A family of Chebyshev type methods
in the complex plane

By M.A. HERNÁNDEZ (Logroño) and M.A. SALANOVA (Logroño)

Abstract. A family of third order iterative processes that includes the Chebyshev method is studied in the Complex Plane. Results on convergence and uniqueness of the solution are given, as well as error estimates.

1. Introduction

The degree of logarithmic convexity, introduced in [3], is a measure of convexity. Let $g : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be a convex, twice differentiable function on an interval $[a, b]$ and $t_0 \in [a, b]$ such that $g'(t_0) \neq 0$. The degree of logarithmic convexity of g at t_0 is

$$L_g(t_0) = \frac{g(t_0)g''(t_0)}{g'(t_0)^2}.$$

One of the most interesting applications of the degree of logarithmic convexity is its relation with the convergence of iterative processes of third order [2]. In [4] we study the convergence of iterative processes given by

the expression

(1)
$$t_{n,\alpha} = G_\alpha(t_{n-1,\alpha})$$
$$= t_{n-1,\alpha} - \frac{g(t_{n-1,\alpha})}{g'(t_{n-1,\alpha})}\left[1 + \frac{\alpha L_g(t_{n-1,\alpha})}{2\alpha - L_g(t_{n-1,\alpha})}\right], \quad \alpha \in \mathbb{R}$$

for scalar equations. In this paper, we extend this family of iterative processes to the Complex Plane for $\alpha < 0$, and obtain convergence results by means of the degree of logarithmic convexity.

The purpose of this paper is to solve a nonlinear complex equation

(2)
$$f(z) = 0$$

using an iterative process with cubical convergence. We present a new procedure for finding majorizing sequences [11] for the family of iterative processes

(3)
$$z_{n,\alpha} = F_\alpha(z_{n-1,\alpha})$$
$$= z_{n-1,\alpha} - \frac{f(z_{n-1,\alpha})}{f'(z_{n-1,\alpha})}\left[1 + \frac{\alpha L_f(z_{n-1,\alpha})}{2\alpha - L_f(z_{n-1,\alpha})}\right], \quad \alpha < 0$$

defined in the Complex Plane.

Notice that if we consider $F_{-\infty} = \lim\limits_{\alpha \to -\infty} F_\alpha$ we obtain the well known Chebyshev method [1]. So we say that (3) is a family of Chebyshev type iterative processes.

Firstly we construct majorizing sequences [4] for $\{z_{n,\alpha}\}$ given by (3) and we apply the original Kantorovich techniques [8] in the Complex Plane. In Section 2 we establish a convergence and uniqueness theorem in the Complex Plane for the family (3).

In the last section we solve the principal problem of this paper, i.e. we show that given a nonlinear complex equation $f(z) = 0$, we can choose an iterative process of (3) with cubical convergence to approximate its solution. The velocity of convergence of the iterative process of this family is analysed by means of the asymptotic error constant [6].

2. Convergence under strong Kantorovich type conditions in the complex plane

Let $f : \Omega \to \mathbb{C}$ be an holomorphic function in an open and convex domain Ω of $\mathbb{C}$. Starting from $z_0 \in \Omega$, we use (3) for solving the equation (2). Most of the autors study the convergence of iterative processes towards a solution of (2) under the conditions of the Kantorovich Theorem, or closely related theorems [10], [12]. In these results it is assumed that f is an holomorphic function under majorant assumptions for f'' and f''', or the weaker assumption of Lipschitz continuity for f'' in Ω. Recently, [5], we have obtained a new type of convergence theorem in the Complex Plane, assuming that $L_{f'}$ verifies a majorant condition. The new result can be used in order to judge whether z_0 is a convergent initial point, i.e., the iteration (3) starting from z_0 converges.

We construct a quadratic polynomial which majorizes f and we establish results on convergence and error estimates for (3), as well as uniqueness of solution for (2). From now, given (2), $\alpha \in (-\infty, 0)$ and $z_0 = z_{0,\alpha} \in \Omega$ we assume that f satisfies the following conditions

(I)
$$\left| \frac{f(z_{0,\alpha})}{f'(z_{0,\alpha})} \right| \leq a.$$

(II)   We consider the equation

(4)
$$g(t) = \frac{1}{2}t^2 - t + a = 0$$

with two positive roots t^* and t^{**}, $(t^* \leq t^{**})$. Equivalently $a \leq \frac{1}{2}$, where the equality holds if and only if $t^* = t^{**}$.

