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On a certain subclass of pseudosymmetric manifolds

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Abstract. This article studies semi-Riemannian manifolds which are pseudosymmetric and satisfy the condition $R\cdot R - Q(S,R) = LQ(g,C)$ simultaneously. We give the necessary and sufficient conditions for a non-conformally flat and non-Einstein manifold to be a manifold of the above type. Using these results we show that such subclasses of semisymmetric manifolds as conformally symmetric manifolds as well as simple conformally recurrent manifolds satisfy this condition.

1. Introduction

Let (M,g) be a connected n-dimensional, $n \geq 3$, semi-Riemannian manifold of class C^∞. We denote by ∇, $\tilde{R}$, R, C, S and κ the Levi-Civita connection, the curvature operator, the Riemann-Christoffel curvature tensor, the Weyl conformal curvature tensor, the Ricci tensor and the scalar curvature of (M,g), respectively.

The semi-Riemannian manifold (M,g) is locally symmetric ([23]) if

$$\nabla R = 0.$$

There exist many various possibilities to obtain curvature conditions weaker than the above one. A semi-Riemannian manifold (M,g) is said to be semisymmetric ([27]) if it satisfies the relation

$$\tilde{R}(X,Y)\cdot R = 0$$

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for all tangent vectors X and Y on M, where the curvature operator $\tilde{R}(X,Y)$ acts on the tensor R as a derivation. In the present paper the above identity is noted shortly: $R \cdot R = 0$.

It is clear that any locally symmetric manifold is semisymmetric. Riemannian semisymmetric manifolds were classified in [27]–[29] and [21]. They are non locally symmetric in general. Very recently theory of Riemannian semisymmetric manifolds has been presented in the monograph [2].

A weaker condition than semisymmetry arose during the study of totally umbilical submanifolds of semisymmetric manifolds as well as during the consideration of geodesic mappings of semisymmetric manifolds (see [14], [30]). A semi-Riemannian manifold (M, g) is said to be pseudosymmetric [16] if at every point of M the following condition is satisfied:

$(*)$ the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent.

This condition is equivalent to the relation

$$R \cdot R = L_R Q(g, R) \tag{1}$$

on the set $\mathcal{U}_R = \{x \in M \mid Z(R) \neq 0 \text{ at } x\}$, where L_R is some function on $\mathcal{U}_R$. The definitions of the tensors used will be given in Section 3. There exist various examples of pseudosymmetric manifolds which are non-semisymmetric (see e.g. [10], [16], [18]) and a review of results on pseudosymmetric manifolds is given in [14] and [30].

Manifolds (M, g) fulfilling

$$R \cdot R = Q(S, R) \tag{2}$$

were considered in [3], [5], [6]. Conformally flat manifolds realizing (2) were investigated in [13]. It is worth noticing that any 3-dimensional semi-Riemannian manifold satisfies (2) ([13], Theorem 3.1). Moreover, any hypersurface M immersed isometrically in an (n+1)-dimensional semi-Euclidean space $\mathbb{E}_s^{n+1}$, $n \geq 4$, fulfils this condition ([19], Corollary 3.1).

It is easy to see that at every point of a pseudosymmetric Einstein manifold the following condition is satisfied:

$(**)$ the tensors $R \cdot R - Q(S, R)$ and $Q(g, C)$ are linearly dependent.

It is obvious that every manifold fulfilling (2) satisfies $(**)$.

The condition $(**)$ is equivalent to the relation

$$R \cdot R - Q(S,R) = LQ(g,C) \tag{3}$$

on the set $\mathcal{U}_C = \{x \in M \mid C \neq 0 \text{ at } x\}$, where L is some function on $\mathcal{U}_C$.

Warped products realizing $(**)$ were considered in [8]. For instance, in [8] it was shown that any warped product $M_1 \times_F M_2$, $\dim M_1 = 1$, $\dim M_2 = 3$, with an arbitrary positive smooth function F on M_1, satisfies $(**)$. In particular, any generalized Robertson–Walker spacetime ([1]) fulfils $(**)$.

In the present paper we investigate semi-Riemannian manifolds which satisfy both conditions $(*)$ and $(**)$ simultaneously. The class of such manifolds contains, as we present in Section 2, various known classes of semi-Riemannian manifolds. In Section 2 we also describe examples of warped product manifolds $M_1 \times_F M_2$, with 1-dimensional base manifold M_1, satisfying $(*)$ and $(**)$.

The next family of examples of manifolds realizing $(*)$ and $(**)$ establishes hypersurfaces. First of all, in [19] it was proved that any hypersurface M in a semi-Riemannian manifold of constant curvature (N, g), $\dim N \geq 4$, satisfies the condition $(**)$. Some of them are such hypersurfaces which at any point have at most 2 distinct principal curvatures, whence, in virtue of Lemma 1 of [20], they are pseudosymmetric. It is worth noticing that examples of hypersurfaces with at most 2 distinct principal curvatures we can obtain using Theorem 5 of [24]. So, the study of pseudosymmetric hypersurfaces in semi-Riemannian spaces of constant curvature we can interpret as a part of the study of semi-Riemannian manifolds realizing $(*)$ and $(**)$. Very recently, results on pseudosymmetric hypersurfaces in semi-Riemannian spaces of constant curvature have been presented in [7], [9] and [15]. The main result of [15] states that a hypersurface M, of dimension ≥ 4, in a semi-Riemannian space of constant curvature is pseudosymmetric if and only if for every point $x \in M$ we have: the shape operator $\mathcal{A}$ of M is of rank 2 at x or the operator $\mathcal{A}^2$ is a linear combination of $\mathcal{A}$ and the identity transformation at x.

