Publ. Math. Debrecen
53 / 1-2 (1998), 49–57

On arrangment of regular cyclic subgroup in symmetric group

By PIOTR W. GAWRON (Gliwice) and
WITOLD TOMASZEWSKI (Gliwice)

Abstract. In this paper we investigate weakly normal and polinormal regular cyclic subgroups in a symmetric group S_n. We give here necessary and sufficient conditions for the subgroup generated by a long cycle $(\overline{0}\,\overline{1}\ldots\overline{n-1})$ in S_n to be weakly normal or polinormal. We also describe a normalizer of this subgroup in S_n.

Let $Z_n = \{\overline{0}\,\overline{1}\ldots\overline{n-1}\}$ be a ring of all integers modulo n (where n is a fixed positive integer) and let S_n be a symmetric group of degree n which acts on Z_n. As usual, Z_n^* denotes a set of invertible elements in Z_n. Each cycle of the length n in S_n is called a long cycle. The subgroup generated by a long cycle is called regular. We denote $x^y = y^{-1}xy$.

Let G be an arbitrary group, and D be its subgroup. We denote by $N_G(D)$ a normalizer of D in G. If $A, B \subseteq G$ then $\langle A, B\rangle$ denotes the subgroup generated by A, B. If $g \in G$ then $D^{\langle g,D\rangle}$ denotes the normal closure of D in the group $\langle g, D\rangle$, that is $D^{\langle g,D\rangle}$ is the subgroup generated by all elements d^h, where $d \in D$ and $h \in \langle g, D\rangle$. The subgroup $D^{D^{\langle g,D\rangle}}$ denotes a normal closure of D in $D^{\langle g,D\rangle}$ and we have $D^{D^{\langle g,D\rangle}} \subseteq D^{\langle g,D\rangle}$.

A subgroup D of an arbitrary group G is called *weakly normal* [3] if $D^g \le N_G(D)$ implies $D^g = D$. In paper [1] Z. I. Borevich and O. Macedońska introduced definition of a polinormal subgroup. The subgroup D is *polinormal* in G if for each $g \in G$ the following property holds: $D^{\langle g,D\rangle} = D^{D^{\langle g,D\rangle}}$. Finally, a subgroup D is called *pronormal* [3]

Mathematics Subject Classification: 20B35.
Key words and phrases: permutation groups, weakly normal subgroups, pronormal subgroups, polinormal subgroups, regular subgroup.

if subgroups D and D^g are conjugated in the subgroup $\langle D, D^g\rangle$, for each $g \in G$. Sylow subgroups in a finite group are pronormal. Every pronormal subgroup is weakly normal and polinormal.

In this paper we investigate weakly normal and polinormal regular cyclic subgroups in a symmetric group S_n. We give here the necessary and sufficient conditions for the subgroup generated by a long cycle $(\overline{0}\,\overline{1}\dots\overline{n-1})$ in S_n to be weakly normal and polinormal. We also describe a normalizer of this subgroup in S_n.

If n is a prime then the regular subgroup generated by a long cycle is a Sylow subgroup in S_n, so it is pronormal. R. I. Tyshkevich proved [5] that if $(n, \varphi(n)) = 1$ (where φ is the Euler function), then the subgroup generated by a long cycle in S_n is pronormal. The same result, by another methods, was obtained by P. P. Palfy in [4].

Let $\gamma = (\overline{0}\,\overline{1}\dots\overline{n-1})$ be a long cycle in S_n and let Γ be the subgroup generated by γ. A permutation η from Γ is a long cycle (i.e. it is conjugated to γ), if $\eta = \gamma^k$, where k is an invertible element of Z_n (that is, $k \in Z_n^*$). We describe a normalizer N of Γ in S_n.

Lemma 1. *The normalizer N of Γ in S_n consists of all such permutations η, which act on Z_n as follows:*

$$\eta(x) = kx + a,$$

where $k \in Z_n^*$, $a \in Z_n$. Moreover:

$$|N| = n \cdot \varphi(n).$$

Proof. The permutation γ acts on Z_n as follows:

$$\gamma(x) = x + \overline{1}.$$

Hence, we have: $\gamma^k(x) = x + k$. A permutation η belongs to N if and only if:

$$\eta\gamma\eta^{-1} = \gamma^k,$$

for some $k \in Z_n^*$.

