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Pseudo-distance dependent dimensions

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Abstract. From 1965 to 1968, Nagami, Roberts and Slaughter introduced several definitions of dimension dependent of the metric of the space and studied the relations between them and to the topological dimensions such as covering dimension. In this paper we extend the definitions and results of [9], [12] and [12] to pseudo-distance spaces (which includes quasi-metrizable spaces) with some specific topological properties such as normality. These new results include some relevant non-metrizable quasi-metrizable spaces as the Sorgenfrey line and the Michael line.

1. Introduction

In 1928, P. Alexandroff (see [1]) proved that a compact subspace X of a euclidean space $\mathbb{R}^n$ has dimension $\leq m$ if and only if for every compact polyhedron K in $\mathbb{R}^n$ of dimension $n-m-1$ and every $\epsilon > 0$, there exists an ϵ-translation $f : X \to \mathbb{R}^n$ such that $f(X) \cap K = \emptyset$. This property of euclidean spaces was used by Alexandroff to define a dimension dependent on the metric. Later, Smirnov proved in [15] that Alexandroff's dimension is equivalent to his $\mu\,\mathrm{dim}$.

In [9], [10] and [12], K. Nagami, J.H. Roberts and F. Slaughter introduced several notions of metric-dependent function (a new type of dimension function definable only in metrizable spaces and depending upon the metric and not only upon the topology of the spaces) and proved several relations between them, to the $\mu\,\mathrm{dim}$ of Smirnov and to the classical topological dimensions.

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This paper extends the definitions and results to distance spaces, which include quasi-pseudo-metrizable spaces, with some specific topological properties such as normality. These new results include some relevant non-metrizable quasi-metrizable spaces as the Sorgenfrey line and the Michael line (see Section 4). The standard terminology about quasi-metrics, distances and so on is taken from [13]. We recall here the notions of pseudo-distance and quasi-metric.

Definition 1.1. A nonnegative real-valued function $d : X \times X \to \mathbb{R}$ is a pseudo-distance function for X if and only if $d(p,p) = 0$ for every $p \in X$; d is a distance function if for every $p, q \in X$, $d(p,q) = 0$ if and only if $p = q$.

A quasi-pseudometric on a set X is a non-negative real valued function d on $X \times X$ such that for all $x, y, z \in X$:

1. $d(x,x) = 0$, and
2. $d(x,y) \leq d(x,z) + d(z,y)$.
3. If d satisfies the additional condition $\mathrm{Sup}\{d(x,y), d(y,x)\} = 0 \Leftrightarrow x = y$, then we shall say that d is a quasi-metric on X.

A quasi-(pseudo) metric ((pseudo-)distance) space is a pair (X,d) such that X is a nonempty set and d is a quasi-(pseudo)metric ((pseudo-)distance) on X.

Each quasi-pseudometric (pseudo-distance) d on X generates a topology $T(d)$ on X which has as a base the family of d-balls $\{S_d(x,r) : x \in X, r > 0\}$, where $S_d(x,r) = \{y \in X : d(x,y) < r\}$.

For any other topological concepts and notations, see [3]. In particular, recall the definition of strongly hereditarily normal (SHN) spaces from 2.1.2. of [3].

Finally, the following lemma collects all the properties of the pseudo-distance spaces that will be needed in the proofs of the theorems of the rest of the paper.

Lemma 1.2. *Let (X,d) be a pseudo-distance space. Let define $d(A,B) = \inf\{d(a,b) : a \in A, b \in B\}$, $d(A) = \sup\{d(a,b) : a, b \in A\}$ and $\mathrm{mesh}\,\mathcal{U} = \sup\{d(U) : U \in \mathcal{U}\}$, where $A, B \subset X$ and $\mathcal{U} \subseteq \mathcal{P}(X)$.*

1. *If F and H are two subsets of X and $d(F,H) > 0$, then $F \cap H = \emptyset$.*
2. *If $f : X \to [0,1]$ is a mapping, $d^*(x,y) = d(x,y) + |f(x) - f(y)|$ is a pseudo-distance equivalent to d.*

Proof. All the proofs are straightforward. $\square$

2. Pseudo-distance dependent functions

This section collects the definitions of all the pseudo-distance dependent functions we are going to use through this paper.

