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On the convergence of the iteration methods to a common fixed point for a pair of mappings

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Abstract. Let T and I be two compatible self-mappings of a closed convex subset C of a normed space X satisfying $I(C) \supseteq (1-k)I(C) + kT(C)$ for all $k \in [0,1]$ and $\|Tx - Ty\| \le \alpha\|Ix - Iy\| + \beta \max[\|Tx - Ix\|, \|Ty - Iy\|] + \gamma \max[\|Ix - Iy\|, \|Tx, -Ix\|, \|Ty - Iy\|]$, where $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma = 1$. Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be real sequences satisfying (i) $0 \le \alpha_n, \beta_n \le 1$, (ii) $\underline{\lim}\alpha_n > 0$, and (iii) $\overline{\lim}\beta_n < 1$. It is proved that if for some $x_0 \in C$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by $Iy_n = (1-\beta_n)Ix_n + \beta_n Tx_n$ and $Ix_{n+1} = (1-\alpha_n)Ix_n + \alpha_n Ty_n (n \ge 0)$ converges to a point z of C and if I is continuous at z then T and I have a unique common fixed point. Further if I is continuous at Tz then T and I have a unique common fixed point at which T is continuous. A similar theorem is proved for involutions of a pair of selfmaps.

Let T and I be two mappings of a metric space (X, d) into itself. Sessa [6] defined T and I to be weakly commuting if $d(TIx, ITx) \le d(Tx, Ix)$ for any $x \in X$. Clearly two commuting mappings weakly commute, but two weakly commuting mappings in general do not commute. Refer to Example 1 in Diviccaro et al. [1]. Gerald Jungck [3] defined T and I to be compatible mappings, if $d(Tx, Ix) \to 0$ implies $d(TIx, ITx) \to 0$. It can be seen that two weakly commuting mappings are compatible but the converse is not true. Examples supporting this fact can be found in [3].

In 1987, Diviccaro et al. [1], established the following result:

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Theorem A. *Let T and I be two weakly commuting mappings of a closed, convex subset C of a Banach space X into itself satisfying the inequality*

$$\text{(I)} \quad \|Tx - Ty\|^p \leq a\|Ix - Iy\|^p + (1-a)\max[\|Tx - Ix\|^p, \|Ty - Iy\|^p]$$

for all x, y in C where $0 < a < 1/2^{p-1}$ and $p \geq 1$. If I is linear, nonexpansive in C and such that $I(C)$ contains $T(C)$, then T and I have a unique common fixed point at which T is continuous.

Later, in 1991, Roy O. Davies [2], showed the following theorems:

Theorem B. *Let T and I be two self-mappings of a non-empty closed convex subset C of a Banach space X, satisfying the inequality*

$$\begin{aligned} \|Tx - Ty\| \leq \alpha\|Ix - Iy\| + \beta \max\left[\|Tx - Ix\|], \|Ty - Iy\|\right] \\ +\gamma \max\left[\|Ix - Iy\|, \|Tx - Ix\|, \|Ty - Iy\|\right] \end{aligned} \tag{1}$$

for all x, y in C where $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma = 1$. Further, let I weakly commute with T and let I be linear and nonexpansive in C. If $I(C)$ contains $T(C)$, then the equations $x = Ty = Iy$ have a unique solution for $x \in C$, and x is a common fixed point of T and I, at which T is continuous.

Theorem C. *Condition* (I) *with $0 < a < 1$ and $p \geq 1$ implies* (1) *for a certain triple $\alpha, \beta, \gamma > 0$ with $\alpha + \beta + \gamma = 1$.*

Thus Theorem B not only implies Theorem A, but also implies that the condition $0 < a < 1/2^{p-1}$ can be relaxed to $0 < a < 1$.

