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(C, α) means of d-dimensional trigonometric-Fourier series

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*Dedicated to the 60th birthday of
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Abstract. The d-dimensional classical Hardy spaces $H_p(\mathbf{T}^d)$ are introduced and it is shown that the maximal operator of the (C,α) $(\alpha = (\alpha_1, \dots, \alpha_d))$ means of a distribution is bounded from $H_p(\mathbf{T}^d)$ to $L_p(\mathbf{T}^d)$ $(d/(d+1), 1/(\alpha_k+1) < p < \infty)$ provided that the supremum in the maximal operator is taken over a positive cone. Moreover, we prove that the (C,α) means are uniformly bounded on the spaces $H_p(\mathbf{T}^d)$ whenever $d/(d+1), 1/(\alpha_k+1) < p < \infty$. Thus, in case $f \in H_p(\mathbf{T})$, the Cesàro means converge to f in $H_p(\mathbf{T}^d)$ norm $(d/(d+1), 1/(\alpha_k+1) < p < \infty)$. The same results are proved for the conjugate (C,α) means, too.

1. Introduction

The Hardy–Lorentz spaces $H_{p,q}(\mathbf{T}^d)$ of distributions are introduced with the $L_{p,q}(\mathbf{T}^d)$ Lorentz norms of the non-tangential maximal function. Of course, $H_p(\mathbf{T}^d) = H_{p,p}(\mathbf{T}^d)$ are the usual Hardy spaces $(0 < p \le \infty)$.

For multi-dimensional trigonometric-Fourier series Marcinkievicz and Zygmund [7] proved that the Fejér means $\sigma_n^1 f$ of a function $f \in L_1(\mathbf{T}^d)$ converge a.e. to f as $\min(n_1, \dots, n_d) \to \infty$ provided that n is in

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a positive cone, i.e., provided that $2^{-\tau} \le n_k/n_j \le 2^{\tau}$ for every $k,j = 1,\dots,d$ and for some $\tau \ge 0$ ($n = (n_1,\dots,n_d) \in \mathbf{N}^d$).

Recently the author [15] obtained the same convergence result for the (C,α) means σ_n^α by proving the weak type inequality

$$\sup_{\rho>0} \rho\lambda(\sigma_*^\alpha f > \rho) \le C\|f\|_1 \qquad (f \in L_1(\mathbf{T}^d))$$

where $\sigma_*^\alpha := \sup_{\substack{2^{-\tau}\le n_k/n_j \le 2^{\tau}\\ k,j=1,\dots,d}} |\sigma_n^\alpha|$, $\alpha = (\alpha_1,\dots,\alpha_d)$ and $0 < \alpha_k \le 1$. Moreover, for the Fejér means (if $\alpha_k = 1$) the author [15] verified that σ_*^1 is bounded from $H_{p,q}(\mathbf{T}^d)$ to $L_{p,q}(\mathbf{T}^d)$ if $(2d+1)/(2d+2) < p \le \infty$ and $0 < q \le \infty$. Under some conditions on α we proved also a similar result for the (C,α) means. The one-dimensional results are described in Weisz [14].

In this paper we sharpen and generalize these results for arbitrary $0 < \alpha_k \le 1$. We will show that the maximal operator σ_*^α is bounded from $H_{p,q}(\mathbf{T}^d)$ to $L_{p,q}(\mathbf{T}^d)$ whenever $d/(d+1), 1/(\alpha_k+1) < p \le \infty, 0 < q \le \infty$. We introduce the conjugate distributions (or Riesz transforms) $\tilde{f}^{(i)} = R_i f$, the conjugate (C,α) means $\tilde{\sigma}_n^{(i);\alpha}$ and the conjugate maximal operators $\tilde{\sigma}_*^{(i);\alpha}$ where $i = 0,1,\dots,d$. We obtain that the operator $\tilde{\sigma}_*^{(i);\alpha}$ is also of type $(H_{p,q}(\mathbf{T}^d), L_{p,q}(\mathbf{T}^d))$ if $d/(d+1)$, $1/(\alpha_k+1) < p \le \infty$, $0 < q \le \infty$ and of weak type $(1,1)$.

A usual density argument implies then that, besides the convergence results mentioned above, the conjugate (C,α) means $\tilde{\sigma}_n^{(i);\alpha} f$ converge a.e. to $\tilde{f}^{(i)}$ as $n \to \infty$ and $2^{-\tau} \le n_k/n_j \le 2^{\tau}$, provided that $f \in L_1(\mathbf{T}^d)$. Note that $\tilde{f}^{(i)}$ is not necessarily integrable whenever f is.

We will prove also that the operators σ_n^α and $\tilde{\sigma}_n^{(i);\alpha}$ ($n \in \mathbf{N}$) are uniformly bounded from $H_{p,q}(\mathbf{T}^d)$ to $H_{p,q}(\mathbf{T}^d)$ if $d/(d+1)$, $1/(\alpha_k+1) < p \le \infty$, $0 < q \le \infty$. From this it follows that $\sigma_n^\alpha f \to f$ and $\tilde{\sigma}_n^{(i);\alpha} f \to \tilde{f}^{(i)}$ in $H_{p,q}(\mathbf{T}^d)$ norm as $n \to \infty$, $2^{-\tau} \le n_k/n_j \le 2^{\tau}$, whenever $f \in H_{p,q}(\mathbf{T})$ and $d/(d+1)$, $1/(\alpha_k+1) < p \le \infty$, $0 < q \le \infty$.