In Section 3 of this paper we fix the notations and present auxiliary lemmas. In Section 4 we give necessary and sufficient conditions for a non-conformally flat and non-Einstein semi-Riemannian manifold to be a manifold satisfying $(*)$ and $(**)$. In the Riemannian case conditions $(*)$ and $(**)$ force the very special form of the curvature tensor, namely

$$R(X,Y,Z,W) = \phi(S(X,W)S(Y,Z) - S(X,Z)S(Y,W))$$
$$+\eta G(X,Y,Z,W) + \mu(S(X,W)g(Y,Z) + S(Y,Z)g(X,W)$$
$$-S(X,Z)g(Y,W) - S(Y,W)g(X,Z)).$$

Basing on Theorem 4.2 we show that every essentially conformally symmetric manifold as well as every simple conformally recurrent manifold with non-parallel Ricci tensor satisfy the condition $(**)$. Results obtained in the present paper will make essentially easier investigations of warped product spacetimes realizing $(*)$ and $(**)$. Such manifolds will be considered in a subsequent paper of the authors.

Throughout this paper all manifolds are assumed to be connected paracompact manifolds of class C^∞.

2. Examples

First we show that the class of manifolds satisfying $(*)$ and $(**)$ contains various known classes of semi-Riemannian manifolds.

Example 2.1. A semi-Riemannian manifold (M,g) is said to be a K^*-space [31] if it is either recurrent ($\nabla R = R \otimes \omega$) or locally symmetric and satisfies the following condition:

$$\sum_{X,Y,Z} a(X)R(Y,Z) = 0, \tag{4}$$

where a is a 1-form on M and Σ denotes the cyclic sum. We note that if M is recurrent then (4) also holds and a is the recurrence 1-form.

Every K^*-space is semisymmetric. Using now Lemma 3.3, in virtue of (4) we have $Q(S,R) = 0$. Thus the relation (3) is satisfied with $L = 0$.

Example 2.2. Let (M,g) be an essentially conformally symmetric manifold [11] (e.c.s. in short), i.e., such a conformally symmetric manifold ($\nabla C = 0$) which is neither conformally flat nor locally symmetric.

Every e.c.s. manifold is semisymmetric ([11], Theorem 9) and satisfies the condition ([12], Theorem 7)

$$\sum_{X,Y,Z} S(W,X)C(Y,Z) = 0.$$

Moreover, every e.c.s. manifold fulfils the relation $\operatorname{rank} S \leq 2$ ([12], Theorem 5). It is worth noticing that if M is Ricci-recurrent then $\operatorname{rank} S \leq 1$. Thus in the case $S = ea \otimes a$ where $e = \pm 1$, we have $\sum_{X,Y,Z} a(X)C(Y,Z)=0$ and, in virtue of $\kappa = 0$ ([11], Theorem 7), also (4). Hence, in the same manner as in the Example 2.1 we see that M satisfies $(**)$.

Example 2.3. A conformally recurrent manifold (M, g) ($\nabla C = C \otimes \omega$) is said to be simple [25] (s.c.r. in short) if its metric is locally conformal to a non-conformally flat conformally symmetric one. The curvature tensor of every s.c.r. manifold satisfies $R \cdot R = 0$ ([25], Proposition 2). Every s.c.r. manifold with non-parallel Ricci tensor satisfies $\operatorname{rank} S \leq 2$. Moreover, if M is Ricci-recurrent then $\operatorname{rank} S \leq 1$ ([25], Theorem 6). If the Ricci tensor of a non-locally symmetric s.c.r. manifold is of the form $S = ea \otimes a$, then the equation (4) holds ([26], Lemma 12). This, as in the previous examples, leads to $(**)$.

In Section 4 we show that every e.c.s. manifold as well as every s.c.r. manifold with non-parallel Ricci tensor satisfy the condition $(**)$.

Now we present examples of warped product manifolds $M_1 \times_F M_2$, with 1-dimensional base manifold M_1, realizing $(*)$ and $(**)$.

Example 2.4. Let (M_1, g_1), $\dim M_1 = 1$, and (M_2, g_2), $\dim M_2 = n-1 \geq 3$, be semi-Riemannian manifolds. We denote by $M_1 \times_F M_2$ the warped product manifold of (M_1, g_1) and (M_2, g_2). It is well known that if (M_2, g_2) is a space of constant curvature then the warped product $M_1 \times_F M_2$ is a conformally flat pseudosymmetric manifold ([10], Lemma 3.1). Moreover, from Lemma 4.1 of [8] it follows that $M_1 \times_F M_2$ realizes $(**)$.

Example 2.5. Let the warped product $M_1 \times_F M_2$, $\dim M_1 = 1$, $\dim M_2 = 3$, be a pseudosymmetric manifold. Of course, such manifolds exist (e.g. see [16]). Theorem 4.2 of [8] states that the manifold $M_1 \times_F M_2$ realizes $(**)$.