Recently, H. K. Pathak and R. George [5] established the following Theorem D which omits linearity and nonexpansiveness of the map I, and the proof of Theorem A is made under considerably weaker conditions on the mappings, i.e. replacing a weakly commuting pair of maps (T, I) with compatible maps, and using an iteration method of Mann type. Also the range of p has been extended to the case when $0 < p < 1$. He proved the following

Theorem D. *Let T and I be two compatible selfmaps of a closed convex bounded subset C of a normed space X such that $I(C) \supseteq (1 - k)I(C) + kT(C)$ where $0 < k < 1$ is fixed and satisfies* (I) *with $0 < a < 1$ and $p > 0$. If for some $x_0 \in X$, the sequence $\{x_n\}$ defined by*

$$Ix_{n+1} = (1-k)Ix_n + kIx_n, \qquad \forall n \geq 0$$

converges to a point z of C and if I is continuous at z then T and I have a unique common fixed point. Further if I is continuous at Tz then T and I have a unique common fixed point at which T is continous.

Following the basic ideas of Theorem D of H. K. Pathak and R. George [5] and drawing inspiration from Theorem B of Roy O. Davies [2], we shall use an iteration method of Ishikawa type to establish the following result:

Theorem 1. *Let T and I be two compatible self-mappings of a closed convex subset C of a normed space X such that*

$$I(C) \supseteq (1-k)I(C) + kT(C) \tag{2}$$

for all $k \in [0,1]$ and satisfying (1) with $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma = 1$.

Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be real sequences satisfying

$$\begin{aligned} &\text{(i)} && 0 \le \alpha_n,\ \beta_n \le 1, \\ &\text{(ii)} && \underline{\lim}\alpha_n > 0, \text{ and} \\ &\text{(iii)} && \overline{\lim}\beta_n < 1. \end{aligned}$$

If for some $x_0 \in C$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$Iy_n = (1-\beta_n)Ix_n + \beta_n Tx_n, \quad n \ge 0 \tag{3}$$

$$Ix_{n+1} = (1-\alpha_n)Ix_n + \alpha_n Ty_n, \quad n \ge 0 \tag{4}$$

converges to a point z of C and if I is continuous at z then T and I have a unique common fiexed point. Futher if I is continuous at Tz then T and I have a unique common fixed point at which T is continuous.

Proof. Since $\underline{\lim}\alpha_n > 0$ and $\overline{\lim}\beta_n < 1$, there exist $a > 0$, $b < 1$, and an integer $N \ge 1$ such that $a \le \alpha_n$ and $\beta_n \le b$ for all $n \ge N$. Hence, for $n \ge N$, we have

$$\|Ix_{n+1} - Ix_n\| = \alpha_n \|Ty_n - Ix_n\| \ge a\|Ty_n - Ix_n\|.$$

By hypothesis, this implies that $\|Ty_n - Ix_n\| \to 0$. Since $\|Ty_n - Iz\| \le \|Ty_n - Ix_n\| + \|Ix_n - Iz\|$, we have $\|Ty_n - Iz\| \to 0$. Observe that

$$\|Tx_n - Ix_n\| \le \|Tx_n - Ty_n\| + \|Ty_n - Ix_n\|, \tag{5}$$

$$\begin{aligned} \|Iy_n - Ix_n\| &= \beta_n \|Tx_n - Ix_n\| \qquad \text{(by (3))} \\ &\le \beta_n[\|Tx_n - Ty_n\| + \|Ty_n - Ix_n\|], \end{aligned} \tag{6}$$

and

$$\|Ty_n - Iy_n\| \leq (1-\beta_n)\|Ty_n - Ix_n\| + \beta_n\|Ty_n - Tx_n\| \quad \text{(by(3))}. \tag{7}$$

From (1), we have

$$\begin{aligned} \|Tx_n - Ty_n\| &\leq \alpha\|Ix_n - Iy_n\| \\ &\quad + \beta \max\left[\|Tx_n - Ix_n\|, \|Ty_n - Iy_n\|\right] \\ &\quad + \gamma \max\left[\|Ix_n - Iy_n\|, \|Tx_n - Ix_n\|, \|Ty_n - Iy_n\|\right]. \end{aligned} \tag{8}$$