(III) $|L_f(z)| \leq \dfrac{L_g(t)}{1 + 2\sqrt{2}}$ when $|z - z_0| \leq t - t_0$.

(IV) $L_{f'}(z) \in \overline{B\left(0, 3\left(1 - \dfrac{1}{2\alpha}\right)\right)}$ for $z \in \overline{B(z_0, t^*)}$.

Let $\alpha < 0$ and let g be the polynomial given in (4). As $L_{g'}(t) = 0$ the scalar sequence $\{t_{n,\alpha}\}$ given by (1) converges to $t^* = 1 - \sqrt{1 - 2a}$, a solution of the equation $g(t) = 0$ [4].

To establish the convergence of (3) and the uniqueness of the solution we need the following results:

Lemma 2.1. *Let $f : \Omega \to \mathbb{C}$ be an holomorphic function in an open and convex domain Ω of $\mathbb{C}$.*

(i) *If $|L_f(z)| \leq \dfrac{L_g(t)}{1 + 2\sqrt{2}}$ when $|z - z_0| \leq t - t_0$, and $\alpha \leq \frac{-1}{4}$, then*

$$\left| \frac{L_f(z)}{2\alpha - L_f(z)} \right|^2 \leq \frac{1}{2} \left[\frac{L_g(t)}{2\alpha - L_g(t)} \right]^2.$$

(ii) *For $\alpha \leq -\dfrac{1}{4}$*

(a) $|F'_\alpha(z)| \leq \dfrac{1}{2} \left[\dfrac{L_g(t)}{2\alpha - L_g(t)} \right]^2 \Big(6\alpha^2 + (1 - \alpha)L_g(t)$

$\qquad\qquad + 2\alpha^2 |L_{f'}(z)| - 3\alpha \Big)$ *when $|z - z_0| \leq t - t_0$.*

(b) $G'_\alpha(t) = \left[\dfrac{L_g(t)}{2\alpha - L_g(t)} \right]^2 \big(6\alpha^2 + (1 - \alpha)L_g(t) - 3\alpha \big),$ *for $t \in [0, t^*]$.*

PROOF. To prove (i), notice that $|L_f(z)| \leq \dfrac{L_g(t)}{1 + 2\sqrt{2}} < -2\alpha$ since $L_g(t) \leq \frac{1}{2}$ and $\alpha \in (-\infty, -\frac{1}{4}]$. Therefore

$$\left| \frac{L_f(z)}{2\alpha - L_f(z)} \right|^2 \leq \left[\frac{L_g(t)}{-2(1 + 2\sqrt{2})\alpha - L_g(t)} \right]^2.$$

On the other hand, $-2(1 + 2\sqrt{2})\alpha - L_g(t) \geq \sqrt{2}(-2\alpha + L_g(t))$ for $t \in [0, t^*]$. So (i) follows.

Finally, we derive (a) from (i) and, as $L_{g'}(t) = 0$, (b) is proved (see e.g. [4], p. 61). □

Theorem 2.2. *Let us assume (I), (II), (III) and (IV). Then, for each $\alpha \in (-\infty, -\frac{1}{4}]$, the iterative process given by (3) converges to the root z^* of (2) in $\overline{B(z_0, t^*)} \cap \Omega$. If $t^* < t^{**}$ then the solution z^* is unique in $B\left(z_0, \frac{t^* + t^{**}}{2}\right) \cap \Omega$. If $t^* = t^{**}$ then the solution is unique in $\overline{B(z_0, t^*)} \cap \Omega$. Besides $|z^* - z_{n,\alpha}| \leq t^* - t_{n,\alpha}$.*