Example 2.6. We denote by (M_2, g_2) the Cartesian product of the standard spheres S^1 and S^{n-2}, $n \geq 5$. It is clear that (M_2, g_2) is a semisymmetric and conformally flat manifold. The Ricci tensor $\tilde{S}$ of (M_2, g_2) has two distinct eigenvalues of multiplicity 1 and $n-2$, respectively. Now from Theorem 3.5 of [15] it follows that

$$\tilde{R} \cdot \tilde{R} = Q(\tilde{S}, \tilde{R}) \tag{5}$$

holds on M_2. Further, let $(M_1, \bar{g})$, $\bar{g}_{11} = \varepsilon$, $\varepsilon = \pm 1$, be a 1-dimensional manifold and F a function on M_1 defined by $F(x^1) = a \exp(bx^1)$, $a, b \in \mathbb{R} - \{0\}$. Corollary 4.2 of [16] states that the warped product $M_1 \times_F M_2$ is a non-semisymmetric pseudosymmetric manifold. Since (5) holds on M_2, Theorem 4.2 of [8] states that $(**)$ is fulfilled on $M_1 \times_F M_2$.

3. Preliminaries

Let (M, g) be an n-dimensional, $n \geq 3$, semi-Riemannian manifold. A tensor $\tilde{B}$ of type $(1,3)$ on M is said to be a generalized curvature tensor [22], if

$$\sum_{X_1, X_2, X_3} \tilde{B}(X_1, X_2)X_3 = 0,$$

$$\tilde{B}(X_1, X_2) + \tilde{B}(X_2, X_1) = 0,$$

$$B(X_1, X_2, X_3, X_4) = B(X_3, X_4, X_1, X_2),$$

where

$$B(X_1, X_2, X_3, X_4) = g(\tilde{B}(X_1, X_2)X_3, X_4).$$

The Ricci tensor $\operatorname{Ric}(\tilde{B})$ of $\tilde{B}$ is the trace of the linear mapping $X_1 \to \tilde{B}(X_1, X_2)X_3$. For a generalized curvature tensor $\tilde{B}$ we define the scalar curvature $\kappa(\tilde{B})$ and the tensor $Z(B)$ by

$$\kappa(\tilde{B}) = \sum_{i=1}^{n} \epsilon_i \operatorname{Ric}(\tilde{B})(E_i, E_i), \quad \epsilon_i = g(E_i, E_i),$$

and

$$Z(B) = B - \frac{\kappa(\tilde{B})}{n(n-1)} G$$

respectively, where $E_1, \dots, E_n$ is an orthonormal basis and the tensor G is defined by

$$G(X_1, X_2, X_3, X_4) = g((X_1 \wedge X_2)X_3, X_4),$$
$$(X_1 \wedge X_2)X_3 = g(X_2, X_3)X_1 - g(X_1, X_3)X_2.$$

Further, we define the Weyl curvature tensor $C(\tilde{B})$ associated with $\tilde{B}$ by

$$\begin{aligned} C(\tilde{B})(X_1, X_2, X_3, X_4) &= B(X_1, X_2, X_3, X_4) \\ +\frac{\kappa(\tilde{B})}{(n-1)(n-2)} G(X_1, X_2, X_3, X_4) &- \frac{1}{n-2}\big(g((\widetilde{\mathrm{Ric}}(\tilde{B})X_1 \wedge X_2)X_3, X_4) \\ &- g((\widetilde{\mathrm{Ric}}(\tilde{B})X_1 \wedge X_2)X_4, X_3)\big), \end{aligned}$$

where the tensor field $\widetilde{\mathrm{Ric}}(\tilde{B})$ is defined by

$$\mathrm{Ric}(\tilde{B})(X, Y) = g(\widetilde{\mathrm{Ric}}(\tilde{B})X, Y).$$

For an $(0,2)$-tensor field A on (M, g) we define the endomorphism $X \wedge_A Y$ of $\Xi(M)$ by $(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y$, where $X, Y, Z \in \Xi(M)$. In particular we have $X \wedge_g Y = X \wedge Y$.

For an $(0,k)$-tensor field T, $k \geq 1$, an $(0,2)$-tensor field A and a generalized curvature tensor $\tilde{B}$ on (M, g) we define the tensors $B \cdot T$ and $Q(A, T)$ by

$$\begin{aligned} (B \cdot T)(X_1, \dots, X_k; X, Y) &= -T(\tilde{B}((X, Y)X_1, X_2, \dots, X_k) \\ &\quad - \dots - T(X_1, \dots, X_{k-1}, \tilde{B}(X, Y)X_k), \\ Q(A, T)(X_1, \dots, X_k; X, Y) &= -T((X \wedge_A Y)X_1, X_2, \dots, X_k) \\ &\quad - \dots - T(X_1, \dots, X_{k-1}, (X \wedge_A Y)X_k), \end{aligned}$$

where $X, Y, Z, X_1, X_2, \ldots \in \Xi(M)$. Putting in the above formulas

$$\tilde{B}(X, Y)Z = \tilde{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

$T = R$, $A = g$ or $A = S$, we obtain the tensors $R \cdot R$, $R \cdot S$, $Q(g, R)$, $Q(g, S)$, $Q(S, R)$ and $Z(R)$, respectively.