Let $c_n := \|Tx_n - Ty_n\|$ and $d_n := \|Ty_n - Ix_n\|$. Introducing (5), (6) and (7) into (8), we obtain

$$\begin{aligned} c_n &\leq \alpha\beta_n(c_n + d_n) \\ &\quad + \beta \max\{c_n + d_n, (1-\beta_n)d_n + \beta_n c_n\} \\ &\quad + \gamma \max\{\beta_n(c_n + d_n), c_n + d_n, (1-\beta_n)d_n + \beta_n c_n\}. \end{aligned} \tag{9}$$

From (9), we consider the following cases:

Case 1. For all $n \geq N$, we have

$$c_n \leq \alpha\beta_n(c_n + d_n) + \beta(c_n + d_n) + \gamma\beta_n(c_n + d_n).$$

Since $1 - \alpha\beta_n - \beta - \gamma\beta_n = (1-\beta) - \beta_n(\alpha+\gamma) = (1-\beta) - \beta_n(1-\beta) = (1-\beta)(1-\beta_n) > (1-\beta)(1-b) > 0$, we have

$$c_n \leq \frac{1}{(1-\beta)(1-b)}\, d_n. \tag{10}$$

Case 2. For all $n \geq N$, we have

$$c_n \leq \alpha\beta_n(c_n + d_n) + \beta(c_n + d_n) + \gamma(c_n + d_n).$$

Now $1 - \alpha\beta_n - \beta - \gamma = \alpha - \alpha\beta_n = \alpha(1-\beta_n) > \alpha(1-b) > 0$. Thus

$$c_n \leq \frac{1}{\alpha(1-b)} d_n. \tag{11}$$

Case 3. For all $n \geq N$, we have

$$c_n \leq \alpha\beta_n(c_n + d_n) + \beta(c_n + d_n) + \gamma[(1-\beta_n)d_n + \beta_n c_n].$$

Now $1-\alpha\beta_n-\beta-\gamma\beta_n>(1-\beta)(1-b)>0$. Thus

$$c_n \le \frac{1}{(1-\beta)(1-b)}d_n. \tag{12}$$

Case 4. For all $n \ge N$, we have

$$c_n \le \alpha\beta_n(c_n+d_n)+\beta[(1-\beta_n)d_n+\beta_n c_n]+\gamma\beta_n(c_n+d_n).$$

Now $1-\alpha\beta_n-\beta\beta_n-\gamma\beta_n=1-\beta_n(\alpha+\beta+\gamma)=1-\beta_n>1-b>0$. Thus

$$c_n \le \frac{1}{1-b}d_n. \tag{13}$$

Case 5. For all $n \ge N$, we have

$$c_n \le \alpha\beta_n(c_n+d_n)+\beta[(1-\beta_n)d_n+\beta_n c_n]+\gamma(c_n+d_n).$$

Now $1-\alpha\beta_n-\beta\beta_n-\gamma=1-\gamma-\beta_n(\alpha+\beta)=1-\gamma-\beta_n(1-\gamma)=(1-\gamma)(1-\beta_n)>(1-\gamma)(1-b)>0$. Thus

$$c_n \le \frac{1}{(1-\gamma)(1-b)}d_n. \tag{14}$$

Case 6. For all $n \ge N$, we have

$$c_n \le \alpha\beta_n(c_n+d_n)+\beta[(1-\beta_n)d_n+\beta_n c_n]+\gamma[(1-\beta_n)d_n+\beta_n c_n].$$