PROOF. Let $\{t_{n,\alpha}\}$ be the real sequence obtained by (1) for $\alpha \in (-\infty, -\frac{1}{4}]$, with g given by (4). Then, $\{t_{n,\alpha}\}$ converges to $t^* = 1 - \sqrt{1 - 2a}$ [4]. Now, we are going to prove that in the previous conditions the sequence

$\{t_{n,\alpha}\}$ majorizes $\{z_{n,\alpha}\}$ and the results of the theorem are attained. It is known [8] that the sequence $\{t_{n,\alpha}\}$ majorizes $\{z_{n,\alpha}\}$ if the following conditions are satisfied

(A) $$|F_\alpha(z_0) - z_0| \leq G_\alpha(t_0) - t_0.$$

(B) $$|F'_\alpha(z)| \leq G'_\alpha(t) \text{ when } |z - z_0| \leq t - t_0.$$

The condition (A) follows immediately applying Lemma 2.1 (i).

Taking into account the Lemma 2.1 (a) and (b), (B) holds if

$$\frac{1}{2}\left(6\alpha^2 + (1-\alpha)L_g(t) + 2\alpha^2|L_{f'}(z)| - 3\alpha\right) \leq 6\alpha^2 + (1-\alpha)L_g(t) - 3\alpha.$$

Then, (B) follows from (IV).

The uniqueness is a consequence from the well known classical theorem on the existence and uniqueness of the solutions of equation (2) via majorizing sequences [7]. $\hfill\square$

Now, we ask if we can extend the values of α on $\left(-\frac{1}{4}, 0\right)$. For this, let $b > 0$, and we consider

(5) $$h(s) = \frac{b}{2}s^2 - s + a = 0.$$

This equation has two positive roots s^* and s^{**}, $(s^* \leq s^{**})$ if and only if $ab \leq \frac{1}{2}$, where the equality holds if and only if $s^* = s^{**}$.

For $\alpha < 0$, by $L_{h'}(s) = 0$, we have (see [4]) that the scalar sequence

$$s_{n,\alpha} = H_\alpha(s_{n-1,\alpha}) = s_{n-1,\alpha} - \frac{h(s_{n-1,\alpha})}{h'(s_{n-1,\alpha})}\left[1 + \frac{\alpha L_h(s_{n-1,\alpha})}{2\alpha - L_h(s_{n-1,\alpha})}\right],$$

$$s_0 = s_{0,\alpha} = 0, \ n \geq 1$$

converges to $s^* = \dfrac{1 - \sqrt{1 - 2ab}}{b}$, a root of $h(s) = 0$.

Theorem 2.3. *Let us assume* (I), (IV) *and*

(II*) $$ab \leq \frac{1}{2}$$

(III*) $$|L_f(z)| \leq \frac{L_h(s)}{1 + 2\sqrt{2}} \quad when \ |z - z_0| \leq s - s_0.$$

Then, for each $\alpha \in \left(-\infty, -\frac{ab}{2}\right]$, the iterative process given by (3) converges to the root z^* of (2) in $\overline{B(z_0, s^*)} \cap \Omega$. If $s^* < s^{**}$ then the solution z^* is unique in $B(z_0, \frac{1}{b}) \cap \Omega$. If $s^* = s^{**}$ then the solution is unique in $\overline{B(z_0, s^*)} \cap \Omega$. Besides $|z^* - z_{n,\alpha}| \leq s^* - s_{n,\alpha}$.

PROOF. Notice that, as the Lemma 2.1 we obtain that $L_h(s) \leq \frac{1}{2}$ in $[0, s^*]$. Moreover $L'_h(s) = \frac{h''(s)}{h'(s)}(1 - 2L_h(s))$, then L_h is a decreasing function in $[0, s^*]$. Therefore $L_h(s) \leq ab$.

Then, taking into account that $|L_f(z)| \leq \dfrac{ab}{1 + 2\sqrt{2}} \leq -2\alpha$, we obtain, as in the last lemma

$$\left|\frac{L_f(z)}{2\alpha - L_f(z)}\right|^2 \leq \frac{1}{2}\left[\frac{L_h(s)}{2\alpha - L_h(s)}\right]^2 \quad \text{when} \quad |z - z_0| \leq s - s_0.$$

From Lemma 2.1 (ii) and in a similar way that in Theorem 2.2, the result follows inmediately.   □

Notice that if $\alpha \in \left(\frac{1}{4}, 0\right)$, then it is suficient to consider $b = -2\alpha > 0$ and therefore we extend the values of α.