Let (M, g) be a semi-Riemannian manifold covered by a system of charts $\{U; x^k\}$. We denote by g_{ij}, R_{hijk}, S_{ij}, $S_i^{\ j} = g^{jk} S_{ik}$, $G_{hijk} = g_{hk} g_{ij} - g_{hj} g_{ik}$ and

$$
\begin{aligned}
(6) \qquad C_{hijk} = R_{hijk} &- \frac{1}{n-2}(g_{hk}S_{ij} - g_{hj}S_{ik} + g_{ij}S_{hk} - g_{ik}S_{hj}) \\
&+ \frac{\kappa}{(n-1)(n-2)} G_{hijk}
\end{aligned}
$$

the local components of the metric tensor g, the Riemann-Christoffel curvature tensor R, the Ricci tensor S, the Ricci operator $\tilde{S}$, the tensor G and the Weyl tensor C, respectively.

At the end of this section we present some results which will be used in the next section.

Lemma 3.1 ([8], Lemma 2.1). *Let $\tilde{B}$ be a generalized curvature tensor on a semi-Riemannian manifold (M, g), $n \geq 3$. Then the following identity is fulfilled on M:*

$$
Q(g, C(\tilde{B})) = Q(g, B) + \frac{1}{n-2} Q(\mathrm{Ric}(\tilde{B}), G).
$$

Lemma 3.2 ([8], Corollary 2.3). *The following two conditions are equivalent on any semi-Riemannian manifold (M, g), $n \geq 3$:*

$$
R \cdot R = Q(S, R) + LQ(g, C),
$$

$$
R \cdot R = Q\left(S + Lg, R + \frac{L}{n-2} G\right).
$$

Lemma 3.3 ([17], Theorem 1). *Let $\tilde{B}$ be a generalized curvature tensor at $x \in M$ such that the condition*

$$
\sum_{X,Y,Z} \omega(X) \tilde{B}(Y, Z) = 0
$$

is satisfied for $\tilde{B}$ and a covector ω at x, where $X, Y, Z \in T_x(M)$, and Σ denotes the cyclic sum. If $\omega \neq 0$ then the following relation holds at x:

$$
B \cdot B = Q(\mathrm{Ric}(\tilde{B}), B).
$$

Lemma 3.4 ([3], Proposition 4.1)**.** *Let (M, g), $\dim M \geq 3$, be a semi-Riemannian manifold. Let A be a non-zero symmetric $(0, 2)$-tensor and $\tilde{B}$ a generalized curvature tensor at a point x of M satisfying the condition*

$$Q(A, B) = 0.$$

Moreover, let V be a vector at x such that the scalar $\rho = a(V)$ is non-zero, where a is a covector defined by $a(X) = A(X, V)$, $X \in T_x(M)$.

(i) If the tensor $A - \frac{1}{\rho} a \otimes a$ vanishes then the relation

$$\sum_{X,Y,Z} a(X)\tilde{B}(Y, Z) = 0$$

holds at x, where $X, Y, Z \in T_x(M)$.

(ii) If the tensor $A - \frac{1}{\rho} a \otimes a$ is non-zero then the relation

$$\rho B(X, Y, Z, W) = \lambda(A(X, W)A(Y, Z) - A(X, Z)A(Y, W))$$

holds at x, where $\lambda \in \mathbb{R}$ and $X, Y, Z, W \in T_x(M)$.

Moreover, in both cases the following condition holds at x:

$$B \cdot B = Q(\mathrm{Ric}(\tilde{B}), B).$$

Using algebraic properties of a (0,2)-symmetric tensor and generalized curvature tensors we can prove the following

Lemma 3.5. *Let A be a symmetric (0,2)-tensor at $x \in M$. Moreover, let $\tilde{B}_1$ and $\tilde{B}_2$ be generalized curvature tensors at $x \in M$ defined by*

$$B_1(X, Y, Z, W) = A(X, W)A(Y, Z) - A(X, Z)A(Y, W),$$

and

$$\begin{aligned} B_2(X, Y, Z, W) &= A(X, W)g(Y, Z) + A(Y, Z)g(X, W) \\ &\quad - A(X, Z)g(Y, W) - A(Y, W)g(Y, Z), \end{aligned}$$

respectively, where $X, Y, Z, W \in T_x(M)$. Then the relations

$$Q(A, G) = -Q(g, B_2),$$

$$Q(A, B_2) = -Q(g, B_1)$$

hold at x.

Lemma 3.6. *Let $\tilde{B}$ be a generalized curvature tensor at $x \in M$ such that the condition*

$$\sum_{X,Y,Z} a(X)\tilde{B}(Y,Z) = 0$$

is satisfied for a covector a at x, where $X, Y, Z \in T_x(M)$. Then the relation

$$Q(a \otimes a, B) = 0$$

holds at x.