Let η be in N, then we have:

$$\eta(x + \overline{1}) = \eta(\gamma(x)) = \eta\gamma\eta^{-1}\eta(x) = \gamma^k\eta(x) = \eta(x) + k,$$

thus by induction

$$\eta(x+y) = \eta(x) + ky.$$

If

$$\eta(\overline{0}) = a, \text{ then } \eta(y) = ky + a, \text{ and } \eta^{-1}(y) = k^{-1}y - k^{-1}a.$$

Conversely, let $\eta(x)$ be equal to $kx + a$, where $k \in Z_n^*$, $a \in Z_n$. Hence

$$\begin{aligned}\eta\gamma\eta^{-1}(x) &= \eta\gamma(k^{-1}x - k^{-1}a + \overline{1}) \\ &= \eta(k^{-1}x - k^{-1}a + \overline{1}) = x + k = \gamma^k(x),\end{aligned}$$

which finishes the proof. □

A weak normality of Γ in S_n is equivalent to the following condition: if η is a long cycle that belongs to N, then η belongs to Γ. If n is a prime, then Γ has $n-1$ long cycles and there are no long cycles in $N \setminus \Gamma$.

If $n = p_1^{\alpha_1} \dots p_m^{\alpha_m}$ is a factorization of n, then the ring Z_n is isomorphic to a direct product of rings Z_{q_i}, where $q_i = p_i^{\alpha_i}$. Let S_{q_i} be a symmetric group acting on Z_{q_i}, let Γ_i be a subgroup generated in S_{q_i} by a cycle $(\overline{0}\,\overline{1}\dots\overline{q_i - 1})$ and let N_i be the normalizer of Γ_i in S_{q_i}.

Lemma 2. *The normalizer N of a subgroup Γ in S_n is isomorphic to the direct product $N_1 \times \ldots \times N_m$.*

Proof. Let ψ be a mapping:

$$\psi : N \to N_1 \times \cdots \times N_m : \psi(\eta) = (\eta_1, \ldots, \eta_m),$$

such, that if $\eta(x) = kx + a$ for $k \in Z_n^*$, $a \in Z_n$, then for all $i \in \{1, \ldots, m\}$ $\eta_i(x) = kx + a$, where all numbers are taken modulo q_i. Clearly, ψ is a homomorphism. Let η, $\mu \in N$ and

$$\eta(x) = kx + a, \quad \mu(x) = lx + b.$$

If $\psi(\eta) = \psi(\mu)$ then $kx+a \equiv lx+b \pmod{q_i}$ for $i = 1 \ldots m$, $x \in Z$. Hence $k \equiv l$, $a \equiv b \pmod{q_i}$, so $k = l$, $a = b$ in Z_n and this means that ψ is a monomorphism. We know from Lemma 1 that $|N| = n \cdot \varphi(n)$. Let us compute:

$$|N_1 \times \ldots \times N_m| = q_1 \ldots q_m \cdot \varphi(q_1) \ldots \varphi(q_m) = n \cdot \varphi(n),$$

where the last equation holds from multiplicative property of the Euler function. It means that φ is a monomorphism of two groups, which have the same number of elements, so φ is an isomorphism. □

In the notations of Lemma 2 we get:

Lemma 3. *A permutation $\eta \in N$ is a long cycle, if and only if, each η_i is a long cycle in N_i. A number of long cycles in N is a product of numbers of long cycles in all N_i.*

PROOF. If η is a long cycle, which belongs to N, then n is equal to the least common multiple of orders of all permutations η_i. Lengths of different η_i are coprime, so each η_i must be a long cycle in N_i. The converse statement is clear. Thus in N we have as many long cycles as rows of the type $(\eta_1, \ldots, \eta_m)$, where η_i is a long cycle in N_i. □

The following theorem was firstly presented without proof in [2].

Theorem 1. *A subgroup Γ generated by a long cycle in S_n is weakly normal if and only if n is not divisible by 8 and is not divisible by a square of an odd prime.*

POOF. We note, that if $n = 4$, then there are only two long cycles in N: γ and γ^3, and these cycles belong to Γ. Let n be not divisible by 8 and by a square of an odd prime. Then for an arbitrary diviser p_i of n, there exist $\varphi(q_i)$ long cycles in a subgroup N_i. Hence there are $\varphi(q_1) \ldots \varphi(q_m) = \varphi(n)$ long cycles in N and each of them belongs to Γ. So we proved that Γ is weakly normal in S_n.