2. Hardy spaces and Riesz transforms

For a set $\mathbf{X} \ne \emptyset$ let $\mathbf{X}^d$ be its Cartesian product taken with itself d times ($d \in \mathbf{N}$), moreover, let $\mathbf{T} := [-\pi,\pi)$ and λ be the d-dimensional Lebesgue measure. We also use the notation $|I|$ for the Lebesgue measure of the set I. We briefly write L_p instead of the real $L_p(\mathbf{T}^d,\lambda)$ space while the norm (or quasinorm) of this space is defined by $\|f\|_p := (\int_{\mathbf{T}^d} |f|^p \, d\lambda)^{1/p}$ $(0 < p \le \infty)$. For simplicity, we assume that for a function $f \in L_1$ we have $\int_{\mathbf{T}^d} f \, d\lambda = 0$.

The distribution function of a Lebesgue-measurable function f is defined by

$$\lambda(\{|f| > \rho\}) := \lambda(\{x : |f(x)| > \rho\}) \qquad (\rho \ge 0).$$

The *weak* L_p space L_p^* $(0 < p < \infty)$ consists of all measurable functions f for which

$$\|f\|_{L_p^*} := \sup_{\rho > 0} \rho \lambda(\{|f| > \rho\})^{1/p} < \infty$$

while we set $L_\infty^* = L_\infty$.

The spaces L_p^* are special cases of the more general Lorentz spaces $L_{p,q}$. In their definition another concept is used. For a measurable function f the *non-increasing rearrangement* is defined by

$$\check{f}(t) := \inf\{\rho : \lambda(\{|f| > \rho\}) \le t\}.$$

Lorentz space $L_{p,q}$ is defined as follows: for $0 < p < \infty, 0 < q < \infty$

$$\|f\|_{p,q} := \left(\int_0^\infty \check{f}(t)^q t^{q/p} \, \frac{dt}{t} \right)^{1/q}$$

while for $0 < p \le \infty$

$$\|f\|_{p,\infty} := \sup_{t > 0} t^{1/p} \check{f}(t).$$

Let

$$L_{p,q} := L_{p,q}(\mathbf{T}^d, \lambda) := \{f : \|f\|_{p,q} < \infty\}.$$

One can show the following equalities:

$$L_{p,p} = L_p, \quad L_{p,\infty} = L_p^* \qquad (0 < p \le \infty)$$

(see e.g. Bennett, Sharpley [1] or Bergh, Löfström [2]).

We introduce the $H_p(\mathbf{T}^d)$ Hardy space in a similar way as in Weisz [15]. Let us fix $d \ge 1$. For $n = (n_1, \dots, n_d) \in \mathbf{Z}^d$ and $x = (x_1, \dots, x_d) \in \mathbf{T}^d$ set $n \cdot x := \sum_{k=1}^d n_k x_k$ and $|n| := (\sum_{k=1}^d |n_k|^2)^{1/2}$. Let f be a distribution on $C^\infty(\mathbf{T}^d)$ (briefly $f \in \mathcal{D}'(\mathbf{T}^d) = \mathcal{D}'$). The nth Fourier coefficient is defined by $\hat{f}(n) := f(e^{-\imath n\cdot x})$ where $\imath = \sqrt{-1}$ and $n \in \mathbf{Z}^d$. In special case, if f is an integrable function then

$$\hat{f}(n) = \frac{1}{(2\pi)^d} \int_{\mathbf{T}^d} f(x) e^{-\imath n\cdot x}\, dx.$$

For simplicity, we assume that, for a distribution $f \in \mathcal{D}'$, we have $\hat{f}(0) = 0$.

For $f \in \mathcal{D}'$ and $t > 0$ define the harmonic function u by

$$u(x,t) := (f * P_t)(x)$$

where $*$ denotes the convolution and

$$P_t(x) := \sum_{k\in\mathbf{Z}^d} e^{-t|k|} e^{\imath k\cdot x} \qquad (x \in \mathbf{T}^d)$$

is the Poisson kernel. Let $\Gamma := \{(x,t) : |x| < t\}$, a cone whose vertex is the origin. We denote by $\Gamma'(x)$ ($x \in \mathbf{T}^d$) the translate of Γ so that its vertex is x. Set

$$\Gamma(x) = \bigcup_{k\in\mathbf{Z}^d} \Gamma'\big((x_i + k_i 2\pi)\big) \cap \mathbf{T}^d.$$

The non-tangential maximal function is defined by

$$u^*(x) := \sup_{(x',t)\in\Gamma(x)} |u(x',t)| \qquad (\alpha > 0).$$

For $0 < p, q \le \infty$ the *Hardy-Lorentz space* $H_{p,q}(\mathbf{T}^d) = H_{p,q}$ consists of all distributions f for which $u^* \in L_{p,q}$ and set

$$\|f\|_{H_{p,q}} := \|u^*\|_{p,q}.$$

Note that in case $p = q$ the usual definition of Hardy spaces $H_{p,p} = H_p$ are obtained. It is known that if $f \in H_p$ ($0 < p < \infty$) then $f(x) = \lim_{t\to 0} u(x,t)$ in the sense of distributions (see Fefferman, Stein [6]). Recall that $L_1 \subset H_{1,\infty}$, more exactly,

$$\|f\|_{H_{1,\infty}} = \sup_{\rho>0} \rho\lambda(u^* > \rho) \le C\|f\|_1 \qquad (f \in L_1). \tag{1}$$

Moreover, $H_{p,q} \sim L_{p,q}$ for $1 < p < \infty$, $0 < q \le \infty$ (see Fefferman, Stein [6], Stein [10], Fefferman, Riviere, Sagher [5]).