Proof. We can write the left hand side of the above equality in the form

$$\begin{aligned} Q(a \otimes a, B)_{hijklm} &= a_h a_l B_{mijk} - a_h a_m B_{lijk} + a_i a_l B_{hmjk} - a_i a_m B_{hljk} \\ &\quad + a_j a_l B_{himk} - a_j a_m B_{hilk} + a_k a_l B_{hijm} - a_k a_m B_{hijl} \\ &= a_l(a_h B_{mijk} + a_i B_{hmjk}) + a_l(a_j B_{himk} + a_k B_{hijm}) \\ &\quad - a_m(a_h B_{lijk} + a_i B_{hljk}) - a_m(a_j B_{hilk} + a_k B_{hijl}), \end{aligned}$$

where a_i and B_{hijk} denote the local components of the covector a and the tensor B, respectively. In local components our assumption takes the form

$$a_h B_{mijk} + a_m B_{ihjk} + a_i B_{hmjk} = 0$$

which immediately gives

$$a_h B_{mijk} + a_i B_{hmjk} = a_m B_{hijk}.$$

Taking into account this equality we easily obtain our assertion.

4. Main results

Assume that (M, g) is a non-conformally flat and non-Einstein semi-Riemannian manifold. We restrict our considerations to the set $\mathcal{U} = \mathcal{U}_C \cap \mathcal{U}_S$, where $\mathcal{U}_S = \{x \in M \mid S - \frac{\kappa}{n} g \neq 0 \text{ at } x\}$. It is easy to verify that $\mathcal{U} \subset \mathcal{U}_R$.

Propopsition 4.1. *Let* $(M,g), \dim M \geq 4$, *be a semi-Riemannian manifold which is pseudosymmetric and satisfies the relation* $(**)$. *Then the equalities*

$$\begin{aligned}R(\tilde{S}X,Y,Z,W) = \kappa R(X,Y,Z,W) + S(X,Z)S(Y,W) - S(X,W)S(Y,Z)\\ +(n-1)(LC(X,Y,Z,W) - L_R R(X,Y,Z,W))\\ -L_R(S(X,Z)g(Y,W) - S(X,W)g(Y,Z)),\end{aligned}$$

$$R(\tilde{S}X,Y,Z,W) + R(\tilde{S}Z,Y,W,X) + R(\tilde{S}W,Y,X,Z) = 0 \tag{7}$$

hold on $\mathcal{U}$, *where* $X,Y,Z,W \in \Xi(M)$.

Proof. First of all we note that from $(*)$ and $(**)$ it follows that the equality

$$Q(S,R) = Q(g, L_R R - LC)$$

is fulfilled on $\mathcal{U}$. In local coordinates this equation takes the form

$$\begin{aligned}S_{hl}R_{mijk} - S_{hm}R_{lijk} + S_{il}R_{hmjk} - S_{im}R_{hljk} + S_{jl}R_{himk} - S_{jm}R_{hilk}\\ +S_{kl}R_{hijm} - S_{km}R_{hijl} = g_{hl}(L_R R_{mijk} - LC_{mijk})\\ -g_{hm}(L_R R_{lijk} - LC_{lijk}) + g_{il}(L_R R_{hmjk} - LC_{hmjk})\\ -g_{im}(L_R R_{hljk} - LC_{hljk}) + g_{jl}(L_R R_{himk} - LC_{himk}) - g_{jm}(L_R R_{hilk} - LC_{hilk})\\ +g_{kl}(L_R R_{hijm} - LC_{hijm}) - g_{km}(L_R R_{hijl} - LC_{hijl}).\end{aligned}$$

Contracting the above equality with g^{hl} we find

$$\begin{aligned}\kappa R_{mijk} - D_{mijk} + D_{imjk} + D_{jimk} + D_{kijm} + S_{jm}S_{ik} - S_{km}S_{ij}\\ = (n-1)(L_R R_{mijk} - LC_{mijk}) + L_R(g_{jm}S_{ik} - g_{km}S_{ij}),\end{aligned} \tag{8}$$

where $D_{lijk} = S_l{}^r R_{rijk}$, whence, by cyclic permutation of m,j,k we obtain (7).

Substituting (7) into (8) we have

$$\begin{aligned}D_{imjk} = \kappa R_{imjk} + S_{ij}S_{mk} - S_{ik}S_{mj} + (n-1)(LC_{imjk} - L_R R_{imjk})\\ -L_R(S_{ij}g_{mk} - S_{ik}g_{mj}),\end{aligned}$$

which completes the proof.

Assume now that (M, g) is a non-conformally flat and non-Einstein pseudosymmetric manifold satisfying $(**)$. Applying (1), (3) and Lemma 3.2 we easily obtain the following equality:

$$Q(L_R g, R) = Q\left(S + Lg, R + \frac{L}{n-2}G\right).$$

On the other hand $Q(L_R g, R) = Q(L_R g, R + \frac{L}{n-2}G)$. Thus we have the following relation:

$$Q\left(S + (L - L_R)g, R + \frac{L}{n-2}G\right) = 0.$$

Let V be a vector at $x \in \mathcal{U}$ such that the scalar $\rho = a(V)$ is non-zero, where a is a covector defined by

$$a(X) = S(X, V) + (L - L_R)g(X, V),$$

$X \in T_x(M)$. Using now Lemma 3.4 we have two cases:

(i) If the tensor $S + (L - L_R)g - \frac{1}{\rho}a \otimes a$ vanishes then the relation

$$\sum_{X,Y,Z} a(X)\tilde{B}(Y, Z) = 0 \tag{9}$$

holds at x, where $B = R + \frac{L}{n-2}G$ and $X, Y, Z \in T_x(M)$.

