^v|\right)$$

true for all $u,v\in\mathbb{C}$. Therefore

$$R_{23}\ll\frac{\Psi(x)}{\pi(x)}\sum_{p\in P_{r3}}\big|\log\mathcal{G}_r^*(p)-P(r,x)\big|\ll R_{231}+R_{232}+R_{233}, \tag{21}$$

where

$$R_{231} = \frac{\Psi(x)}{\pi(x)} \sum_{l=1}^{s} \sum_{p \in P_{r3}} \Bigg| \sum_{\substack{q | a_l p + b_l \\ q > r}} h_l(q) - \sum_{r < q \le x} \frac{h_l(q)}{q} \Bigg|,$$

$$R_{232} = \frac{\Psi(x)}{\pi(x)} \sum_{p \in P_{r3}} \Bigg| \sum_{r < q \le x} \frac{h_1(q) + \cdots + h_s(q)}{q} - \log P(r,x) \Bigg|,$$

$$R_{233} = \frac{\Psi(x)}{\pi(x)} \sum_{l=1}^{s} \sum_{p \in P_{r3}} \sum_{\substack{q | a_l p + b_l \\ q > r}} \big| h_l(q) \big|^2.$$

The sum R_{231} does not exceed

$$R_{2311} + R_{2312} + R_{2313} + R_{2314},$$

where

$$R_{2311} = \frac{\Psi(x)}{\pi(x)} \sum_{l=1}^{s} \sum_{p \le x} \Bigg| \sum_{\substack{q | a_l p + b_l \\ r < q \le \sqrt{x}}} h_l(q) - \sum_{r < q \le \sqrt{x}} \frac{h_l(q)}{q} \Bigg|,$$

$$R_{2312} = \frac{\Psi(x)}{\pi(x)} \sum_{l=1}^{s} \sum_{p \le x} \sum_{\substack{q | a_l p + b_l \\ \sqrt{x} < q \le x^{1-\delta}}} |h_l(q)|,$$

$$R_{2313} = \frac{\Psi(x)}{\pi(x)} \sum_{l=1}^{s} \sum_{p \le x} \sum_{\substack{q | a_l p + b_l \\ q > x^{1-\delta}}} 1,$$

$$R_{2314} = \Psi(x) \sum_{l=1}^{s} \sum_{\sqrt{x} < q \le x} \frac{|h_l(q)|}{q}$$

with $\delta = \big(S(\sqrt{x}, x)\big)^{1/4}$.

By the Cauchy inequality, the inner sum from R_{2311} does not exceed

$$\Bigg(\sum_{p \le x} \Bigg| \sum_{\substack{q | a_l p + b_l \\ r < q \le \sqrt{x}}} h_l(p) - \sum_{r < q \le \sqrt{x}} \frac{h_l(q)}{q} \Bigg|^2 \Bigg)^{1/2} \Bigg(\sum_{p \le x} 1 \Bigg)^{1/2},$$

and then by Lemma 1, it is

$$\ll \pi(x)\Big(\sum_{r<q\le\sqrt{x}} \frac{|h_l(q)|^2}{q}\Big)^{1/2}.$$

Therefore

$$R_{2311} \ll \Psi(x)\big(s\,S(r,\sqrt{x})\big)^{1/2}.$$

Let us estimate R_{2312}. Changing the order of summation, we obtain that

$$R_{2312} = \frac{\Psi(x)}{\pi(x)} \sum_{l=1}^{s} \sum_{\sqrt{x}<q\le x^{1-\delta}} |h_l(q)| \sum_{\substack{p\le x\\ a_l p+b_l\equiv 0 \mod q}} 1.$$

By Lemma 7, the inner sum is

$$\ll \frac{x}{q\log\frac{x}{q}} \ll \frac{x}{\delta q\log x}.$$

It follows now from the Cauchy inequality that

$$R_{2312} \ll \frac{\Psi(x)}{\delta}\Big(\sum_{\sqrt{x}<q\le x^{1-\delta}} \frac{\big(|h_1(q)|+\cdots+|h_s(q)|\big)^2}{q}\Big)^{1/2}\Big(\sum_{\sqrt{x}<q\le x^{1-\delta}} \frac{1}{q}\Big)^{1/2}$$
$$\ll \Psi(x)\sqrt{s}\big(S(\sqrt{x},x)\big)^{1/4}.$$

Similarly as in (17) and (18) (the only difference is that in place of y we take $x^{1-\delta}$), we have

$$R_{2313} \ll \Psi(x)s\log\log(a|b|+2)\log x\,\frac{1}{\log^2\dfrac{x^{2-\delta}}{ax+b}}\,\log\frac{ax+b}{x^{1-\delta}}$$
$$\ll \Psi(x)s\log\log(a|b|+2)\bigg(\big(S(\sqrt{x},x)\big)^{1/4}+\frac{\log\big(a+\frac{b}{x}\big)}{\log x}\bigg).$$

In the same way as in the estimation of R_{2312}, it follows from the Cauchy inequality that

$$R_{2314} \ll \Psi(x)\big(s\,S(\sqrt{x},x)\big)^{1/2},$$

and therefore

$$R_{231} \ll \Psi(x) s \log\log(a|b|+2)$$
$$\times\Bigg(\big(S(r,\sqrt{x})\big)^{1/2} + \big(S(\sqrt{x},x)\big)^{1/4} + \frac{\log\big(a+\frac{b}{x}\big)}{\log x}\Bigg).$$