3. Practical remarks

In this section we are going to study the principal purpose of this paper. Given the equation (2) verifiying the initial condition (I), we can always find an iterative process of the family (3) to solve this equation. Besides, we derive an optimization result considering the velocity of convergence, and we obtain error estimates. First, we analyse the condition (IV).

Theorem 3.1. *Let us assume* (I), *then there exist* $b \in \mathbb{R}$, $b > 0$ *and* $\alpha < 0$ *such that the conditions* (II*) *and* (IV) *are verified for* $h(s)$ *given by* (5).

PROOF. In this proof we find b and α such that the conditions of Theorem 2.2 are verified.

Denote $M = \max\{|L_{f'}(z)| ; z \in \overline{B(z_0, s^*)}\}$, then

(i) If $M \leq 3$, then $L_{f'}(z) \in \overline{B(0, 3)}$. Let $b > 0$ be such that $ab \leq \frac{1}{2}$. Then, for every $b > \alpha \in \left(-\infty, -\frac{ab}{2}\right]$, we have $\overline{B(0, 3)} \subseteq \overline{B\left(0, 3\left(1 - \frac{1}{2\alpha}\right)\right)}$ and the conditions (II*) and (IV) hold.

(ii) If $3 < M \leq 9$ then we can choose every $b > 0$ such that $ab \leq \frac{1}{2}$ and $\alpha \in \left[\frac{3}{2(3-M)}, -\frac{ab}{2} \right]$. Notice that $\frac{3}{2(3-M)} \leq -\frac{ab}{2}$. Then, as $L_{f'}(z) \in \overline{B(0,M)}$ the conditions (II*) and (IV) are derived.

(iii) If $M > 9$, as $\frac{3}{2(3-M)} \geq -\frac{ab}{2(1+2\sqrt{2})}$ for $ab \leq \frac{1}{2}$, then there is only one $\alpha = \frac{3}{2(3-M)}$ and $b = \frac{3}{a(M-3)} > 0$ such that the conditions (II*) and (IV) hold. $\qquad\square$

Figure 1.

Notice that if $M \leq 9$ then there exists a domain of values for α such that the sequence given by (3) converges, but we wonder what values of α make that the sequence converge more rapidly. Now, we give an optimization result by means of the asymptotic error constants [6]. We denote

$$C_\alpha = \left| \frac{F_\alpha'''(z^*)}{6} \right| \quad \text{and} \quad C_\beta = \left| \frac{F_\beta'''(z^*)}{6} \right|$$

the asymptotic error constants for $z_{n,\alpha} = F_\alpha(z_{n-1,\alpha})$ and $z_{n,\beta} = F_\beta(z_{n-1,\beta})$ respectively. So, we have

Theorem 3.2. *Under the conditions of the last theorem, let $\alpha, \beta \in \left(-\infty, -\frac{ab}{2}\right], (\alpha < \beta)$. Then $C_\alpha < C_\beta$.*

PROOF. We can obtain $C_\alpha = \left| \frac{f''(z^*)^2(2\alpha N - 3)}{12 f'(z^*)^2 \alpha} \right|$, with $N = 3 - L_{f'}(z^*) = N_1 + iN_2$. On the other hand, it follows that $0 \leq 2N_1 - \frac{3}{\alpha} < 2N_1 - \frac{3}{\beta}$, so we have $\left| 2N - \frac{3}{\alpha} \right| < \left| 2N - \frac{3}{\beta} \right|$, and therefore $C_\alpha < C_\beta$. $\square$

Then, for a suitable $z_0 \in D$, with $z_0 = z_{0,\alpha} = z_{0,\beta}$, we obtain that the sequence $\{z_{n,\alpha}\}$ converges to z^* faster when the value of α is smaller, at each case.