Let now n be divisible by p^2 or by 8, where p is an odd prime. We want to show that Γ is not weakly normal. For this purpose we use the permutation acting on Z_n:

$$\sigma(x) = x + \frac{x(x+\overline{1})}{\overline{2}} \frac{n}{k},$$

where $k = p$ if n is divisible by p^2 and $k = 2$ if n is divisible by 8. Then the inverse permutation is:

$$\sigma^{-1}(x) = x - \frac{x(x+\overline{1})}{\overline{2}} \frac{n}{k}.$$

The permutation $\eta = \sigma\gamma\sigma^{-1}$ is a long cycle and

$$\eta(x) = \left(\overline{1} + \frac{n}{k}\right) x + \overline{1} + \frac{n}{k}.$$

By Lemma 1, permutation belongs to N and does not belong to Γ. So Γ is not a weakly normal subgroup of S_n, which finishes the proof. □

Corollary 1. *If n is divisible by 8 or by a square of an odd prime, then Γ is not a polinormal subgroup of S_n.*

Proof. Let σ and η be the same permutations as in the proof of Theorem 1. We show that a permutation σ normalizes the subgroup $\langle\gamma,\eta\rangle$. Indeed by computation we get $\sigma\eta\sigma^{-1} = \gamma^{(-1+\frac{n}{p})}\eta^2 \in \langle\gamma,\eta\rangle$ and by definition of η we have $\sigma\gamma\sigma^{-1} = \eta \in \langle\gamma,\eta\rangle$. Hence $\langle\gamma,\eta\rangle \subseteq \Gamma^{\langle\sigma,\Gamma\rangle}$. So we get $\Gamma^{\langle\sigma,\Gamma\rangle} = \langle\gamma^h; h \in \langle\sigma\rangle\rangle = \langle\gamma,\eta\rangle$. The subgroup $\Gamma^{\Gamma^{\langle\sigma,\Gamma\rangle}}$ is equal to $\Gamma^{\langle\gamma,\eta\rangle}$ and by Theorem 1 $\langle\gamma,\eta\rangle \subseteq N$, so we have $\Gamma^{\langle\gamma,\eta\rangle} = \Gamma$, which means that Γ is not polinormal in S_n. □

The above theorem implies straightforward, such a corollary:

Corollary 2. *If Γ is polinormal in S_n then it is also weakly normal.*

By simple calculation it follows that for $n = 6$ the subgroup Γ is not polinormal in S_n. So the converse to Corollary 2 is not true. We note that $(6, \varphi(6)) = 2 > 1$, and we can prove:

Theorem 2. *A subgroup Γ is polinormal in S_n if and only if $(n, \varphi(n)) = 1$ or $n = 4$.*

Before we prove the above Theorem, we will give some auxiliary lemmas. Let, for now, n be an even integer grater than 4. So there exists k, such that $n = 2k$. We take $\gamma = (\overline{0}\ \overline{1} \dots \overline{2k-1})$ and $g = (\overline{3}\ \overline{2k-1})(\overline{5}\ \overline{2k-3}) \dots$. The permutation g is an involution. We also introduce two symbols x and y for following cycles:

$$x = (\overline{0}\,\overline{2} \dots \overline{2k-2}),\ y = (\overline{1}\,\overline{3} \dots \overline{2k-1}).$$

Permutations x and y commute and have the same order k. We, also, have a relation $\gamma^2 = xy$. Let Γ be the cyclic subgroup generated by γ. The normal closure of $\Gamma = \langle\gamma\rangle$ in the group $\langle\gamma, g\rangle$ is generated by all conjugates of γ, so $\langle\gamma\rangle^{\langle\gamma,g\rangle} = \langle\gamma^h;\ h \in \langle\gamma, g\rangle\rangle$. It is enough to take as generators of this normal closure all elements γ^h for $h \in \langle g\rangle$. The element g is an involution, so:

$$\Gamma^{\langle g,\Gamma\rangle} = \langle\gamma\rangle^{\langle g,\gamma\rangle} = \langle\gamma^h \mid h \in \langle g\rangle\rangle = \langle\gamma, \gamma^g\rangle.$$

Now we describe the subgroup $\Gamma^{\Gamma^{\langle g,\Gamma\rangle}}$ which, by the above is:

$$\Gamma^{\Gamma^{\langle g,\Gamma\rangle}} = \langle\gamma\rangle^{\langle\gamma,\gamma^g\rangle}, \tag{1}$$

and we prove, it is generated by γ and γ^{γ^g}.