Using (15), we easily get that

$$R_{232} \ll \Psi(x)\bigg(\frac{s(\log r)^{A-1}}{r} + S(r,x)\bigg).$$

Analogously as in the estimation of R_{21}, we obtain that

$$R_{233} \ll \Psi(x) s \log\log(a|b|+2)\left(\big(S(r,x)\big)^{1/2} + \frac{\log\big(a+\frac{b}{x}\big)}{\log x}\right).$$

Collecting the latter estimates into (21), we get the estimate of R_{23}. Then it follows from (19), (20), and (21) that

$$R_2 \ll \Psi(x) s\Bigg(\frac{(\log r)^{A-1}}{r} + \log\log(a|b|+2) \times\Bigg(\big(S(r,x)\big)^{1/2} + \big(S(\sqrt{x},x)\big)^{1/4} + \frac{\log\big(a+\frac{b}{x}\big)}{\log x}\Bigg)\Bigg). \tag{22}$$

Finally, putting (13) and (22) into (12) and remembering (15), we obtain that

$$\begin{aligned} M_x(\mathcal{G}) - P(x) \ll \bigg(&\frac{1}{(\log x)^B} + \frac{(ax+b)^{s(1-\alpha)}}{x}\log x\bigg) \times \exp\left(\frac{csr^{\alpha}(\log r)^{A-1}}{(1-\alpha)^{A+1}}\right) \\ &+\Psi(x) s\Bigg(\frac{(\log r)^{A-1}}{r} + \log\log(a|b|+2)\Big(\big(S(r,x)\big)^{1/2} \\ &+\big(S(\sqrt{x},x)\big)^{1/4} + \frac{\log\big(a+\frac{b}{x}\big)}{\log x}\Big)\Bigg), \end{aligned} \tag{23}$$

and our Theorem is proved.

Proof of Corollary 1. In case when $g_l(n) = \varphi(n)/n$, we have $h_l(p) = h(p) = -1/p$ and $h_l(p^m) = 0$ for $m \geq 2$. Hence

$$w_p = \sum_{\substack{m_1=0 \\ (p^{m_j}, p^{m_k}) | (k-j), 1 \leq j < k \leq s \\ (p^{m_l}, l)=1, \forall l}}^{1} \cdots \sum_{m_s=0}^{1} \frac{h(p^{m_1} \dots h(p^{m_s})}{\varphi([p^{m_1}, \dots, p^{m_s}])} .$$

If $p > s$, it is clear that

$$w_p = 1 - \frac{s}{p(p-1)}.$$

Let $p \leq s$. Split the numbers $1, \dots, s$ into residue classes mod p. There are $p-1$ residue classes the members of which are coprime to p. Among these classes there are η residue classes with $\xi + 1$ members and $p - \eta - 1$ residue classes with ξ members. Let us observe also that $p|(k-j)$ only if j and k belong to the same residue class mod p. Therefore

$$\begin{aligned} w_p = 1 + \frac{1}{\varphi(p)} \Bigg(& \Big(\eta C_{\xi+1}^1 + (p-\eta-1) C_\xi^1 \Big) h(p) \\ & + \Big(\eta C_{\xi+1}^2 + (p-\eta-1) C_\xi^2 \Big) h^2(p) \\ & + \dots + \Big(\eta C_{\xi+1}^\xi + (p-\eta-1) C_\xi^\xi \Big) h^\xi(p) + \eta C_{\xi+1}^{\xi+1} h^{\xi+1}(p) \Bigg) \\ = 1 + \frac{\eta}{p-1} & \left(\left(1 - \frac{1}{p}\right)^{\xi+1} - 1 \right) + \frac{p-\eta-1}{p-1} \left(\left(1 - \frac{1}{p}\right)^{\xi} - 1 \right). \end{aligned}$$

The values of v_p can be evaluated in a similar way.

The estimates of the remainder terms in Corollary 1 can be got from (23) by choosing for example

$$\alpha = \frac{s - 1 + c_{20}}{s},$$
$$r = \begin{cases} c_{21} (\log\log x)^{1/3} \log\log\log x & \text{in the common case,} \\ c_{21} (\log\log x \log\log\log x)^{1/\alpha} & \text{if } s \text{ is fixed,} \end{cases}$$

with sufficiently small constants c_{20}, c_{21} and by making some simple calculations.

The proof of Corollary 2 is based on the method of characteristic functions. This method is well-known. It is explained for example in [6].

The proof of Corollary 4 is based on Weyl's well-known theorem [10] about the uniform distribution mod 1 of a number sequence. This proof can be realized in the same way as in [9].

For *the proof of Corollary 5* it is enough to repeat the proof of the analogous result from [9].

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GEDIMINAS STEPANAUSKAS
DEPARTMENT OF PROBABILITY THEORY AND NUMBER THEORY
VILNIUS UNIVERSITY
NAUGARDUKO 24
2600 VILNIUS
LITHUANIA

E-mail: Gediminas.Stepanauskas@maf.vu.lt

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