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Generalized pseudo-contractions and nonlinear variational inequalities

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Abstract. Based on a modified iterative algorithm, the solvability of a class of nonlinear variational inequality problems involving Lipschitzian generalized pseudo-contractions is presented on convex sets in Hilbert spaces.

1. Introduction

General variational inequalities have been applied to many problems in applied mathematics, physics, engineering sciences, and others. A closely associated notion of the complementarity involves several problems in mathematical programming, game theory, economics, and mechanics. There are situations where both concepts are equivalent, especially on a colsed convex cone. For more details on variational inequalities, we advise to consult [2–4, 7–12].

Let H be a real Hilbert space and let K be a nonempty closed convex subset of H. Let $\langle u, v\rangle$ and $\|u\|$ denote, respectively, the inner product and norm on H for u, v in H. Let P_K be the projection of H onto K. For an operator $T : K \to H$, we consider the nonlinear variational inequality (NVI) problem (Pl): Find an element x in K such that

$$\langle (I - T)x,\, y - x\rangle \geq 0 \quad \text{for all } y \text{ in } K, \tag{1}$$

where I is the identity.

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The NVI problem (1) is equivalent to a complementarity problem when K is a closed convex cone ([9]).

Next, we consider an important concept of the generalized pseudo-contractivity – a mild generalization of the pseudo-contractivity introduced by Browder and Petryshyn in [1]. Generalized pseudo-contractions are more general than Lipschitz continuous operators and unify certain class of operators.

Definition 1.1. An operator $T : H \to H$ is said to be a generalized pseudo-contraction if, for all x, y in H, there exists a constant $r > 0$ such that

$$\|Tx - Ty\|^2 \leq r^2\|x - y\|^2 + \|Tx - Ty - r(x - y)\|^2. \tag{2}$$

It is easy to check that (2) is mutually equivalent to

$$\langle Tx - Ty,\, x - y\rangle \leq r\,\|x - y\|^2. \tag{3}$$

Clearly, this implies that

$$\langle (I - T)x - (I - T)y,\, x - y\rangle \geq (1 - r)\,\|x - y\|^2, \tag{4}$$

that is, $I - T$ is strongly monotone for $r < 1$. Here I is the identity.

For $r = 1$ in (2), we arrive at the usual concept of the pseudo-contractivity of T introduced by Browder and Petryshyn in [1], that is,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - Ty - (x - y)\|^2. \tag{5}$$

An operator $T : H \to H$ is called Lipschitz continuous if there is a constant $s > 0$ such that

$$\|Tx - Ty\| \leq s\,\|x - y\| \quad \text{for all } x, y \text{ in } H. \tag{6}$$

Clearly, (6) implies

$$\langle Tx - Ty,\, x - y\rangle \leq s\,\|x - y\|^2. \tag{7}$$

Remark 1.1. We note that (2) and (3) are mutually equivalent, whereas (6) and (7) are not (since (7) does not imply (6)). That is why the generalized pseudo-contractions are more general than the Lipschitz continuous operators.

Here our aim is to present, based on a modified iterative algorithm, the solution of the NVI problem (1) involving the generalized psedudo-contractions which are Lipschitz continuous. The obtained results generalize, especially the results on pseudo-contractive and Lipschitz continuous operators in Hilbert space settings. For selected recent research works on the pseudo-contractivity, we advise [5, 6].

2. Nonlinear variational inequalities

We are just about ready to present the result on the solvability of the NVI problem (1).

Lemma 2.1 [4]**.** *Let K be a nonempty closed convex subset of a real Hilbert space H. Then for an element z in H, an element x in K satisfies*

$$\langle x - z, y - x\rangle \geq 0 \quad \text{for all } y \text{ in } K \qquad \text{iff} \quad x = P_K z. \tag{8}$$

Theorem 2.1. *Let K be a nonemty closed convex subset of a Hilbert space H. Then NVI problem* (1) *has a solution x in K iff x in K satisfies the relation*

$$x = P_K[x - t(x - Tx)], \tag{9}$$

where $t > 0$ is arbitrary.

Proof. Assume an element u in K is a solution of the NVI problem (1). Then u in K is such that

$$\langle u - Tu, y - u\rangle \geq 0 \quad \text{for all } y \text{ in } K. \tag{10}$$

Now for any $t > 0$, it follows that

$$\langle u - (u - t(u - Tu)), y - u\rangle \geq 0 \quad \text{for all } y \text{ in } K. \tag{11}$$

By Lemma 2.1, we find that

$$u = P_K[u - t(u - Tu)]. \tag{12}$$

Conversely, if u satisfies the relation

$$u = P_K[u - t(u - Tu)],$$

then u belongs to K and, by Lemma 2.1, we obtain

$$\langle u - (u - t(u - Tu)),\, y - u\rangle \geq 0 \quad \text{for all } y \text{ in } K.$$

Since $t > 0$, this implies that

$$\langle u - Tu,\, y - u\rangle \geq 0 \quad \text{for all } y \text{ in } K.$$

Hence u is a solution of the NVI problem (1).