Now $1-\alpha\beta_n-\beta\beta_n-\gamma\beta_n=1-\beta_n(\alpha+\beta+\gamma)=1-\beta_n>1-b>0$. Thus

$$c_n \le \frac{1}{1-b}d_n. \tag{15}$$

By (10)–(15), we obtain

$$c_n \le \max\left\{\frac{1}{(1-\beta)(1-b)},\ \frac{1}{\alpha(1-b)},\ \frac{1}{(1-\gamma)(1-b)}\right\}d_n. \tag{16}$$

Now $d_n:=\|Ty_n-Ix_n\|\to 0$. Thus $c_n:=\|Tx_n-Ty_n\|\to 0$ by (16). Also, $\|Tx_n-Iz\|\le\|Tx_n-Ty_n\|+\|Ty_n-Iz\|$, this implies that $\|Tx_n-Iz\|\to 0$, and hence $\|Tx_n - Ix_n\| \to 0$. In order to prove that $Tz = Iz$ we remark that

$$\begin{aligned} \|Iz - Tz\| &\leq \|Iz - Tx_n\| + \|Tx_n - Tz\| \\ &\leq \|Iz - Tx_n\| + \alpha\|Ix_n - Iz\| + \beta \max\left[\|Tx_n - Ix_n\|,\, \|Tz - Iz\|\right] \\ &\quad +\gamma \max\left[\|Ix_n - Iz\|,\, \|Tx_n - Ix_n\|,\, \|Tz - Iz\|\right] \qquad (\text{by}(1)). \end{aligned} \tag{17}$$

Taking the limit of (17) as $n \to \infty$ gives $\|Iz - Tz\| \leq (\beta + \gamma)\|Tz - Iz\|$, which implies that $Iz = Tz$.

From Jungck [4], T and I commute at the coincidence point, i.e., $TIz = ITz$. Hence, by using (1) we have

$$\begin{aligned} \|T^2z - Tz\| &\leq \alpha\|ITz - Iz\| + \beta \max\left[\|T^2z - ITz\|,\, \|Tz - Iz\|\right] \\ &\quad + \gamma \max\left[\|ITz - Iz\|,\, \|T^2z - ITz\|,\, \|Tz - Iz\|\right] \\ &= (\alpha + \gamma)\|T^2z - Tz\|, \end{aligned}$$

whence $T^2z = Tz$, i.e., Tz is a fixed point of T. Now $ITz = TIz = T^2z = Tz$, i.e. Tz is also a fixed point of I. To prove uniqueness, suppose that u is also a common fixed point of T and I. From (1) we have

$$\begin{aligned} \|u - Tz\| &= \|Tu - T^2z\| \\ &\leq \alpha\|Iu - ITz\| + \beta \max\left[\|Tu - Iu\|,\, \|T^2z - ITz\|\right] \\ &\quad + \gamma \max\left[\|Iu - ITz\|,\, \|Tu - Iu\|,\, \|T^2z - ITz\|\right] \\ &= (\alpha + \gamma)\|u - Tz\|. \end{aligned}$$

Thus $u = Tz$.

Finally, let $\{z_n\}$ be a sequence of points of C, with limit Tz. Observe that

$$\begin{aligned} \|Tz_n - Iz_n\| &\leq \|Tz_n - ITz\| + \|ITz - Iz_n\| \\ &= \|Tz_n - T^2z\| + \|ITz - Iz_n\| \end{aligned} \tag{18}$$

and (1) yields

$$\begin{aligned} \|Tz_n - T^2z\| &\leq \|\alpha Iz_n - ITz\| + \beta \max\left[\|Tz_n - Iz_n\|, \|T^2z - ITz\|\right] \\ &\quad +\gamma \max\left[\|Iz_n - ITz\|, \|Tz_n - Iz_n\|, \|T^2z - ITz\|\right] \\ &\leq \alpha\|Iz_n - ITz\| + (\beta + \gamma)\left[\|Tz_n - T^2z\| + \|ITz - Iz_n\|\right] \quad (\text{by}(18)). \end{aligned} \tag{19}$$

Thus $\|Tz_n - T^2z\| \le \frac{1}{\alpha}\|Iz_n - ITz\|$. Since I is continuous at Tz, we have $\|Tz_n - T^2z\| \to 0$ and this means that T is continuous at Tz.