The following interpolation result concerning Hardy-Lorentz spaces will be used several times in this paper (see Fefferman, Riviere, Sagher [5] and also Weisz [16]).

Theorem A. *If a sublinear (resp. linear) operator T is bounded from H_{p_0} to L_{p_0} (resp. to H_{p_0}) and from L_{p_1} to L_{p_1} ($p_0 \le 1 < p_1 \le \infty$) then it is also bounded from $H_{p,q}$ to $L_{p,q}$ (resp. to $H_{p,q}$) if $p_0 < p < p_1$ and $0 < q \le \infty$.*

For a distribution

$$f \sim \sum_{n \in \mathbf{Z}^d} \hat{f}(n) e^{\imath n \cdot x}$$

the *Riesz transforms* or the *conjugate distributions* are defined by

$$\tilde{f}^{(i)} := R_i f \sim \sum_{n \in \mathbf{Z}^d} -\imath \, \frac{n_i}{|n|} \hat{f}(n) e^{\imath n \cdot x} \qquad (i = 1, \dots, d).$$

We use the notation $\tilde{f}^{(0)} := f$.

As is well known, if f is an integrable function then the conjugate functions $\tilde{f}^{(i)}$ ($i = 1, \dots, d$) do exist almost everywhere, but they are not integrable in general.

Fefferman and Stein [6] and Uchiyama [13] verified that if $f \in H_p$ ($0 < p < \infty$) then all conjugate distributions are also in H_p and

$$\|\tilde{f}^{(i)}\|_{H_p} \le C_p \|f\|_{H_p} \qquad (i = 1, \dots, d). \tag{2}$$

Furthermore, if $(d-1)/d < p < \infty$ then the following equivalence holds:

$$\|f\|_{H_p} \sim \|f\|_p + \sum_{i=1}^{d} \|\tilde{f}^{(i)}\|_p. \tag{3}$$

3. (C,α) summability of d-dimensional trigonometric-Fourier series

Denote by $s_n f$ and $\tilde{s}_n^{(i)} f$ the nth partial sum and conjugate partial sum of the Fourier series of a distribution f, respectively, namely,

$$s_n f(x) := \sum_{j=1}^{d} \sum_{k_j = -n_j}^{n_j} \hat{f}(k) e^{\imath k \cdot x}$$

and

$$\tilde{s}_n^{(i)} f(x) := \sum_{j=1}^{d} \sum_{k_j = -n_j}^{n_j} -\imath \, \frac{k_i}{|k|} \hat{f}(k) e^{\imath k \cdot x}.$$

Let $\alpha=(\alpha_1,\ldots,\alpha_d)\in \mathbf{N}^d$ with $0<\alpha_k\le 1$ $(k=1,\ldots,d)$ and let

$$A_j^\gamma := \binom{j+\gamma}{j} = \frac{(\gamma+1)(\gamma+2)\ldots(\gamma+j)}{j!} = O(j^\gamma) \quad (j\in\mathbf{N}, 0<\gamma\le 1)$$

(see ZYGMUND [17]). The (C,α) *means* of a distribution f are defined by

$$\begin{aligned}\sigma_n^\alpha f :&= \prod_{i=1}^d \frac{1}{A_{n_i}^{\alpha_i}} \sum_{i=1}^d \sum_{k_i=0}^{n_i} A_{n_i-k_i}^{\alpha_i-1} s_k f \\ &= \prod_{i=1}^d \frac{1}{A_{n_i}^{\alpha_i}} \sum_{i=1}^d \sum_{k_i=-n_i}^{n_i} A_{n_i-|k_i|}^{\alpha_i} \hat{f}(k) e^{\imath k\cdot x} = f * (K_{n_1}^{\alpha_1}\times\ldots\times K_{n_d}^{\alpha_d})\end{aligned}$$

where the K_j^γ kernel satisfies the conditions

$$|K_j^\gamma(t)| \le C_\gamma j \qquad (0<|t|<\pi) \tag{4}$$

and

$$|K_j^\gamma(t)| \le \frac{C_\gamma}{j^\gamma |t|^{\gamma+1}} \qquad (0<|t|<\pi) \tag{5}$$

for $j\in\mathbf{N}$ and $0<\gamma\le 1$. From this it follows that

$$|K_j^\gamma(t)| \le \frac{C_\gamma j^{\eta+\gamma(\eta-1)}}{|t|^{(\gamma+1)(1-\eta)}} \qquad (0<|t|<\pi) \tag{6}$$