Theorem 2.2. *Let H be a real Hilbert space and K be a nonempty closed convex subset of H. Let $T : K \to H$ be generalized pseudo-contractive (with constant $r > 0$) and Lipschitz continuous (with constant $s \geq 1$). Let $\{a_n\}$ be an increasing sequence in $[0, 1)$ such that*

$$\sum_{n=0}^{\infty} a_n = \infty \quad \text{for all } n \geq 0. \tag{13}$$

If, for an element x_0 in K, the sequence $\{x_n\}$ is generated by an iterative algorithm

$$x_{n+1} = (1 - a_n)x_n + a_n P_K\left[(1 - t)x_n + t\,Tx_n\right] \quad \text{for all } n \geq 0, \tag{14}$$

then the sequence $\{x_n\}$ converges to a unique solution of the NVI problem (1) for $0 < t < 2(1 - r)/(1 - 2r + s^2)$, and $r < 1$.

For $\{a_n\} = 1$, Theorem 2.2 reduces to

Corollary 2.1. *Let $T : K \to H$ be generalized pseudo-contractive and Lipschitz continuous, and let $r > 0$ and $s > 1$ be constants of the generalized pseudo-contractivity and Lipschitz continuity of T, respectively. Then the sequence $\{x_n\}$, generated by an iterative algorithm*

$$x_{n+1} = P_K[(1 - t)x_n + t\,Tx_n] \quad \text{for an element } x_0 \text{ in } K \tag{15}$$

and for all t such that $0 < t < 2(1 - r)/(1 - 2r + s^2)$, converges to a unique solution of the NVI problem (1).

Proof of Theorem 2.2. Suppose that z is a solution of the NVI problem (1). Then by Theorem 2.1, we have

$$z = P_K[(1 - t)z + t\,Tz].$$

Since P_K is nonexpansive, we find that

$$\begin{aligned} \|x_{n+1} - z\| &= \|(1-a_n)x_n + a_n P_K\left[(1-t)x_n + t\,Tx_n\right] - z\| \\ &\leq (1-a_n)\|x_n - z\| + a_n\|t(Tx_n - Tz) + (1-t)(x_n - z)\|. \end{aligned} \tag{16}$$

Now, since T is generalized pseudo-contractive (and hence equivalent to (3)) and Lipschitz continuous, it follows that

$$\begin{aligned} &\|t(Tx_n - Tz) + (1-t)(x_n - z)\|^2 \\ &= (1-t)^2\|x_n - z\|^2 + 2t(1-t)\langle Tx_n - Tz,\, x_n - z\rangle + t^2\|Tx_n - Tz\|^2 \\ &\leq (1-t)^2\|x_n - z\|^2 + 2t(1-t)r\|x_n - z\|^2 + t^2 s^2\|x_n - z\|^2 \\ &= \left[(1-t)^2 + 2t(1-t)r + t^2 s^2\right]\|x_n - z\|^2. \end{aligned} \tag{17}$$

Applying (17) to (16), we get

$$\begin{aligned} \|x_{n+1} - z\| &\leq \left[1 - a_n + a_n\left((1-t)^2 + 2t(1-t)r + t^2 s^2\right)^{1/2}\right]\|x_n - z\| \\ &= \left[1 - (1-k)a_n\right]\|x_n - z\| \leq \prod_{j=0}^{n}\left[1 - (1-k)a_j\right]\|x_0 - z\|, \end{aligned} \tag{18}$$

where $0 < k = \left[(1-t)^2 + 2t(1-t)r + t^2 s^2\right]^{1/2} < 1$ for all t such that $0 < t < 2(1-r)/((1-2r+s^2)$, $r < 1$ and $s \geq 1$. Since $\sum_{j=0}^{\infty} a_j = \infty$ and $k < 1$, this implies that $\lim_{n\to\infty}\prod_{j=0}^{n}[1-(1-k)a_j] = 0$. Hence $\{x_n\}$ converges to z.

To show the uniqueness of the solution, let x_1 and x_2 be two solutions of the NVI problem (1). Then we have

$$\langle (I-T)x_1,\, y - x_1\rangle \geq 0 \quad \text{for all } y \text{ in } K, \tag{19}$$

and

$$\langle (I-T)x_2,\, y - x_2\rangle \geq 0 \quad \text{for all } y \text{ in } K. \tag{20}$$

If we replace y in (19) by x_2 and y in (20) by x_1, and add, we obtain

$$\langle (I-T)x_1 - (I-T)x_2,\, x_1 - x_2\rangle \leq 0. \tag{21}$$

Since $I - T$ is strongly monotone with constant $1 - r$, we find on applying (21) that

$$(1-r)\|x_1 - x_2\|^2 \leq \langle (I-T)x_1 - (I-T)x_2,\, x_1 - x_2 \rangle \leq 0. \tag{22}$$

This implies that $x_1 = x_2$, and this copmpletes the proof.

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