Publ. Math. Debrecen
52 / 3-4 (1998), 699–704

On common eigenfunctions of difference operators

By LÁSZLÓ SZÉKELYHIDI (Debrecen)

Dedicated to Professors Zoltán Daróczy and Imre Kátai

Abstract. Exponential functions on the reals can be characterized as normed eigenfunctions of the differential operator belonging to nonzero eigenvalues. Exponential functions on abelian groups are defined as homomorphisms into the multiplicative group of nonzero complex numbers. In other words they are common normed eigenfunctions of all translation operators. In this paper nonconstant exponential functions on some abelian groups are characterized as those common normed eigenfunctions of all difference operators of order n, which do not belong to their common kernel.

In [3] we studied functional equations of the type

$$f(x+ny)+\sum_{k=0}^{n-1} c_k(y) f(x+ky)=0 \tag{1}$$

on locally compact abelian groups (see also [4], Chapter 3, Section 13). As a main result we proved that on finitely generated discrete abelian groups, or on compactly generated locally compact abelian groups, in which the set of all compact elements is connected, every continuous complex valued function f satisfying an equation of the above form with some complex valued functions c_k is an exponential polynomial. In this paper we consider the special case (2) (see below) of the above functional equation and we

---

Mathematics Subject Classification: Primary 39B40, Secondary 39B50.
Key words and phrases: functional equation, spectral synthesis.
Research supported by FKFP 0310/1997 and OTKA Grant T-016846

characterize all solutions of it on some types of abelian groups. It turns out that this leads to a new characterization of exponential functions.

If G is a locally compact abelian group, then the *difference operator* Δ_y^n *of order* n is defined for any y in G and for any positive integer n in the usual way: if x, y are in G, then

$$\Delta_y^n f(x) = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} f(x+iy),$$

for any complex valued function f on G. The continuous complex valued function f is called a *generalized polynomial of degree at most* n, if $\Delta_y^{n+1} f(x) = 0$ holds for any x, y in G. In other words, polynomials of degree at most n are the elements of the common kernel of the difference operators of order $n+1$. For instance, if G is the additive group of the reals equipped with the usual topology, then generalized polynomials of degree at most n are exactly the ordinary polynomials of degree at most n. For more about generalized polynomials see e.g. [4].

If G is a locally compact abelian group, then any continuous homomorphism of G into the additive group of the reals is called an *additive function*, and any continuous homomorphism of G into the multiplicative group of nonzero complex numbers is called an *exponential function*. A product of additive functions is called a *monomial* and a product of a monomial and an exponential function is called an *exponential monomial*. Complex linear combinations of monomials, resp. exponential monomials are called *polynomials*, resp. *exponential polynomials*. Any polynomial is a generalized polynomial in the above sense. For more about polynomials and exponential polynomials on abelian groups see e.g. [4].

Exponential functions are common eigenfunctions of all translation operators. As any difference operator is a linear combination of translation operators, exponential functions are also common eigenfunctions of all difference operators of any order. The natural question arises: which functions are the common eigenfunctions of all difference operators of a fixed order? If the fixed order is one, then the common eigenfunctions are the constants, forming the common kernel, and constant multiples of exponentials. The subject of this paper is to prove a partial converse of this statement for higher order difference operators.

For any locally compact abelian group G the symbol $\mathcal{C}(G)$ denotes the locally convex topological vectorspace of all continuous complex valued functions equipped with the pointwise operations and the topology of uniform convergence on all compact sets. A closed linear translation invariant subspace of $\mathcal{C}(G)$ is called a *variety*. For instance, the closed linear subspace generated by all translates of a function in $\mathcal{C}(G)$ is a variety, which is called the *variety generated* by the given function. The set of all exponential functions, resp. exponential monomials contained in a variety is called the *spectrum*, resp. the *spectral set* of the variety. We say that *spectral synthesis holds* for a given variety, if this variety is equal to the closure of the linear hull of its spectral set. If spectral synthesis holds for any variety in $\mathcal{C}(G)$, then we simply say that spectral synthesis holds for G. For instance, spectral synthesis holds for $\mathbb{R}$, and for any finitely generated discrete abelian group. For more about spectral synthesis see e.g. [1], [2], [4].

As it has been pointed out in [4] spectral synthesis can be utilized for the solution of some types of functional equations. Here we consider the functional equation

$$\Delta_y^n f(x) = f(x)g(y), \tag{2}$$

where we suppose that G is a locally compact abelian group, $f, g : G \to \mathbb{C}$ are continuous functions, n is a fixed positive integer and (2) is supposed to hold for any x, y in G. In this case we say that *f, g form a solution* of (2). Obviously (2) is a special case of (1), but the results of [3] could be used only to derive that f in (2) must be an exponential polynomial.

Our main result establishes that nonconstant complex exponential functions on G can be characterized as common continuous normed eigenfunctions of all difference operators of order n belonging to nonzero eigenvalues, if G is an n-divisible locally compact abelian group for which spectral synthesis holds. Here n is a fixed positive integer, "normed" means that $f(0) = 1$ and an abelian group is called *n-divisible*, if the mapping $x \mapsto nx$ is surjective.

Theorem. *Let n be a positive integer and let G be an n-divisible locally compact abelian group for which spectral synthesis holds. Let $f, g : G \to \mathbb{C}$ be continuous functions, where f is nonidentically zero. The functions f, g form a solution of* (2) *if and only if*

(i) *either g is identically zero and f is a generalized polynomial of degree at most $n - 1$,*

(ii) *or there exists an exponential function m on G and a complex number c such that*

$$f(x) = cm(x),$$
$$g(x) = (m(x) - 1)^n$$

holds for every x in G.