where $0\le\eta\le 1$ is arbitrary. We have similar estimates for the derivative of the kernel:

$$|(K_j^\gamma)'(t)| \le C_\gamma j^2 \qquad (0<|t|<\pi) \tag{7}$$

and

$$|(K_j^\gamma)'(t)| \le \frac{C_\gamma}{j^{\gamma-1}|t|^{\gamma+1}} \qquad (0<|t|<\pi) \tag{8}$$

for $j\in\mathbf{N}$ and $0<\gamma\le 1$ (see ZYGMUND [17], Vol. II, p. 60). Thus, for $0\le\theta\le 1$,

$$|(K_j^\gamma)'(t)| \le \frac{C_\gamma j^{\theta+1+\gamma(\theta-1)}}{|t|^{(\gamma+1)(1-\theta)}} \qquad (0<|t|<\pi). \tag{9}$$

It is easy to see that (4) and (5) imply

$$\int_{\mathbf{T}} |K_j^\gamma|\, d\lambda \le C_\gamma \qquad (j \in \mathbf{N}). \tag{10}$$

The *conjugate* (C,α) *means* of a distribution f are introduced by

$$\begin{aligned}\tilde\sigma_n^{(i);\alpha} f :&= \prod_{i=1}^d \frac{1}{A_{n_i}^{\alpha_i}} \sum_{i=1}^d \sum_{k_i=0}^{n_i} A_{n_i-k_i}^{\alpha_i-1} \tilde s_k^{(i)} f \\ &= \prod_{i=1}^d \frac{1}{A_{n_i}^{\alpha_i}} \sum_{i=1}^d \sum_{k_i=-n_i}^{n_i} A_{n_i-|k_i|}^{\alpha_i} (-\imath\, \frac{k_i}{|k|}) \hat f(k) e^{\imath k\cdot x} \\ &= \tilde f^{(i)} * (K_{n_1}^{\alpha_1} \times \ldots \times K_{n_d}^{\alpha_d}).\end{aligned}$$

For a fixed $\tau \ge 0$ the *restricted maximal* and *restricted maximal conjugate* (C,α) *operators* are defined by

$$\sigma_*^\alpha f := \sup_{\substack{2^{-\tau} \le n_k/n_j \le 2^\tau \\ k,j=1,\ldots,d}} |\sigma_n^\alpha f|$$

and

$$\tilde\sigma_*^{(i);\alpha} f := \sup_{\substack{2^{-\tau} \le n_k/n_j \le 2^\tau \\ k,j=1,\ldots,d}} |\tilde\sigma_n^{(i);\alpha} f|.$$

Obviously,

$$\tilde\sigma_n^{(i);\alpha} f = \sigma_n^\alpha \tilde f^{(i)} \quad \text{and } \tilde\sigma_*^{(i);\alpha} f = \sigma_*^\alpha \tilde f^{(i)} \qquad (i = 0,1,\ldots,d). \tag{11}$$

4. The boundedness of the maximal (C,α) operator

A *generalized interval* on $\mathbf{T}$ is either an interval $I \subset \mathbf{T}$ or $I = [-\pi, x) \cup [y, \pi)$. A *generalized cube* on $\mathbf{T}^d$ is the Cartesian product $I_1 \times \ldots \times I_d$ of d generalized intervals with $|I_1| = \ldots = |I_d|$. A bounded measurable function a is a p-*atom* if there exists a generalized cube R such that

(i) $\int_R a(x) x^\beta\, dx = 0$ for all multi-indices $\beta = (\beta_1, \ldots, \beta_d) \in \mathbf{N}^d$ with $|\beta| \le [d(1/p-1)]$, the integer part of $d(1/p-1)$,

(ii) $\|a\|_\infty \le |R|^{-1/p}$,

(iii) $\{a \ne 0\} \subset R$.

If I is a generalized interval then let $4I$ be the generalized interval with the same center as I and with lenght $4|I|$. For a generalized cube $R = I_1 \times \ldots \times I_d$ let $4R = 4I_1 \times \ldots \times 4I_d$.

An operator T which maps the set of distributions into the collection of measurable functions, will be called *p-quasi-local* if there exists a constant $C_p > 0$ such that

$$\int_{\mathbf{T}^d \setminus 4R} |Ta|^p \, d\lambda \leq C_p$$

for every p-atom a where R is the support of the atom. The following result can be found in WEISZ [15]:

Theorem B. *Suppose that the operator T is sublinear and p-quasi-local for some $0 < p \leq 1$. If T is bounded from L_{p_1} to L_{p_1} for a fixed $1 < p_1 \leq \infty$ then*

$$\|Tf\|_p \leq C_p \|f\|_{H_p} \qquad (f \in H_p).$$

Now we can formulate our main result.

Theorem 1. *Suppose that* $\max\{d/(d+1), 1/(\alpha_k+1), k = 1, \ldots, d\} =: p_0 < p < \infty$, $0 < q \leq \infty$ *and* $0 < \alpha_k \leq 1$ $(k = 1, \ldots, d)$. *Then*

$$\|\sigma_*^{\alpha} f\|_{p,q} \leq C_{p,q} \|f\|_{H_{p,q}} \quad (f \in H_{p,q}). \tag{12}$$

PROOF. For simplicity we prove the result for $d = 2$, only. For $d > 2$ the verification is very similar. Now we denote the elements of $\mathbf{N}^2$ by (n, m) and we write (α, β) instead of (α_1, α_2).