(ii) If the tensor $S + (L - L_R)g - \frac{1}{\rho}a \otimes a$ is non-zero then the relation

$$\rho B(X, Y, Z, W) = \lambda(A(X, W)A(Y, Z) - A(X, Z)A(Y, W)) \tag{10}$$

holds at x, where $B = R + \frac{L}{n-2}G$, $A = S + (L - L_R)g$, $\lambda \in \mathbb{R}$ and $X, Y, Z, W \in T_x(M)$.

Now we consider the case (i). In local coordinates we have the following relations:

$$S_{ij} = (L_R - L)g_{ij} + \frac{1}{\rho}a_i a_j, \tag{11}$$

$$\begin{aligned} a_m\left(R_{lijk} + \frac{L}{n-2}G_{lijk}\right) + a_j\left(R_{likm} + \frac{L}{n-2}G_{likm}\right) \\ + a_k\left(R_{limj} + \frac{L}{n-2}G_{limj}\right) = 0. \end{aligned} \tag{12}$$

Contracting (12) with g^{hm} we have

$$a_r R^r{}_{ijk} + a_j S_{ik} - a_k S_{ij} = L(a_k g_{ij} - a_j g_{ik})$$

and using (11) we obtain

$$a_r R^r{}_{ijk} = L_R(a_k g_{ij} - a_j g_{ik}). \tag{13}$$

Contracting (13) with g^{ij} we find $a_r S^r{}_k = (n-1) L_R a_k$. On the other hand, from (11) it follows that $a_r S^r{}_k = (L_R - L) a_k + \frac{1}{\rho} a^r a_r$, where $a^r = g^{rs} a_s$. Comparing the two last equalities we obtain

$$L + (n-2) L_R = \frac{1}{\rho} a^r a_r$$

and taking into account that $\kappa = n(L_R - L) + \frac{1}{\rho} a^r a_r$, we have also

$$2L_R - L = \frac{\kappa}{n-1}.$$

Thus we have proved the following

Proposition 4.2. *If (M, g) is a pseudosymmetric manifold satisfying $(**)$ such that $S = (L_R - L) g + \frac{1}{\rho} a \otimes a$ at $x \in \mathcal{U}$, then the equalities*

$$a_r R^r{}_{ijk} = L_R(a_k g_{ij} - a_j g_{ik}),$$
$$L + (n-2) L_R = \frac{1}{\rho} a^r a_r,$$
$$2L_R - L = \frac{\kappa}{n-1}$$

hold at x.

Corollary 4.1. *If $a^r a_r = 0$ at $x \in \mathcal{U}$ then $L + (n-2) L_R = 0$, $L_R = \frac{\kappa}{n(n-1)}$ and $S = \frac{\kappa}{n} g + \frac{1}{\rho} a \otimes a$ hold at x.*

We present now the converse statement.

Theorem 4.1. *Let x be a point of a semi-Riemannian manifold (M, g), $\dim M \geq 4$, such that the following conditions are fulfilled:*

$$S = \alpha g + \beta a \otimes a, \tag{14}$$

$$\sum_{X,Y,Z} a(X) \tilde{B}(Y, Z) = 0, \tag{15}$$

for some non-zero covector a, where $B = R - \gamma G$, $\alpha, \beta, \gamma \in \mathbb{R}$. Then the equalities:

$$R \cdot R = \left(\frac{\kappa}{n-1} - \alpha\right) Q(g, R), \tag{16}$$

$$R \cdot R = Q(S, R) + \left(\frac{\kappa}{n-1} - 2\alpha\right) Q(g, C), \tag{17}$$

$$\sum_{X,Y,Z} a(X)\tilde{C}(Y, Z) = 0 \tag{18}$$

hold at x. Moreover, if $x \in \mathcal{U}$ then $a^r a_r = 0$ at x.

Proof. In the same manner as in the proof of Proposition 4.1 we obtain

$$a_r R^r{}_{ijk} = (\alpha - (n-2)\gamma)(a_k g_{ij} - a_j g_{ik}),$$

$$\beta a^r a_r = \kappa - n\alpha, \tag{19}$$

$$2\alpha - (n-2)\gamma = \frac{\kappa}{n-1}. \tag{20}$$

Now we observe that

$$(R - \gamma G) \cdot (R - \gamma G) = (R - \gamma G) \cdot R = R \cdot R - \gamma Q(g, R).$$

Using Lemma 3.3, we have $B \cdot B = Q(\mathrm{Ric}(\tilde{B}), B)$. Now, for $B = R - \gamma G$ we get $\mathrm{Ric}(\tilde{B}) = S - (n-1)g$ and next, in virtue of Lemma 3.6,

$$Q(\mathrm{Ric}(\tilde{B}), B) = Q((\alpha - (n-1)\gamma)g, R - \gamma G) = (\alpha - (n-1)\gamma)Q(g, R).$$

Thus we have

$$R \cdot R - \gamma Q(g, R) = (\alpha - (n-1)\gamma)Q(g, R)$$

and taking into account (20) we obtain (16).