Remark 1. In our Theorem 1, assume that I is continuous at a point of C instead of nonexpansive in C as in Theorem B, and the weakly commuting pair of maps (T, I) can be replaced by compatible maps. Also, the hypothesis of the linearity of I is not necessary in our result.

Theorem 2. *Let T and I be two compatible selfmappings of a closed convex subset C of a normed space X such that $I(C) \supseteq (1-k)I(C)+kT(C)$ where $0 < k < 1$ is fixed and satisfying (1) with $\alpha, \beta, \gamma > 0$ and $\alpha+\beta+\gamma=1$. If for some $x_0 \in C$, the sequence $\{x_n\}$ defined by*

$$Ix_{n+1} = (1-k)Ix_n + kTx_n, \qquad \forall_n \ge 0$$

converges to a point z of C and if I is continuous at z then T and I have a unique common fixed point. Further if I is continuous at Tz then T and I have a unique common fixed point at which T is continuous.

Proof. We have $\|Ix_{n+1} - Ix_n\| = k\|Tx_n - Ix_n\|$. Thus, by hypothesis, $\|Tx_n - Ix_n\| \to 0$, and hence $\|Tx_n - Iz\| \to 0$. Now, taking the limit of the following inequality as $n \to \infty$:

$$\begin{aligned} \|Iz - Tz\| &\le \|Iz - Tx_n\| + \|Tx_n - Tz\| \\ &\le \|Iz-Tx_n\|+\alpha\|Ix_n-Iz\|+\beta \max\left[\|Tx_n-Ix_n\|, \|Tz-Iz\|\right] \\ &\quad + \gamma \max\left[\|Ix_n - Iz\|, \|Tx_n - Ix_n\|, \|Tz - Iz\|\right] \end{aligned}$$

yields that $\|Iz - Tz\| \le (\beta+\gamma)\|Tz - Iz\|$. Then $Iz = Tz$. The sequel of the proof is the same as that of Theorem 1.

Remark 2. By Theorem C, Theorem 2 generalizes Theorem D for $p \ge 1$, and we note that the boundedness of C in Theorem D is not a necessary condition in the case $p \ge 1$. Also Corollary 2 in [5] is a special case of Theorem 2.

Assuming I to be the identity map of X in Theorem 1, we have the following

Corollary 1. *Let T be a self-mapping of a closed convex subset C of a normed space X satisfying*

$$\begin{aligned}\|Tx-Ty\| &\leq \alpha\|x-y\| + \beta \max\left[\|Tx-x\|, \|Ty-y\|\right] \\ &+\gamma \max\left[\|x-y\|,\ \|Tx-x\|,\ \|Ty-y\|\right]\end{aligned}$$

for all x, y in C, where $\alpha,\ \beta,\ \gamma > 0$ and $\alpha+\beta+\gamma = 1$.

Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be real sequences satisfying

$$\text{(i) } 0 \leq \alpha_n,\ \beta_n \leq 1,\ \text{(ii) } \underline{\lim}\alpha_n > 0,\ \text{and (iii) } \overline{\lim}\beta_n < 1.$$

If for some $x_0 \in C$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$y_n = (1-\beta_n)x_n + \beta_n T x_n, \qquad n \geq 0$$

and

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \qquad n \geq 0$$

converges to a point z in C then T has a unique fixed point at which T is continuous.

Remark 3. The case $p \geq 1$ of Corollary 1 in [5] is a special case of Corollary 1 by Theorem C.

In following result, assume that T and I are involutions instead of assuming that I is continuous as in Theorem 1.