By Theorems A and B the proof of Theorem 1 will be complete if we show that the operator $\sigma_*^{\alpha,\beta}$ is p-quasi-local for each $p_0 < p \leq 1$ and is bounded from L_∞ to L_∞.

The boundedness follows from (10). Let a be an arbitrary p-atom with support $R = I \times J$ and $2^{-K-1} < |I|/\pi = |J|/\pi \leq 2^{-K}$ $(K \in \mathbf{N})$. We can suppose that the center of R is zero. In this case

$$[-\pi 2^{-K-2}, \pi 2^{-K-2}] \subset I, J \subset [-\pi 2^{-K-1}, \pi 2^{-K-1}].$$

Choose $r \in \mathbf{N}$ such that $r - 1 < \tau \leq r$. It is easy to see that if $n \geq k$ or $m \geq k$ for a fixed $k \in \mathbf{N}$ then we have $n, m \geq k 2^{-r}$. Indeed, since (n, m) is in a cone, $n \geq k$ implies

$$m \geq 2^{-\tau} n \geq k 2^{-r}.$$

To prove the quasi-locality of the operator $\sigma_*^{\alpha,\beta}$ we have to integrate $|\sigma_*^{\alpha,\beta}a|^p$ over $\mathbf{T}^2 \setminus 4R$. We do this in three steps.

Step 1. Integrating over $(\mathbf{T} \setminus 4I) \times 4J$. Obviously,

$$\begin{aligned}
(13) \qquad & \int_{\mathbf{T}\setminus 4I} \int_{4J} |\sigma_*^{\alpha,\beta} a(x,y)|^p \, dx \, dy \\
& \le \sum_{|i|=1}^{2^K-1} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \int_{4J} |\sigma_*^{\alpha,\beta} a(x,y)|^p \, dx \, dy \\
& \le \sum_{|i|=1}^{2^K-1} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \int_{4J} \sup_{n,m \ge r_i 2^{-r}} |\sigma_{n,m}^{\alpha,\beta} a(x,y)|^p \, dx \, dy \\
& \quad + \sum_{|i|=1}^{2^K-1} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \int_{4J} \sup_{n,m < r_i} |\sigma_{n,m}^{\alpha,\beta} a(x,y)|^p \, dx \, dy \\
& = (A) + (B)
\end{aligned}$$

where $r_i := \left[\frac{2^K}{|i|^\delta}\right]$ $(i \in \mathbf{N})$ with $\delta > 0$ chosen later. We can suppose that $i > 0$.

The term (A) was estimated for $\alpha = \beta = 1$ in Weisz [15]. For the sake of the completeness we give the details in the general case. Using (5), (10) and the definition of the atom we conclude

$$\begin{aligned}
|\sigma_{n,m}^{\alpha,\beta} a(x,y)| &= \frac{1}{(2\pi)^2} \left| \int_I \int_J a(t,u) K_n^\alpha(x-t) K_m^\beta(y-u) \, dt \, du \right| \\
&\le C_p 2^{2K/p} \int_I \frac{1}{n^\alpha |x-t|^{\alpha+1}} \, dt.
\end{aligned}$$

If $x \in [\pi i 2^{-K}, \pi(i+1)2^{-K})$ $(i \ge 1)$ and $t \in I$ then

$$(14) \qquad \frac{1}{|x-t|^\nu} \le \frac{1}{(\pi i 2^{-K} - \pi 2^{-K-1})^\nu} \le \frac{C 2^{K\nu}}{i^\nu} \quad (\nu > 0).$$

Hence

$$|\sigma_{n,m}^{\alpha,\beta} a(x,y)| \le C_p 2^{2K/p + K\alpha} \frac{1}{n^\alpha i^{\alpha+1}}.$$

Since $n \geq r_i 2^{-r}$, we obtain

$$(A) \leq C_p \sum_{i=1}^{2^K-1} 2^{-2K} 2^{2K+K\alpha p} \frac{1}{r_i^{\alpha p} i^{(\alpha+1)p}} \leq C_p \sum_{i=1}^{2^K-1} \frac{1}{i^{(\alpha+1)p-\alpha\delta p}}$$

which is a convergent series if

$$\delta < \frac{(\alpha+1)p-1}{\alpha p}. \tag{15}$$

Now we consider (B). Let

$$A_1(x,u) := \int_{-\pi}^{x} a(t,u)\,dt \quad (x,u \in \mathbf{T})$$

and

$$A(x,y) := \int_{-\pi}^{y} A_1(x,u)\,du \quad (x,y \in \mathbf{T}).$$

Observe that

$$|A_1(x,u)| \leq |I|^{1-2/p}, \quad |A(x,y)| \leq |I|^{2-2/p}. \tag{16}$$

Integrating by parts we can see that

$$\begin{aligned} (17) \qquad & \int_I a(t,u) K_n^\alpha(x-t)\,dt \\ & = \Big[A_1(t,u)K_n^\alpha(x-t)\Big]_{-\mu}^{\mu} - \int_I A_1(t,u)(K_n^\alpha)'(x-t)\,dt \\ & = A_1(\mu,u)K_n^\alpha(x-\mu) - \int_I A_1(t,u)(K_n^\alpha)'(x-t)\,dt \end{aligned}$$