In virtue of (14) the equality (6) leads to

$$C_{hijk} = R_{hijk} + \frac{1}{n-2}\left(\frac{\kappa}{n-1} - 2\alpha\right) G_{hijk} - \frac{\beta}{n-2} T_{hijk}, \tag{21}$$

where

$$T_{hijk} = g_{hk}a_i a_j - g_{hj}a_i a_k + g_{ij}a_h a_k - g_{ik}a_h a_j.$$

Thus $Q(g,C) = Q(g,R) - \frac{\beta}{n-2}Q(g,T)$ and using the equality $Q(g,T) = -Q(a\otimes a, G)$ (see Lemma 3.5), we obtain

$$Q(g,C) = Q(g,R) + \frac{\beta}{n-2}Q(a\otimes a, G). \tag{22}$$

On the other hand, using (20) and Lemma 3.6, we find

$$\begin{aligned} Q(S,R) &- \left(\frac{\kappa}{n-1} - \alpha\right) Q(g,R) \\ &= \alpha Q(g,R) + \beta Q(a\otimes a, R) - \left(\frac{\kappa}{n-1} - \alpha\right) Q(g,R) \\ &= \left(2\alpha - \frac{\kappa}{n-1}\right) Q(g,R) + \beta\gamma Q(a\otimes a, G) \\ &= (n-2)\gamma \left(Q(g,R) + \frac{\beta}{n-2}Q(a\otimes a, G)\right). \end{aligned}$$

Comparing this equation with (22) and using (20) we obtain

$$Q(S,R) + \left(\frac{\kappa}{n-1} - 2\alpha\right) Q(g,C) = \left(\frac{\kappa}{n-1} - \alpha\right) Q(g,R).$$

This, by making use of (16), yields (17). Next, using (15), (21) and (20), after standard calculation we obtain (18).

Now suppose that $a^r a_r \neq 0$ at $x \in \mathcal{U}$. Transvection of the equality

$$a_m R_{lijk} + a_j R_{likm} + a_k R_{limj} = \gamma(a_m G_{lijk} + a_j G_{likm} + a_k G_{limj})$$

with a^m yields

$$a^r a_r R_{lijk} - \left(\frac{\kappa}{n-1} - \alpha - \gamma\right) T_{lijk} = \gamma a^r a_r G_{lijk}.$$

But using (19) and (20) we get

$$\frac{\kappa}{n-1} - \alpha - \gamma = \frac{\beta}{n-2} a^r a_r$$

which turns the previous equality into

$$a^r a_r R_{lijk} = \frac{\beta}{n-2} a^r a_r T_{lijk} + \gamma a^r a_r G_{lijk}$$

and, in virtue of $a^r a_r \neq 0$, into

$$R_{lijk} = \frac{\beta}{n-2} T_{lijk} + \gamma G_{lijk}.$$

Substituting this relation into (21) and using (20) we get $C = 0$ at $x \in \mathcal{U}$, a contradiction. Our theorem is thus proved.

We consider now the case when $S + (L - L_R)g \neq \frac{1}{\rho} a \otimes a$. Thus, in view of Lemma 3.4, we have

$$R_{lijk} + \frac{L}{n-2} G_{lijk} = \phi(A_{lk}A_{ij} - A_{lj}A_{ik}),$$

where $A_{ij} = S_{ij} + (L - L_R)g_{ij}$, $\phi \in \mathbb{R}$, from which we conclude the following

Lemma 4.1. *If (M, g) is a pseudosymmetric manifold satisfying $(**)$ such that the tensor $S + (L - L_R)g - \frac{1}{\rho} a \otimes a$ is non-zero at $x \in \mathcal{U}$, then the curvature tensor has the following form at x:*

$$R_{hijk} = \phi(S_{hk}S_{ij} - S_{hj}S_{ik}) + \phi(L - L_R)(S_{hk}g_{ij} + S_{ij}g_{hk}$$
$$-S_{hj}g_{ik} - S_{ik}g_{hj}) + (\phi(L - L_R)^2 - \frac{L}{n-2})G_{hijk}.$$

Theorem 4.2. *Let (M, g) be a semi-Riemannian manifold with curvature tensor of the form*

$$R(X, Y, Z, W) = \phi(S(X, W)S(Y, Z) - S(X, Z)S(Y, W))$$
$$+\eta G(X, Y, Z, W) + \mu(S(X, W)g(Y, Z) + S(Y, Z)g(X, W)$$
$$-S(X, Z)g(Y, W) - S(Y, W)g(X, Z))$$

at $x \in \mathcal{U}$, where $X, Y, Z, W \in T_x(M)$ and $\phi, \mu, \eta \in \mathbb{R}$. Then the equalities

(23) $$R \cdot R = L_R Q(g, R),$$

(24) $$R \cdot R = Q(S, R) + (L_R + \frac{\mu}{\phi})Q(g, C)$$

hold at x, where

$$L_R = \frac{\mu}{\phi}((n-2)\mu - 1) - \eta(n-2).$$

PROOF. First we observe that $\phi \neq 0$ at x (the equality $\phi = 0$, by standard calculation gives $C = 0$). Contracting the equality

$$\begin{aligned} R_{hijk} &= \phi(S_{hk}S_{ij} - S_{hj}S_{ik}) \\ &\quad + \mu(S_{hk}g_{ij} + S_{ij}g_{hk} - S_{hj}g_{ik} - S_{ik}g_{hj}) + \eta G_{hijk} \end{aligned} \tag{25}$$

with g^{hk} we have

$$\phi S_{ir}S^r{}_j = (\phi\kappa - 1 + \mu(n-2))S_{ij} + (\mu\kappa + \eta(n-1))g_{ij}.$$