Lemma 4. *The permutations g and γ satisfy the relation:*

$$(\gamma^g)^2 = \gamma \cdot \gamma^{\gamma^g} \tag{2}$$

and for all numbers $l \in Z$ we have:

$$\gamma^{(\gamma^g)^l} \in \langle \gamma, \gamma^{\gamma^g} \rangle. \tag{3}$$

Proof. By calculation we get $(\gamma)^2 = xy^{-1} = \gamma \cdot \gamma^{\gamma^g}$ and it proves the first part of the lemma. We prove the second part by induction on l. Let us start from $l = 2$. By equation (2):

$$\gamma^{(\gamma^g)^2} = \gamma^{\gamma \cdot \gamma^{\gamma^g}} \in \langle \gamma, \gamma^{\gamma^g} \rangle.$$

Let, now (3) be true for all numbers less than l, and let $l \geq 2$ then $\gamma^{(\gamma^g)^l} = (\gamma^{(\gamma^g)^{l-2}})^{(\gamma^g)^2}$, by inductive assumption $(\gamma^{(\gamma^g)^{l-2}}) \in \langle \gamma, \gamma^{\gamma^g} \rangle$ and by the equation (2) $(\gamma^g)^2 \in \langle \gamma, \gamma^{\gamma^g} \rangle$, which finishes the proof. □

We have to show that Γ is not polinormal. Due to the second part of the Lemma 4 and from the equation (1) we get:

$$\Gamma^{\Gamma^{\langle g, \Gamma \rangle}} = \langle \gamma, \gamma^{\gamma^g} \rangle.$$

We need to show, that:

$$\Gamma^{\langle g, \Gamma \rangle} \not\subseteq \Gamma^{\Gamma^{\langle g, \Gamma \rangle}}.$$

In fact, it is enough to show that:

$$\gamma^g \notin \Gamma^{\Gamma^{\langle , \Gamma \rangle}} = \langle \gamma, \gamma^{\gamma^g} \rangle. \tag{4}$$

Let us denote $\delta = \gamma^{\gamma^g} = (\overline{0}\ \overline{1}\ \overline{2k-2}\ \overline{2k-1} \ldots \overline{5}\ \overline{2}\ \overline{3})$. By computations one can find relations between γ and δ:

$$\gamma^{2k} = \delta^{2k} = 1,$$

$$(\gamma \cdot \delta)^k = (xy^{-1})^k = x^k y^{-k} = 1,$$

$$\gamma^2 \cdot \delta^2 = xy \cdot x^{-1} y^{-1} = 1.$$

Now we want to describe groups with such relations:

Lemma 5. *Let G be a group with presentation:*

$$\langle a, b \mid a^{2k} = b^{2k} = (ab)^k = 1,\ a^2b^2 = 1\rangle \tag{5}$$

Then:

(1) every element of G can be written in the form $a^s c^t$, where: $c = ab$, $s \in \{1, \dots, 2k\}$, $t \in \{1, \dots, k\}$,

(2) $|G| \leq 2k^2$,

(3) if G has such an element d that $a^d = b$ then G has a relation $a^2 = c^s$, where s is an odd integer.

Proof. First, we want to describe the commutator subgroup G' in G. The group G is two-generator, the commutator subgroup G' is generated by elements $[a^s, b^t]$, $s, t \in Z$. From the relation $a^2b^2 = 1$, it follows the relation $a^2b = ba^2$. Indeed it follows from equalities: $a^2b = b^{-2}b = bb^{-2} = ba^2$. So the commutator subgroup is cyclic and generated by $[a, b]$. From the relation $a^2b^2 = 1$ we have the equality $ab = a^{-1}b^{-1}$. Hence

$$[a, b] = a^{-1}b^{-1}ab = (ab)^2, \tag{6}$$

which means that $G' = \langle [a, b] \rangle \subseteq \langle ab \rangle$ and the subgroup $H = \langle ab \rangle$ is normal and has order k. The quotient goup G/H is cyclic generated by the coset aH and has order less or equal $2k$. So every element of G can be written as $a^s(ab)^t$ and G has the order less or equal $2k^2$. It proves (1) and (2).