where $I = J = [-\mu,\mu]$. By (4), (5), (16) and (14),

$$\begin{aligned} & \left| \int_J A_1(\mu,u) K_n^\alpha(x-\mu) K_m^\beta(y-u)\,du \right| \\ & \quad \leq C_p 2^{2K/p-K} 2^{-K} \frac{1}{n^\alpha |x-\mu|^{\alpha+1}} m \leq C_p 2^{2K/p-2K} n^{1-\alpha} \frac{2^{K(\alpha+1)}}{i^{\alpha+1}} \\ & \quad = C_p 2^{2K/p+K\alpha-K} n^{1-\alpha} \frac{1}{i^{\alpha+1}} \end{aligned}$$

whenever $x \in [\pi i 2^{-K}, \pi(i+1)2^{-K})$.

On the other hand, by (5), (10), (16) and (14),

$$\left| \int_J \int_I A_1(t,u)(K_n^\alpha)'(x-t) K_m^\beta(y-u)\, du\, dt \right|$$
$$\le C_p 2^{2K/p-K} \int_I \frac{1}{n^{\alpha-1}|x-t|^{\alpha+1}}\, dt \le C_p 2^{2K/p+K\alpha-K} n^{1-\alpha} \frac{1}{i^{\alpha+1}}$$

in case $x \in [\pi i 2^{-K}, \pi(i+1)2^{-K})$. The inequality $n < r_i$ imply

$$(B) \le C_p \sum_{i=1}^{2^K-1} 2^{-2K} 2^{2K+K\alpha p-Kp} r_i^{(1-\alpha)p} \frac{1}{i^{(\alpha+1)p}}$$
$$\le C_p \sum_{i=1}^{2^K-1} \frac{1}{i^{(\alpha+1)p+(1-\alpha)\delta p}}$$

which is independent of K if

$$\delta > \frac{1-(\alpha+1)p}{(1-\alpha)p}.$$

This together with (15) yields that $p > 1/(\alpha+1)$. Hence we have proved that

$$\int_{\mathbf{T}\setminus 4I} \int_{4J} |\sigma_*^{\alpha,\beta} a(x,y)|^p\, dx\, dy \le C_p \tag{18}$$

for $p > 1/(\alpha+1)$ where C_p depends only on p, τ, α and β.

Step 2. Integrating over $(\mathbf{T}\setminus 4I) \times (\mathbf{T}\setminus 4J)$. Similarly to (13),

$$\int_{\mathbf{T}\setminus 4I} \int_{\mathbf{T}\setminus 4J} |\sigma_*^{\alpha,\beta} a(x,y)|^p\, dx\, dy$$
$$\le \sum_{|i|=1}^{2^K-1} \sum_{|j|=1}^{2^K-1} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \int_{\pi j 2^{-K}}^{\pi(j+1)2^{-K}} \sup_{n,m \ge r_{i,j} 2^{-r}} |\sigma_{n,m}^{\alpha,\beta} a(x,y)|^p\, dx\, dy$$
$$+ \sum_{|i|=1}^{2^K-1} \sum_{|j|=1}^{2^K-1} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \int_{\pi j 2^{-K}}^{\pi(j+1)2^{-K}} \sup_{n,m < r_{i,j}} |\sigma_{n,m}^{\alpha,\beta} a(x,y)|^p\, dx\, dy$$
$$= (C) + (D)$$

where $r_{i,j} := \left[\frac{2^K}{|ij|^{\delta/(\alpha+\beta)}}\right]$ with $\delta > 0$ chosen later. We suppose again that $i, j > 0$.

The term (C) was estimated in [15] for $\alpha = \beta = 1$. For arbitrary α and β we have by (5) and (14),

$$
\begin{aligned}
|\sigma_{n,m}^{\alpha,\beta} a(x,y)| &\le C_p 2^{2K/p} \int_I \frac{1}{n^\alpha |x-t|^{\alpha+1}}\, dt \int_J \frac{1}{m^\beta |y-u|^{\beta+1}}\, du \\
&\le C_p \frac{2^{2K/p+K\alpha+K\beta}}{n^\alpha m^\beta i^{\alpha+1} j^{\beta+1}}
\end{aligned}
$$

whenever $x \in [\pi i 2^{-K}, \pi(i+1)2^{-K})$ and $y \in [\pi j 2^{-K}, \pi(j+1)2^{-K})$. Therefore

$$
\begin{aligned}
(C) &\le C_p \sum_{i=1}^{2^K-1} \sum_{j=1}^{2^K-1} 2^{-2K} \frac{2^{2K+K\alpha p+K\beta p}}{r_{ij}^{\alpha p+\beta p} i^{(\alpha+1)p} j^{(\beta+1)p}} \\
&\le C_p \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{i^{(\alpha+1)p-\delta p} j^{(\beta+1)p-\delta p}}
\end{aligned}
$$

which converges if

$$
\delta < \frac{(\alpha+1)p-1}{p} \quad \text{and } \delta < \frac{(\beta+1)p-1}{p}. \tag{19}
$$