We beging with that what was defined first.

Definition 2.1. Let (X, d) be a pseudo-distance space. We define $\mu\dim(X, d)$ as follows:

1. If $X = \emptyset$, then $\mu\dim(X, d) = -1$.
2. If there exists a sequence $\mathcal{U}_i$ of locally finite cozero coverings of X such that $\mathrm{Ord}\,\mathcal{U}_i \leq n + 1$ for each i and $\lim \mathrm{mesh}\,\mathcal{U}_i = 0$, then we say that $\mu\dim(X, d) \leq n$.
3. If $\mu\dim(X, d) \leq n$ and the statement $\mu\dim(X, d) \leq n - 1$ is false, then $\mu\dim(X, d) = n$.
4. If $\mu\dim(X, d)$ is not less than n for any n, then $\mu\dim(X, d) = \infty$.

Let us recall how was this dimension introduced.

For a subspace X of $\mathbb{R}^n$ and $\epsilon > 0$, a continuous map $f : X \to \mathbb{R}^n$ is called an ϵ-translation if $d(x, f(x)) < \epsilon$ for each $x \in X$. The metric dimension $\mu\dim$ was defined by Alexandroff as the least integer m such that for every $\epsilon > 0$, X admits an ϵ-translation into a polyhedron which is an underlying space of a locally finite simplicial complex of dimension $= m$. That definition was proved to be equivalent in $\mathbb{R}^n$ to the preceding one by Smirnov in [15].

The following four dimensions were introduced in [9] and [10] for metric spaces. We are not going to prove any result about the first, but we define it for the pseudo-distance case.

Definition 2.2. Let (X, d) be a pseudo-distance space. We define $d_1(X, d)$ as follows:

1. If $X = \emptyset$, then $d_1(X, d) = -1$.
2. If for every pair of closed sets F, H of X with $d(F, H) > 0$ there exists a closed set B, separating F and H, with $d_1(B, d) \leq n - 1$, then we say $d_1(X, d) \leq n$.
3. If $d_1(X, d) \leq n$ and the statement $d_1(X, d) \leq n - 1$ is false, then $d_1(X, d) = n$.
4. If there is no such integer n, then we say $d_1(X, d) = \infty$.

Definition 2.3. Let (X,d) be a pseudo-distance space. We define $d_2(X,d)$ as follows:

1. If $X=\emptyset$, then $d_2(X,d)=-1$.
2. If for any $n+1$ pairs of zero sets C_i, C'_i $i=1,\ldots,n+1$ of X with $d(C_i,C'_i)>0$ for every $i=1,\ldots,n+1$, there exist closed sets B_i, $i=1,\ldots,n+1$ such that $\bigcap_{i\in\mathbb{N}} B_i=\emptyset$ and such that B_i separates C_i from C'_i for every $i=1,\ldots,n+1$, then we say $d_2(X,d)\leq n$.
3. If $d_2(X,d)\leq n$ and the statement $d_2(X,d)\leq n-1$ is false, then $d_2(X,d)=n$.
4. If there is no such integer n, then we say $d_2(X,d)=\infty$.

Definition 2.4. Let (X,d) be a pseudo-distance space. We define $d_3(X,d)$ as follows:

1. If $X=\emptyset$, then $d_3(X,d)=-1$.
2. If for any finite number of pairs of zero sets C_i, C'_i $i=1,\ldots,m$ of X with $d(C_i,C'_i)>0$ for every $i=1,\ldots,m$, there exist closed sets B_i, $i=1,\ldots,m$ such that B_i separates C_i from C'_i for every $i=1,\ldots,m$ and $\mathrm{Ord}\{B_i : i=1,\ldots,m\}\leq n$, then we say $d_3(X,d)\leq n$.
3. If $d_3(X,d)\leq n$ and the statement $d_3(X,d)\leq n-1$ is false, then $d_3(X,d)=n$.
4. If there is no such integer n, then we say $d_3(X,d)=\infty$.