Theorem 3. *Let T, I be selfmaps of a closed convex subset C of a normed space X satisfying*

(a) T and I are compatible,
(b) $T^2 = I^2 :=$ the identity map,
(c) T and I satisfy (1) with $\alpha, \beta, \gamma > 0$ and $\alpha+\beta+\gamma = 1$,
(d) $I(C) \supseteq (1-k)I(C) + kT(C)$ for all $k \in [0, 1]$.

Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be real sequences satisfying

$$\textit{(i) } 0 \leq \alpha_n,\ \beta_n \leq 1,\ \textit{(ii) } \underline{\lim}\alpha_n > 0,\ \textit{(iii) } \overline{\lim}\beta_n < 1.$$

If for some $x_0 \in C$, there is a sequence $\{x_n\}$ defined by

$$\begin{aligned} Iy_n &= (1-\beta_n)Ix_n + \beta_n T x_n, \qquad n \geq 0 \\ Ix_{n+1} &= (1-\alpha_n)Ix_n + \alpha_n T y_n, \qquad n \geq 0 \end{aligned}$$

and for which $\{Ix_n\}$ *converges to a point* u *of* C*, then* u *is a unique common fixed point of* T *and* I*.*

PROOF. As in the proof of Theorem 1, the conditions on $\{Ix_n\}$ imply that $\|Ty_n - Ix_n\| \to 0$. Also, $\|Ty_n - u\| \le \|Ty_n - Ix_n\| + \|Ix_n - u\|$. Then $\|Ty_n - u\| \to 0$. By (5)–(16), we have $\|Tx_n - Ty_n\| \to 0$. Thus $\|Tx_n - u\| \to 0$, and hence $\|Tx_n - Ix_n\| \to 0$. In (c) set $x = x_n$, $y = Iu$ to get

$$\begin{aligned}\|Tx_n - TIu\| &\le \alpha\|Ix_n - I^2u\| + \beta \max\left[\|Tx_n - Ix_n\|, \|TIu - I^2u\|\right]\\ &+\gamma \max\left[\|Ix_n - I^2u\|, \|Tx_n - Ix_n\|, \|TIu - I^2u\|\right].\end{aligned}$$

Taking the limit as $n \to \infty$ and using (b), one obtains $\|u - TIu\| \le (\beta+\gamma)\|TIu - u\|$, which implies that $u = TIu$. Thus $Tu = T^2Iu = Iu$, and p is a coincidence point for T and I. From JUNGCK [4] T and I commute at coincidence points.

Using (c) with $x = u$, $y = Tu$,

$$\begin{aligned}\|Tu - T^2u\| &\le \alpha\|Iu - ITu\| + \beta \max\left[\|Tu - Iu\|, \|T^2u - ITu\|\right]\\ &+\gamma \max\left[\|Iu - ITu\|, \|Tu - Iu\|, \|T^2u - ITu\|\right],\end{aligned}$$

thus $\|Tu - u\| \le (\alpha+\gamma)\|Tu - u\|$, and $u = Tu = Iu$. Uniqueness follows from (c).

References

[1] M. L. DIVICCARO, B. FISHER and S. SESSA, A common fixed point theorem of Gregus type, *Publ. Math. Debrecen* **34** (1987), 83–89.

[2] ROY O. DAVIES, Another version of a common fixed point theorem, *Publ. Math.* **38** (1991), 237–243.

[3] GERALD JUNGCK, Common fixed points for commuting and compatible maps on compacta, *Proc. Amer. Math. Soc.* **103** (3) (1988), 977–983.

[4] GERALD JUNGCK, Compatible mappings and common fixed points, *Internat. J. Math. and Math. Sci.* **9** (1986), 771–779.

[5] H. K. PATHAK and R. GEORGE, A common fixed point theorem of Gregus type for compatible mappings and its applications, *Publ. Math. Debrecen* **44** (1994), 205–213.

[6] S. SESSA, On a weak commutativity condition of mappings in fixed point considerations, *Publ. Inst. Math.* **36** (46) (1982), 149–153.

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