Using (17) and integrating by parts in both variables we get that

$$
\begin{aligned}
&\int_I \int_J a(t,u) K_n^\alpha(x-t) K_m^\beta(y-u)\, dt\, du \\
&\qquad = -\int_J A(\mu,u) K_n^\alpha(x-\mu) (K_m^\beta)'(y-u)\, du \\
&\qquad\quad + \int_I A(t,\mu) (K_n^\alpha)'(x-t) K_m^\beta(y-\mu)\, dt \\
&\qquad\quad - \int_I \int_J A(t,u) (K_n^\alpha)'(x-t) (K_m^\beta)'(y-u)\, dt\, du \\
&\qquad =: D_{n,m}^1(x,y) + D_{n,m}^2(x,y) + D_{n,m}^3(x,y)
\end{aligned}
$$

because $A(\mu,-\mu) = A(\mu,\mu) = 0$.

Applying (6), (9), (16) and (14) we derive

$$\begin{aligned}
&|D^1_{n,m}(x,y)| \\
&\le C_p 2^{2K/p-2K} \frac{n^{\eta+\alpha(\eta-1)}}{|x-\mu|^{(\alpha+1)(1-\eta)}} 2^{-K} \frac{m^{\theta+1+\beta(\theta-1)}}{|y-u|^{(\beta+1)(1-\theta)}} \\
&\le C_p 2^{2K/p-3K} n^{\eta+\alpha(\eta-1)} \left(\frac{2^K}{i}\right)^{(\alpha+1)(1-\eta)} m^{\theta+1+\beta(\theta-1)} \left(\frac{2^K}{j}\right)^{(\beta+1)(1-\theta)}
\end{aligned}$$

provided that $x \in [\pi i 2^{-K}, \pi(i+1)2^{-K})$ and $y \in [\pi j 2^{-K}, \pi(j+1)2^{-K})$. Choosing

$$\eta := \frac{2\alpha-1}{2(\alpha+1)} \quad \text{and} \quad \theta := \frac{2\beta-1}{2(\beta+1)}$$

we obtain

$$|D^1_{n,m}(x,y)| \le C_p 2^{2K/p} \frac{1}{i^{3/2}} \frac{1}{j^{3/2}}.$$

Thus

$$\int_{\mathbf{T}\setminus 4I} \int_{\mathbf{T}\setminus 4J} \sup_{n,m<r_{i,j}} |D^1_{n,m}(x,y)|^p \, dx \, dy \le C_p \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} 2^{-2K} 2^{2K} \frac{1}{i^{3p/2}} \frac{1}{j^{3p/2}}$$

and this is convergent if $p > 2/3$. Note that this is the best possible result which can be obtained in this way. The analogous estimation for $|D^2_{n,m}(x,y)|$ can be proved similarly.

To estimate $|D^3_{n,m}(x,y)|$ use (8), (14) and (16) and observe that

$$\begin{aligned}
|D^3_{n,m}(x,y)| &\le C_p 2^{2K/p-2K} \int_I \frac{1}{n^{\alpha-1}|x-t|^{\alpha+1}} \, dt \int_J \frac{1}{m^{\beta-1}|y-u|^{\beta+1}} \, du \\
&\le C_p \frac{2^{2K/p-2K+K\alpha+K\beta} n^{1-\alpha} m^{1-\beta}}{i^{\alpha+1} j^{\beta+1}}
\end{aligned}$$

for $x \in [\pi i 2^{-K}, \pi(i+1)2^{-K})$ and $y \in [\pi j 2^{-K}, \pi(j+1)2^{-K})$. So

$$\begin{aligned}
&\int_{\mathbf{T}\setminus 4I} \int_{\mathbf{T}\setminus 4J} \sup_{n,m<r_{i,j}} |D^3_{n,m}(x,y)|^p \, dx \, dy \\
&\qquad \le C_p \sum_{i=1}^{2^K-1} \sum_{j=1}^{2^K-1} 2^{-2K} \frac{2^{2K-2Kp+K\alpha p+K\beta p} r_{ij}^{(2-\alpha-\beta)p}}{i^{(\alpha+1)p} j^{(\beta+1)p}} \\
&\qquad \le C_p \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{i^{\frac{\delta(2-\alpha-\beta)p}{\alpha+\beta}+(\alpha+1)p}} \frac{1}{j^{\frac{\delta(2-\alpha-\beta)p}{\alpha+\beta}+(\beta+1)p}}.
\end{aligned}$$

This series converges if

$$\delta > (\alpha+\beta)\frac{1-(\alpha+1)p}{(2-\alpha-\beta)p} \quad \text{and} \quad \delta > (\alpha+\beta)\frac{1-(\beta+1)p}{(2-\alpha-\beta)p} \tag{20}$$

if $\alpha+\beta \neq 2$. In case $\alpha=\beta=1$ we have $p>1/(\alpha+1)$. It is easy to see that (19) and (20) imply $p>1/(\alpha+1)$ and $p>1/(\beta+1)$. In this case we have shown that

$$\int_{\mathbf{T}\setminus 4I}\int_{\mathbf{T}\setminus 4J} |\sigma_*^{\alpha,\beta} a(x,y)|^p \, dx\, dy \le C_p. \tag{21}$$

Step 3. Integrating over $4I \times (\mathbf{T}\setminus 4J)$. This case is analogous to Step 1.