Definition 2.5. Let (X,d) be a pseudo-distance space. We define $d_4(X,d)$ as follows:

1. If $X=\emptyset$, then $d_4(X,d)=-1$.
2. If for any sequence of pairs of zero sets C_i, C'_i $i\in\mathbb{N}$ of X with $d(C_i,C'_i)>0$ for every $i\in\mathbb{N}$, there exist closed sets B_i, $i\in\mathbb{N}$ such that B_i separates C_i from C'_i for every $i\in\mathbb{N}$ and $\mathrm{Ord}\{B_i : i\in\mathbb{N}\}\leq n$, then we say $d_4(X,d)\leq n$.
3. If $d_4(X,d)\leq n$ and the statement $d_4(X,d)\leq n-1$ is false, then $d_4(X,d)=n$.
4. If there is no such integer n, then we say $d_4(X,d)=\infty$.

Note that, since every open (closed) set in a metric space is a cozero (zero) set, these definitions generalize those given in metric spaces.

3. Relation between dimensions and pseudo-distance functions

Clearly $d_2(X,d) \leq d_3(X,d) \leq d_4(X,d)$.

The first two results of this section give relations of two of our pseudo-distance dimensions with the covering dimension of the underlying topological space.

Theorem 3.1. *For any pseudo-distance space* (X,d),

$$d_4(X,d) \leq \dim X.$$

Proof. It is proved in [5] that the modification of covering dimension due to Katetov and Morita involving basic covers instead of open covers is equivalent to the following assertion: $\dim X \leq n$ if and only if for any $n+1$ pairs of disjoint zero sets C_i, C_i' $i = 1,\dots,n+1$ of X there exist zero sets B_i, $i = 1,\dots,n+1$ such that $\mathrm{Ord}\{B_i : i = 1,\dots,n+1\}$ and such that B_i separates C_i from C_i' for every $i = 1,\dots,n+1$. From this result the theorem is clear. □

The next lemma shows how the pseudo-distance can be modified in order to obtain another pseudo-distance equivalent to d_2 coinciding with the covering dimension (and hence with the other two dimensions d_2 and d_3).

Theorem 3.2. *If* (X,ρ) *is a pseudo-distance space with* $\dim X = n$, *then there exists a pseudo-distance* ρ' *equivalent to* ρ *such that* $d_2(X,\rho')=n$.

Proof. Since $\dim X = n$ (see [5]) there exists an essential system of n pairs $C_1, C_1', \dots, C_n, C_n'$. Let $f_i : X \to [0,1]$, $i = 1,\dots,n$ be mappings such that $f_i(C_i) = 0$ and $f_i(C_i') = 1$.

Set $\rho'(x,y) = \rho(x,y) + \sum_{i=1}^{n} |f_i(x) - f_i(y)|$. Then ρ' is a pseudo-distance equivalent to ρ and $\rho'(C_i, C_i') > 0$ for each i. Thus we have $d_2(X,\rho') \geq n$ and from 3.1 we obtain $d_2(X,\rho') = n$. □

Now we are going to relate d_3 with $\mu\dim$.

Theorem 3.3. *For any pseudo-distance normal space*

$$(X,d),\ d_3(X,d) \leq \mu \dim X.$$

Proof. Suppose $\mu\dim X \leq n$. Let C_i, C_i^1 $i = 1, \ldots, m$ be a finite number of pairs of zero sets of X with $d(C_i, C_i^1) > 0$ for every $i = 1, \ldots, m$ such that $m \geq n+1$. Let $\epsilon = \mathrm{Min}\{d(C_i, C_i^1) : i = 1, \ldots, m\} > 0$. By the definition of $\mu\dim X$ there exists a locally finite cozero covering $\mathcal{U}_0$ of X such that $\mathrm{Ord}\,\mathcal{U}_0 \leq n+1$ and $\mathrm{mesh}\,\mathcal{U}_0 < \frac{\epsilon}{2}$.