Now , we are going to obtain error expressions for the sequences $\{z_{n,\alpha}\}$ given by (3). As $h(s)$ is a quadratic polynomial, following OSTROWSKI [9], we deduce the following error bounds , where we denote $\theta = \dfrac{s^*}{s^{**}}$ and $d = s^{**} - s^*$.

Theorem 3.3. *Let $h(s)$ be the quadratic polinomial given by (5). We assume that h has two positive roots $s^* \leq s^{**}$. Let $\{s_{n,\alpha}\}$ be a sequence defined in (3), for $\alpha \leq -\frac{ab}{2}$. Then , when $s^* < s^{**}$ we have*

$$\frac{(\theta\sqrt{R})^{3^n} d}{\sqrt{R} - (\theta\sqrt{R})^{3^n}} \leq s^* - s_{n,\alpha} \leq \frac{\left(\theta\sqrt{2 - \frac{1}{\alpha}}\right)^{3^n} d}{\sqrt{2 - \frac{1}{\alpha}} - \left(\theta\sqrt{2 - \frac{1}{\alpha}}\right)^{3^n}}$$

where $R = H(s^*)$, H is given by

$$(6) \qquad\qquad H(x) = \frac{\alpha x + 2\alpha - 1}{\alpha + (2\alpha - 1)x}$$

and $\theta\sqrt{H(0)} = \theta\sqrt{2 - \frac{1}{\alpha}}$.

If $s^* = s^{**}$, then $s^* - s_{n,\alpha} = (s^* - s_{0,\alpha})\left(\dfrac{1 - 3\alpha}{2(1 - 4\alpha)}\right)^n$.

PROOF. For each $\alpha \leq -\frac{ab}{2}$, let us write $p_n = s^* - s_{n,\alpha}$, $q_n = s^{**} - s_{n,\alpha}$, $n \geq 0$. Thus $h(s_{n,\alpha}) = \frac{b}{2}p_n q_n$, $h'(s_{n,\alpha}) = -\frac{b}{2}(p_n + q_n)$ and $h''(s_n) = b$. By (3) we have

$$p_{n+1} = s^* - s_{n+1,\alpha} = p_n^3 \frac{\alpha p_n + (2\alpha - 1)q_n}{\alpha(p_n + q_n)^3 - p_n q_n(p_n + q_n)},$$

and similarly

$$q_{n+1} = s^{**} - s_{n+1,\alpha} = q_n^3 \frac{\alpha q_n + (2\alpha - 1)p_n}{\alpha(p_n + q_n)^3 - p_n q_n(p_n + q_n)}.$$

If $s^* < s^{**}$, then $\theta < 1$. Denote $\delta_k = \dfrac{p_k}{q_k}$ to obtain

$$\delta_{n+1} = \delta_n^3 \frac{\alpha \delta_n + 2\alpha - 1}{\alpha + (2\alpha - 1)\delta_n}.$$

Taking into account that the function $H(x)$ given by (6) is decreasing we have

(7)
$$\delta_0^{3^{n+1}} R^{\frac{3^{n+1}-1}{2}} \leq \cdots \leq \delta_n^3 R \leq \delta_{n+1}$$
$$\leq \delta_n^3 \left(2 - \frac{1}{\alpha}\right) \leq \cdots \leq \delta_0^{3^{n+1}} \left(2 - \frac{1}{\alpha}\right)^{\frac{3^{n+1}-1}{2}}.$$

Then, as $q_n = s^{**} - s^* + p_n = d + p_n$, by (7) we derive the first part.

If $s^* = s^{**}$, then $p_n = q_n$. Therefore, from (6), we have

$$p_{n+1} = p_n \frac{3\alpha - 1}{8\alpha - 2} = p_n \frac{1 - 3\alpha}{2(1 - 4\alpha)}.$$

By recurrence, the second part holds.   □

Notice that there is only one restrictive condition to prove the convergence of $\{z_{n,\alpha}\}$, that is the condition (III*). Besides, when $M \leq 9$, there are some values of $b > 0$ to choose, so, the condition (III*) is not restrictive.

To finish, we are going to obtain sufficient conditions, easier to apply in the practice, for the condition (III*).