To prove (3) we show, first, that in G holds the relation $ca = ac^{-1}$ (where $c = ab$). We use relations of G and the equality (6), to get:

$$ca = aba = a^2b[b, a] = aab(ab)^{-2} = a(ab)^{-1} = ac^{-1}. \tag{7}$$

Now we prove that for every l the following holds:

$$c^l a = ac^{-l}. \tag{8}$$

Let the equation be true for all numbers less or equal to l, then by inductive assumption and (7): $c^{l+1}a = cac^{-l} = ac^{-1}c^{-l} = c^{-(l+1)}$. Let now there exists $d \in G$, such that $a^d = b$. Then $d = a^s c^t$ and by (8) we have: $a^d = c^{-t}a^{-s}aa^s c^t = c^{-t}ac^t = ac^{2t} = b$. From the last equation we obtain the equality $a^2c^{2t} = ab = c$ and we get the relation $a^2 = c^s$ for s an odd integer. It finishes the third part of Lemma 5. □

Due to the previous Lemma we can prove that:

Corollary 3. *The element γ^g is not in $\langle \gamma, \gamma^{\gamma^g} \rangle$.*

PROOF. The subgroup $\langle \gamma, \gamma^{\gamma^g} \rangle = \langle \gamma, \delta \rangle$ satisfies the same relations as the group G in Lemma 5. So by that Lemma γ^g belongs to $\langle \gamma, \gamma^{\gamma^g} \rangle$ if there is a relation: $\gamma^2 = \delta^{2s+1}$. We show that elements γ and δ do not satisfy such a relation. Indeed, $\gamma^2 = xy$, and $\delta^{2s+1} = (xy^{-1})^s \delta$. Suppose the relation $\gamma^2 = \delta^{2s+1}$ holds then we have: $\delta = x^{1-s} y^{s-1}$, which never holds because δ is a long cycle, while $x^{1-s} y^{s-1}$ is not a long cycle. □

The following Corollary follows straightforward from Corollary 2 and from (4):

Corollary 4. *If n is an even number then the subgroup $\Gamma = \langle (\overline{0}\ \overline{1} \ldots \overline{n-1}) \rangle$ is not polinormal in S_n.*

Now we can prove the main Theorem 2.

PROOF of Theorem 2. Let $n > 4$. If $(n, \varphi(n)) = 1$ then by Palfy–Tyshkievich Theorem ([4], [5]) the subgroup Γ is pronormal, so it is also polinormal. Conversely if Γ is polinormal then by Corollary 2 it is weakly normal. Hence n is not divisible by 8 and by a square of an odd prime number. It means that it is enough to prove that n cannot be an even number, which follows from Corollary 4. □

References

[1] Z. I. Borevich and O. N. Macedońska, On the lattice of subgroups, *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* **103** (1980), 13–19, (LOMI) AN SSSR. (in *Russian*)

[2] P. W. Gawron, On weakly normal subgroups in symmetric groups, In book: XVII Vsesoyuznaya algebraicheskaja konferencya, *Tezisi, Part 1*, *Leningrad*, 1981, 35. (in *Russian*)

[3] K. H. Müller, Schwachnormale Untergruppen: eine gemeinzame Verallgemeinerung der normalen und normalizatorgleichen Untergruppen, *Rend. Semin. Mat. Univ. Padova* **36** N 1 (1966), 129–157.

[4] P. P. Palfy, On regular pronormal subgroups of symmetric groups, *Acta Math. Acad. Sci. Hungar.* **34** (1979), 287–292.

[5] R. I. Tyshkevich, Pronormal regular subgroups of a finite symmetric group, *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* **103** (1980), 132–139, (LOMI) AN SSSR. (in *Russian*)

PIOTR W. GAWRON
INSTITUTE OF MATHEMATICS
SILASIAN TECHNICAL UNIVERSITY
DENT KASZUBSKA 23
44–100 GLIWICE
POLAND

E-mail: pgawron@zeus.polsl.gliwice.pl

WITOLD TOMASZEWSKI
INSTITUTE OF MATHEMATICS
SILASIAN TECHNICAL UNIVERSITY
DENT KASZUBSKA 23
44–100 GLIWICE
POLAND

E-mail: wtomasz@zeus.polsl.gliwice.pl

(Received December 1, 1996; revised April 15, 1997)