By repeated applications of a result from [6] (valid in normal spaces), we obtain closed sets B_i, $i = 1, \ldots, m$ and locally finite open coverings $\mathcal{U}_i$, $i = 1, \ldots, m$ of X which satisfy the following conditions:

1. Each B_i separates C_i and C_i'.
2. $\mathcal{U}_i$ refines $\mathcal{U}_{i-1}$ in a one-one corresponding way.
3. $\mathrm{Ord}_x\,\mathcal{U}_i \leq \mathrm{Ord}_x\,\mathcal{U}_{i-1}$ for any point $x \in B_i$.

To prove $\mathrm{Ord}\,B_i \leq n$, assume the contrary. Then there would be a point x and a sequence $1 \leq i_1 < i_2 < \cdots < i_{n+1} \leq m$ such that $x \in B_{i_j}$, $j = 1, \ldots, n+1$.

By condition (iii), $n+1 \geq \mathrm{Ord}_x\,\mathcal{U}_0 \geq \mathrm{Ord}_x\,\mathcal{U}_1 \geq \cdots > \mathrm{Ord}_x\,\mathcal{U}_{i_1} \geq \cdots > \mathrm{Ord}_x\,\mathcal{U}_{i_2} \geq \cdots > \mathrm{Ord}_x\,\mathcal{U}_{i_{n+1}}$.

Hence $\mathrm{Ord}_x\,\mathcal{U}_{i_{n+1}} < n+1-(n+1) = 0$, a contradiction. Thus we obtain $d_3(X,d) \leq n$. $\square$

The next lemma is a technical step in the proof of the last result of this section.

Lemma 3.4. *Let (X,ρ) be a paracompact Hausdorff distance space with $\mu\dim(X,\rho) = k$; then for every positive number $\epsilon > 0$ there exists a locally finite, σ-discrete, closed covering $\mathcal{H}$ of X such that* $\mathrm{Ord}\,\mathcal{H} \leq k+1$ *and* $\mathrm{mesh}\,\mathcal{H} < \epsilon$.

Proof. Let ϵ be an arbitrary positive number. Then by Definition 2.1 there exists a locally finite open covering $\mathcal{V}$ of X such that $\mathrm{mesh}\,\mathcal{V} < \epsilon$ and $\mathrm{Ord}\,\mathcal{V} \leq k+1$.

As can be seen in Section VIII 3 A) and 3 B) of [8], from the paracompactness of X there exists a polyhedron P with the weak topology such that P approximates $\mathcal{V}$ and the combinatorial dimension of P is at most k.

Let V be the vertex set of P. Then there exists a mapping $f : X \to P$ such that $\mathcal{V}$ is refined by $f^{-1}(\{\mathrm{St}(v) : v \in V\})$. From the locally finite sum Theorem 3.1.10. of [3] in normal spaces for the covering dimension and the relation between the combinatorial dimension of a polyhedron and its covering dimension (see [3], Section 1.10.), we have that $\dim P \leq k$. Since

P is paracompact Hausdorff, the open covering $\{\mathrm{St}(v) : v \in V\}$ has a locally finite open covering $\mathcal{U} = \bigcup \mathcal{U}_i$ such that each $\mathcal{U}_i$ is discrete. By Theorem 1.3 of [7], there exists a normal open covering $\mathcal{G}$ of P such that $\operatorname{Ord} \mathcal{G} \leq k+1$ and $\mathcal{G} < \mathcal{U}$.

Well-order the elements U of $\mathcal{U}$. Let G_U be the union of all elements G of $\mathcal{G}$ such that $G \subset U$ and $G \not\subset U^1$ for any U^1 preceding U. Set $\mathcal{G}_i = \{G_U : U \in \mathcal{U}_i\}$, $i = 1, 2, \ldots$. Then each $\mathcal{G}_i$ is discrete and $\bigcup \mathcal{G}_i$ is locally finite, hence normal. Moreover, $\bigcup \mathcal{G}_i$ covers P, $\bigcup \mathcal{G}_i$ refines $\mathcal{U}$ and $\operatorname{Ord} \bigcup \mathcal{G}_i \leq k+1$.