Transvecting (25) with $S_m^{\ r}$ and using the above equality we obtain

$$\begin{aligned} S_m^{\ r}R_{rijk} &= (\alpha+\mu)(S_{mk}S_{ij} - S_{mj}S_{ik}) \\ &+ \left(\frac{\mu\alpha}{\phi} + \eta\right)(S_{mk}g_{ij} - S_{mj}g_{ik}) + \beta(g_{mk}S_{ij} - g_{mj}S_{ik}) + \frac{\mu\beta}{\phi}G_{mijk}, \end{aligned}$$

where $\alpha = \phi\kappa - 1 + \mu(n-2)$, $\beta = \mu\kappa + \eta(n-1)$. Symmetrizing this equality in m, i, we get

$$S_m^{\ r}R_{rijk} + S_i^{\ r}R_{mjk} = \left(\frac{\mu\alpha}{\phi} + \eta - \beta\right)(S_{mk}g_{ij} - S_{mj}g_{ik} + S_{ik}g_{mj} - S_{ij}g_{mk})$$

i.e.

$$R \cdot S = L_R Q(g, S), \tag{26}$$

where $L_R = \frac{\mu}{\phi}((n-2)\mu - 1) - \eta(n-2)$.

Let $\bar{S}$ and P denote the (0,4)-tensors with local components $\bar{S}_{hijk} = S_{hk}S_{ij} - S_{hj}S_{ik}$ and $P_{hijk} = g_{hk}S_{ij} + g_{ij}S_{hk} - g_{hj}S_{ik} - g_{ik}S_{hj}$, respectively. Thus our assumption can be written in the form

$$R = \phi\bar{S} + \mu P + \eta G.$$

This leads to

$$R \cdot R = \phi R \cdot \bar{S} + \mu R \cdot P.$$

Using (26) we now can verify that $R \cdot P = -L_R Q(S,G)$, which in view of Lemma 3.5 turns into $R \cdot P = L_R Q(g,P)$. In the same manner we can verify that $R \cdot \bar{S} = L_R Q(g,\bar{S})$. Thus we have

$$\begin{aligned} R \cdot R &= \phi L_R Q(g,\bar{S}) + \mu L_R Q(g,P) = L_R Q(g, \phi\bar{S} + \mu P) \\ &= L_R Q(g, \phi\bar{S} + \mu P + \eta G) = L_R Q(g,R). \end{aligned}$$

The equality (23) is thus proved.

Finally, computing in the same manner as in the proof of Theorem 4.1 and using the equality $Q(S,P) = -Q(g,\bar{S})$ (see Lemma 3.5) we get

$$Q(S,R) - L_R Q(g,R) = \left(L_R + \frac{\mu}{\phi}\right) Q(g,C)$$

which, in virtue of (23) leads to (24). This completes the proof.

As an application of the above theorem we have the following

Proposition 4.3. *Let (M,g) be an essentially conformally symmetric manifold (see Example 2.2). If* $\operatorname{rank} S = 2$ *at $x \in M$, then the condition* $(**)$ *is satisfied at x.*

Proof. Any e.c.s. manifold M admits a unique function F such that $FC_{hijk} = S_{hk}S_{ij} - S_{hj}S_{ik}$ ([12], Theorem 3). Clearly, $F(x) = 0$ if and only if $\operatorname{rank} S \leq 1$ at x. Thus we have $F(x) \neq 0$ and $C_{hijk} = \frac{1}{F}(S_{hk}S_{ij} - S_{hj}S_{ik})$ at x. Using now (6), in virtue of $\kappa = 0$ ([11], Theorem 7), we obtain

$$R_{hijk} = \frac{1}{F}(S_{hk}S_{ij} - S_{hj}S_{ik}) + \frac{1}{n-2}(S_{hk}g_{ij} + S_{ij}g_{hk} - S_{hj}g_{ik} - S_{ik}g_{hj}).$$

Applying Theorem 4.2, we get $L_R = 0$ and $Q(S,R) + \frac{F}{n-2}Q(g,C) = 0$. Thus the relation (3) holds at x with $L = \frac{F}{n-2}$. This completes the proof.

From the above proposition and Example 2.2 we can conclude that every e.c.s. manifold satisfies $(*)$ and $(**)$.

Now let (M,g) be an s.c.r. manifold (see Example 2.3) with non-parallel Ricci tensor. In view of Theorems 3 and 4 of [25], we can follow step by step the proof of Proposition 4.3 and taking into account Example 2.3, we obtain the same conclusion for s.c.r. manifolds. We summarise the above in

Theorem 4.3. *Every essentially conformally symmetric manifold as well as every simple conformally recurrent manifold with non-parallel Ricci tensor satisfy* $(*)$ *and* $(**)$*.*

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