Combining (18) and (21) we proved that $\sigma_*^{\alpha,\beta}$ is p-quasi-local for each $p_0 < p \le 1$. Theorems A and B complete the proof of (12). □

Note that Theorem 1 was proved in [15] for greater p_0 and under some strong conditions on α.

We can state the same for the maximal conjugate (C,α) operator.

Theorem 2. *Assume that $i=0,1,\dots,d$ and $0<\alpha_k\le 1$ $(k=1,\dots,d)$. Then*

$$\|\tilde\sigma_*^{(i);\alpha} f\|_{p,q} \le C_{p,q}\|f\|_{H_{p,q}} \quad (f\in H_{p,q})$$

for every $p_0<p<\infty$ and $0<q\le\infty$. Especially, if $f\in L_1$ then

$$\lambda(\tilde\sigma_*^{(i);\alpha} f > \rho) \le \frac{C}{\rho}\|f\|_1 \quad (\rho>0).$$

Proof. By Theorem 1 for $p=q$, (2) and (11) we obtain

$$\|\tilde\sigma_*^{(i);\alpha} f\|_p = \|\sigma_*^{\alpha}\tilde f^{(i)}\|_p \le C_p\|\tilde f^{(i)}\|_{H_p} \le C_p\|f\|_{H_p} \quad (f\in H_p)$$

for every $p_0<p<\infty$. The first inequality of Theorem 2 follows from Theorem A.

Let us point out this inequality for $p=1$ and $q=\infty$. If $f\in L_1$ then (1) implies

$$\|\tilde\sigma_*^{(i);\alpha} f\|_{1,\infty} = \sup_{\rho>0}\rho\lambda(\sigma_*^{(i);\alpha} f>\rho) \le C\|f\|_{H_{1,\infty}} \le C\|f\|_1$$

which shows the weak type inequality in Theorem 2. The proof of the theorem is complete. □

Since the trigonometric polynomials are dense in L_1, the weak type inequality of Theorem 2 and the usual density argument (see MARCINKIEWICZ, ZYGMUND [7]) imply

Corollary 1. *Assume that $i = 0,1,\ldots,d$ and $0 < \alpha_k \le 1$ $(k = 1,\ldots,d)$. If $f \in L_1$ then*

$$\tilde\sigma_n^{(i);\alpha} f \to \tilde f^{(i)} \quad \text{a.e.}$$

as $\min(n_1,\ldots,n_d) \to \infty$ and $2^{-\tau} \le n_k/n_j \le 2^{\tau}$ $(k,j = 1,\ldots,d)$.

Note that Theorem 2 and Corollary 1 for $i = 0$ were proved under some conditions on α (see [15]). For other i's these are new results since $\tilde f^{(i)}$ is not necessarily integrable whenever f is.

Now we consider the norm convergence of $\sigma_n^\alpha f$. It follows from (12) that $\sigma_n^\alpha f \to f$ in L_p norm as $n \to \infty$ if $f \in L_p$ $(1 < p < \infty)$. We are going to generalize this result.

Theorem 3. *Assume that $i = 0,1,\ldots,d$ and $0 < \alpha_k \le 1$ $(k = 1,\ldots,d)$. If $n \in \mathbf{N}^d$ is in the cone, i.e. $2^{-\tau} \le n_k/n_j \le 2^{\tau}$ for all $k,j = 1,\ldots,d$, then*

$$\|\tilde\sigma_n^{(i);\alpha} f\|_{H_{p,q}} \le C_{p,q} \|f\|_{H_{p,q}} \qquad (f \in H_{p,q})$$

whenever $p_0 < p < \infty$ and $0 < q \le \infty$.

PROOF. Since $(\sigma_n^\alpha f)^{\sim(i)} = \tilde\sigma_n^{(i);\alpha} f$, we have by Theorems 1 and 2 that

$$\|(\sigma_n^\alpha f)^{\sim(i)}\|_p \le C_p \|f\|_{H_p} \qquad (f \in H_p).$$

(3) implies that

$$\|\tilde\sigma_n^{(i);\alpha} f\|_{H_p} \le C_p \|f\|_{H_p} \qquad (f \in H_p).$$

Now Theorem A proves Theorem 3. □

Corollary 2. *Suppose that $i = 0,1,\ldots,d$, $0 < \alpha_k \le 1$ $(k = 1,\ldots,d)$, $p_0 < p < \infty$ and $0 < q \le \infty$. If $f \in H_{p,q}$ then*

$$\tilde\sigma_n^{(i);\alpha} f \to \tilde f \quad \text{in } H_{p,q} \text{ norm}$$

as $\min(n_1,\ldots,n_d) \to \infty$ and $2^{-\tau} \le n_k/n_j \le 2^{\tau}$ $(k,j = 1,\ldots,d)$.

We suspect that Theorems 1, 2 and 3 for $p \le p_0$ are not true though we could not find any counterexample.

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