By Theorem 15-10 of [16], $\bigcup \mathcal{G}_i$ is shrinkable. Hence there exists a closed covering $\bigcup \mathcal{F}_i$ of P refining $\bigcup \mathcal{G}_i$ in a one-to-one corresponding way, where each $\mathcal{F}_i$ is correspondent to $\mathcal{G}_i$. then each $\mathcal{F}_i$ is discrete, $\bigcup \mathcal{F}_i$ is locally finite and refines $\mathcal{U}$ and $\operatorname{Ord} \bigcup \mathcal{F}_i \leq k+1$.

Set $\mathcal{H}_i = f^{-1}(\mathcal{F}_i)$; then $\mathcal{H} = \bigcup \mathcal{H}_i$ is a locally finite closed covering of X such that each $\mathcal{H}_i$ is discrete, $\mathcal{H} < \mathcal{V}$ and $\operatorname{Ord} \mathcal{H} \leq k+1$ and then we have $\operatorname{mesh} \mathcal{H} < \epsilon$. □

This lemma shows how changes $\mu \dim$ when one modifies the pseudo-distance.

Lemma 3.5. *Let (X, ρ) be a paracompact Hausdorff pseudo-distance space with $\mu \dim(X, \rho) = k$. Let $f : X \to [0,1]$ be a mapping. Define a pseudo-distance ρ^* equivalent to ρ (see 1.2) by $\rho^*(x,y) = \rho(x,y) + |f(x) - f(y)|$. Then $k \leq \mu \dim(X, \rho^*) \leq k+1$.*

Proof. Since ρ^* enlarges ρ, it is evident that $k \leq \mu \dim(X, \rho^*)$. Let us prove that $\mu \dim(X, \rho^*) \leq k+1$. Let ϵ be an arbitrary positive number. Then by Lemma 3.4, there exists a locally finite closed covering $\mathcal{H} = \bigcup \mathcal{H}_i$ of X such that each $\mathcal{H}_i$ is discrete, $\operatorname{Ord} \mathcal{H} \leq k+1$ and $\rho - \operatorname{mesh} \mathcal{H} < \frac{\epsilon}{2}$.

Set $H_i = \bigcup \{H : H \in \mathcal{H}_i\}$; then $\{H_i\}$ is a closed covering of X of order at most $k+1$. Now pick an infinite sequence of finite sequences $a_{i0} = 0 < a_{i1} < a_{i2} < \cdots < a_{im_i} = 1$ such that $a_{i,j} - a_{i,j-1} < \frac{\epsilon}{2}$ and for different $i \neq k$, $a_{i,j} \neq a_{k,s}$ for all j and s, except, of course, all $a_{i0} = 0$ and all $a_{im_i} = 1$.

For each i, break the closed set H_i into m_i closed sets $H_i \cap \{x : a_{i,j-1} \leq f(x) \leq a_{i,j}\}$, $j = 1, 2, \ldots, m_i$. Let $\mathcal{F}_i$ be the collection of all such pieces. Then $\operatorname{Ord} \bigcup \mathcal{F}_i \leq k+2$, because any x which is on a dividing line $H_i \cap \{x : f(x) = a_{i,j}\}$ $(j \neq 0,\ j \neq m_i)$ for some i will not be on any other dividing line. Set $\mathcal{H} = \bigcup (\mathcal{H}_i \wedge \mathcal{F}_i)$. Then $\mathcal{H}$ is a locally finite closed covering of X satisfying $\rho^* - \operatorname{mesh}\mathcal{H} < \epsilon$ and $\operatorname{Ord}\mathcal{H} \le k+2$. From Definition 2.1, a theorem of [6] and the equivalence between Hausdorff paracompact and Hausdorff fully normal spaces (see [2]), we conclude that $\mu\dim(X,\rho^*) \le k+1$. □

Next theorem shows how to fill in the gaps between $\mu\dim$ and dim, if they exists.