Proof. It is a routine calculation to check that the functions given in (i) and (ii) form a solution of (2). Conversely, suppose that $f \neq 0$ and g are functions forming a solution of (2). If g is identically zero then

$$\Delta_y^n f(x) = 0$$

holds for all x in G, hence f is a generalized polynomial of degree at most $n-1$ (see e.g. [4]). Now suppose that g is nonidentically zero. It is clear that any function in the variety generated by f, together with g forms a solution of (2). As f is nonidentically zero then by spectral synthesis on G there exists an exponential function m such that m, g form a solution of (2). Hence we have for all x, y in G

$$\Delta_y^n m(x) = m(x)(m(y) - 1)^n = m(x)g(y),$$

that is, $g(y) = (m(y) - 1)^n$ holds for all y in G. On the other hand, if m_0 is any exponential function for which m_0, g form a solution of (2), then

$$(m_0(y) - 1)^n = (m(y) - 1)^n$$

holds for any y in G. It follows for any $m(y) \neq 1$ that

$$\begin{aligned} m_0(ny) &= m_0(y)^n (1 - m_0(-y))^n (1 - m(-y))^{-n} \\ &= (m_0(y) - 1)^n (m(y) - 1)^{-n} m(y)^n = m(ny), \end{aligned}$$

and hence $m_0 = m$. This means that the spectrum of the variety generated by f contains the exponential function m only, where $m \neq 1$. Now we determine the spectral set of the variety generated by f. Suppose that the exponential monomial $x \mapsto p(x)m(x)$ belongs to this variety with some monomial p. If p is nonconstant, then by translation invariance we infer that $x \mapsto a(x)m(x)$ also belongs to the variety for some nonzero additive function a (see [4], Lemma 4.8, p. 44), which means, that together with g it forms a solution of (2). By substitution we have for all x, y in G

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} a(x+iy) m(y)^i = a(x)(m(y)-1)^n,$$

which implies

$$a(y) \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} i m(y)^i = 0$$

or any y in G. On the other hand, we have

$$a(y) \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} i m(y)^i = n a(y) m(y)(m(y)-1)^{n-1}$$

for any y in G. This implies that $a(y)(m(y)-1) = 0$, that is $a(y)m(y) = a(y)$ for any y in G. As m is different from the exponential function 1 and different exponential functions are linearly independent over polynomials (see e.g. [4], Lemma 4.3, p. 41), we have a contradiction. That means, the spectral set of the variety generated by f consists of the constant multiples of m. The proof is complete by spectral synthesis.

We can formulate two corollaries.

Corollary 1. *Let $f, g : \mathbb{R} \to \mathbb{C}$ be continuous functions, f is nonidentically zero, and let n be a positive integer. Then f, g form a solution of* (2) *if and only if*

(i) *either g is identically zero and f is a polynomial of degree at most $n-1$,*

(ii) *or there exist complex numbers λ, c such that*

$$f(x) = ce^{\lambda x},$$
$$g(x) = (e^{\lambda x} - 1)^n$$

holds for every x in $\mathbb{R}$.

Corollary 2. *Let n be a positive integer, G a discrete n-divisible finitely generated abelian group and let $f, g : G \to \mathbb{C}$ be functions, where f is nonidentically zero. The functions f, g form a solution of* (2) *if and only if*

(i) *either g is identically zero and f is a generalized polynomial of degree at most $n-1$,*

(ii) *or there exists an exponential function m on G and a complex number c such that*

$$f(x) = cm(x),$$
$$g(x) = (m(x) - 1)^n$$

holds for every x in $\mathbb{R}$.

Here we give an example which shows that the Theorem is no longer valid if the hypothesis on n-divisibility is dropped. Let G be the discrete abelian group $\mathbb{Z}$ and let $n = 4$. Let for any x in $\mathbb{Z}$

$$f(x) = \mathbf{i}^x + (-\mathbf{i})^x,$$
$$g(x) = (\mathbf{i}^x - 1)^4,$$

where $\mathbf{i}$ is the complex imaginary unit. Then f, g form a solution of (2), f, g are nonidentically zero, and f is not a constant multiple of an exponential, as $f(1) = 0$. We note that $\mathbb{Z}$ is finitely generated, but it fails to be 4-divisible.

References

[1] M. Lefranc, L'analyse harmonique dans $\mathbb{Z}^n$, *C. R. Acad. Sci. Paris* **246** (1958), 1951–1953.

[2] L. Schwartz, Théorie génerale des fonctions moyenne-périodiques, *Ann. of Math.* **48** no. 2 (1947), 857–929.

[3] L. Székelyhidi, On a linear functional equation, *Aequationes Math.* **38** (1989), 113–122.

[4] L. Székelyhidi, Convolution Type Functional Equations on Topological Abelian Groups, *World Scientific*, *Singapore, New Jersey, London, Hong Kong*, 1991.

LÁSZLÓ SZÉKELYHIDI
INSTITUTE OF MATHEMATICS AND INFORMATICS
LAJOS KOSSUTH UNIVERSITY
H–4010 DEBRECEN, P.O. BOX 12
HUNGARY

(Received November 24, 1997; revised February 20, 1998)