Theorem 3.4. *If* $\left|\dfrac{f''(z)}{f'(z_{0,\alpha})}\right| \leq b$ *for* $z \in \Omega$ *and*

$$\left|\frac{f(z)}{f'(z_{0,\alpha})}\right| \leq \frac{h(s)}{1 + 2\sqrt{2}} \text{ when } |z - z_{0,\alpha}| \leq s - s_{0,\alpha}$$

then the condition (III*) is verified.

PROOF. Notice that if $|z - z_{0,\alpha}| \leq s - s_{0,\alpha}$ then

$$
\left| \frac{f'(z)}{f'(z_{0,\alpha})} - 1 \right| = \left| \int_{z_{0,\alpha}}^{z} \frac{f''(z)}{f'(z_{0,\alpha})} dz \right| \leq b|z - z_{0,\alpha}| < b(s - s_{0,\alpha}) \leq bs.
$$

Besides $s \in [0, s^*]$ and therefore $bs < 1$. So, we obtain

$$
\left| \frac{f'(z_{0,\alpha})}{f'(z)} \right| \leq \frac{1}{1 - b|z - z_{0,\alpha}|} \leq \frac{1}{1 - bs}
$$

and then, for $|z - z_{0,\alpha}| \leq s - s_{0,\alpha}$

$$
|L_f(z)| = \left| \frac{f(z)}{f'(z_{0,\alpha})} \frac{f'(z_{0,\alpha})}{f'(z)} \frac{f''(z)}{f'(z_{0,\alpha})} \frac{f'(z_{0,\alpha})}{f'(z)} \right|
$$

$$
\leq \frac{b}{(1 - bs)^2} \frac{h(s)}{1 + 2\sqrt{2}} = \frac{L_h(s)}{1 + 2\sqrt{2}}. \qquad \square
$$

References

[1] I. K. ARGYROS and D. CHEN, Results on the Chebyshev Method in Banach Spaces, *Proyecciones* **12**, 2 (1993), 119–128.

[2] M. A. HERNÁNDEZ, A note on Halley's method, *Numer. Math.* **59** (1991), 273–276.

[3] M. A. HERNÁNDEZ, Newton–Raphson's method and convexity, *Zb. Rad. Prirod.-Mat. Fak. Ser. Mat.* **22**, 1 (1992), 159–166.

[4] M. A. HERNÁNDEZ and M. A. SALANOVA, A family of Chebyshev–Halley type methods, *Intern. J. Computer. Math.* **47** (1993), 59–63.

[5] M. A. HERNÁNDEZ and M. A. SALANOVA, A family of Newton type iterative processes, *Intern. J. Computer. Math.* **51** (1994), 205–214.

[6] A. S. HOUSEHOLDER, The Numerical Treatment of a Single Nonlinear Equation, *McGraw Hill*, 1970.

[7] L. V. KANTOROVICH, The Majorant Principle and Newton's Method, *Dokl. Akad. Nauk SSRR* **76** (1951), 17–20.

[8] L. V. KANTOROVICH and G. P. AKILOV, Functional Analysis, *Pergamon Press*, 1982.

[9] A. M. OSTROWSKI, Solution of equations in Euclidean and Banach Spaces, *Academic Press*, 1943.

[10] L. B. RALL, Computational solution of nonlinear operator equations, *Robert E. Krieger Publishing Company, Inc.*, 1979.

[11] W. C. RHEINBOLDT, A Unified Convergence Theory for a Class of Iterative Process, *SIAM J. Numer. Anal.* **5** (1968), 42–63.

[12] T. YAMAMOTO, A Method for Finding Sharp Error Bounds for Newton's Method under the Kantorovich Assumptions, *Numer. Math.* **49** (1986), 203–220.

M.A. HERNÁNDEZ
UNIVERSITY OF LA RIOJA
DPT. OF MATHEMATICS AND COMPUTATION
C/ LUIS DE ULLOA S/N, 26004, LOGROÑO
SPAIN

M.A. SALANOVA
UNIVERSITY OF LA RIOJA
DPT. OF MATHEMATICS AND COMPUTATION
C/ LUIS DE ULLOA S/N, 26004, LOGROÑO
SPAIN