Theorem 3.6. *Let (X,ρ) be a paracompact Hausdorff pseudo-distance space with $\mu\dim(X,\rho) = k < \dim X = n$, then there exist equivalent pseudo-distances $\rho_1,\dots,\rho_{n-k}$ to ρ such that $\mu\dim(X,\rho_i) = k+i$, where $i = 1,\dots,n-k$.*

Proof. Let $C_1, C_1^1, \dots, C_n, C_n^1$ be an essential system of n pairs of zero sets. Let $f_i : X \to [0,1]$ $i = 1,\dots,n$, be mappings such that $f_i(C_i) = 0$ and $f_i(C_i^1) = 1$. Set $\sigma_0 = \rho(x,y)$, $\sigma_i(x,y) = \rho(x,y) + \sum_{j=1}^{i} |f_j(x) - f_j(y)|$, $i = 1,\dots,n$. Then by Lemma 3.5 $\mu\dim(X,\sigma_{i-1}) \le \mu\dim(X,\sigma_i) \le \mu\dim(X,\sigma_{i-1}) + 1$ for $i = 1,\dots,n$.

Since $d_2(X,\sigma_i) \le \mu\dim(X,\sigma_i)$ for i=$0,\dots,n$, $d_2(X,\sigma_{i-1}) \le d_2(X,\sigma_i)$ for $i = 1,\dots,n$ and moreover $d_2(X,\sigma_n) = n$ by Lemma 3.2, then the theorem follows. □

4. Examples and counterexamples

To complete the picture of the relations between the pseudo-distance dependent dimensions we have defined and the classical ones, we quote from [10] several spaces with prescribed values of each of its dimensions. The details of the calculations can be found in the above mentioned paper. Note that all the examples quoted are metrizable.

Example 4.1. There exists a subset of I^4 with the euclidean metric (S,σ) with $d_2(S,\sigma) = 2$ and $\mu\dim(S,\sigma) = \dim(S,\sigma) = 3$.

Example 4.2. There exist spaces (X_n,ρ) with $d_2(X_n,\rho) = [\frac{n}{2}]$ and $\dim X_n \ge n-1$.

Example 4.3. There exist spaces (Y_n,ρ) with $\mu\dim(Y_n,\rho) = [\frac{n}{2}]$ and $\dim Y_n \ge n-1$.

Note that the X_n and Y_n obtained replacing K_n with I^n have $\dim X_n = \dim Y_n = n-1$.

Example 4.4. There exist spaces (Z_n, σ_i), $n \geq 2$, such that $\dim Z_n{=}n$, $d_2(Z_n, \sigma_i) = d_3(Z_n, \sigma_i) = \mu \dim(Z_n, \sigma_i) = i - 1$, for every $i = m, m{+}1, \ldots, n + 1$, where $m = [\frac{n+1}{2}] + 1$.

Example 4.5. There exists a metric space (R, ρ) with $d_2(R, \rho) = 2$, $d_3(R, \rho) = \mu \dim(R, \rho) = 3$ and $\dim R = 4$.

There is also another relation between these dimension functions that is valid in metric spaces but not in quasi-metric spaces. In [10], it is proved that for every precompact metric space (X, ρ) we have $d_3(X, \rho) = \mu \dim(X, \rho)$. However this result is not valid in quasi-metric spaces, as the following example shows.

Example 4.6. Let X be the set of nonnegative integers with the following quasi-metric:

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{y} & \text{if } x = 0 \\ 1 & \text{otherwise.} \end{cases}$$

One can easily check that this quasi-metric space is the one-point compactification of $\mathbb{N}$ with its usual topology (hence a compact metric space). However, it is straightforward that every non-singleton has diameter 1, so one cannot find a family of coverings by open sets with diameters converging to zero, hence $\mu \dim(X, d) = \infty$; however $d_3(X, \rho) = 0$.

Finally we mention four quasi-metrizable non-metrizable spaces with good topological properties such as normality or paracompactness: Sorgenfrey and Michael lines. These spaces are of those to which the results of this paper can be applied and the previous theory not. Their definitions and properties can be found in [4].

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