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A note on multiplicative functions with regularity properties

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J.L. Mauclaire and L. Murata [8] have shown that a multiplicative function $g(n)$ with properties

$$|g(n)| = 1 \quad (n = 1, 2, \dots) \tag{1}$$

and

$$\sum_{n \le x} |g(n+1) - g(n)| = o(x) \quad \text{as} \quad x \to \infty \tag{2}$$

has to be completely multiplicative. It is obvious that (1) and (2) hold for functions of the type

$$g(n) = n^{i\tau},$$

where τ is a real number. I. Kátai [6] conjectured that $g(n) = n^{i\tau}$ are the only multiplicative functions that satisfy the conditions (1) and (2). This conjecture remains open, some partial results are known. For such results we refer to A. Hildebrand [4], [5] and I. Kátai [7].

Our purpose in this note is to prove the following

Theorem. *Let A, B be positive integers and let C be a non-zero complex number. Assume that a complex-valued completely multiplicative function $g(n)$ satisfies the conditions*

$$|g(n)| = 1 \quad (n = 1, 2, \dots) \tag{3}$$

and

$$\sum_{n \le x} |g(An + B) - Cg(n)| = o(x) \quad \text{as} \quad x \to \infty. \tag{4}$$

If there is a positive integer k for which

$$\limsup_{x\to\infty} x^{-1}\left|\sum_{n\le x}(g(n))^k\right| > 0, \tag{5}$$

then there are a real constant τ and a completely multiplicative function $G(n)$ such that

$$g(n) = n^{i\tau}.G(n), \tag{6}$$

and

$$[G(n)]^k = 1 \tag{7}$$

hold for all positive integers n, moreover

$$\sum_{n\le x}|G(n+1)-G(n)| = o(x) \quad \text{as} \quad x\to\infty. \tag{8}$$

Remarks. (i) In the special case when $A=B=C=k=1$, our theorem can be deduced directly from Theorem 2 of A. Hildebrand [3]. In this case, by using Halász' theorem, it follows by (5) that for some real number τ

$$\operatorname{Re}\sum_{P}\frac{1-g(p)p^{-i\tau}}{p} < \infty, \tag{9}$$

the series being taken over all primes p. A. Hildebrand [3] proved that (9) implies

$$\frac{1}{x}\sum_{n\le x}\frac{g(n)}{g(n+1)} \to \prod_p F_p, \tag{10}$$

where

$$\mathbf{F}_p = 1-\frac{2}{p}+2\left(1-\frac{1}{p}\right)\ \operatorname{Re}\frac{g(p)p^{-i\tau}}{p-g(p)p^{-i\tau}}.$$

Thus, (3), (4), and (10) jointly imply that $\mathbf{F}_p = 1$ holds for each prime p, i.e.

$$g(p) = p^{i\tau}.$$

This shows that (6) holds with $G(n)\equiv 1$.

(ii) We hope that the conditions (3) and (4) imply (5), but we are unable to prove it presently. If we write a multiplicative function g satisfying (3) in the form $g = e^{2i\pi f}$, where f is an additive function, then it is known from Chapter 8 of [1] that there are two possibilities: Either (5) holds for some positive integer k or $f(n)$ is uniformly distributed (mod 1).

We shall use some lemmas in the proof of our theorem.

For a given multiplicative function $g(n)$ we denote by $\mathbf{J} = \mathbf{J}(g)$ the set of those pairs (Q, R) of positive integers for which

$$\sum_{n \le x} |g(Qn+R) - g(Qn)| = o(x) \quad \text{as} \quad x \to \infty. \tag{11}$$

Lemma 1. *Assume that a completely multiplicative function $g(n)$ satisfies the conditions (3) and (4). Then $(Q, R) \in \mathbf{J}(g)$ for all fixed integers Q and R which satisfy the condition*

$$0 < R < Q. \tag{12}$$

Proof. We shall prove this lemma by the same method that was used in the proof of Lemma 2 in [10].

Assume that a completely multiplicative function $g(n)$ satisfies the conditions (3) and (4). Then, by using Theorem 1 of [9] and the complete multiplicativity of g, we have

$$g(A) = C. \tag{13}$$

Thus, $(A, B) \in \mathbf{J} = \mathbf{J}(g)$, and so $(A, 1) \in \mathbf{J}$.

We prove next the following assertions:

(a) $(Q, 1) \in \mathbf{J}$ if $(q, 1) \in \mathbf{J}$ and $Q \ge q$

(b) $(Q, R) \in \mathbf{J}$ if $(q, 1) \in \mathbf{J}$ and $0 < R < Q/(q-1)$

(c) $(h, 1) \in \mathbf{J}$ if $(h+1, 1) \in \mathbf{J}$ and $h \ge 2$.

Assume that $(q, 1) \in \mathbf{J}$. By using the complete multiplicativity of g, we have

$$\begin{aligned} g[(q+1)n+1] - g[(q+1)n] = \frac{g(q+1)}{g(q)}\{g(qn+1) - g(qn)\} - \\ - \frac{1}{g(q)}\{g[q((q+1)n+1)+1] - g[q((q+1)n+1)]\}, \end{aligned}$$

and so, by using (3) and the fact $(q, 1) \in \mathbf{J}$, we deduce that $(q+1, 1) \in \mathbf{J}$. By using induction on q we have proved that (a) holds.

Assume again that $(q, 1) \in \mathbf{J}$. We shall prove (b) by using induction on R. From (a) it follows that (b) is satisfied for $R = 1$. Assume that $(Q, R) \in \mathbf{J}$ holds for all integers Q and R satisfying $0 < R < Q/(q-1)$ and $R < R_0$. Let Q_0 be an integer such that

$$0 < R_0 < Q_0/(q-1). \tag{14}$$

In order to show (b) it sufficies to prove that $(Q_0, R_0) \in \mathbf{J}$. Without loss of generality we may assume that $(Q_0, R_0) = 1$.

Let Q and R be positive integers such that

$$R_0 Q = Q_0 R + 1 \quad \text{and} \quad R < R_0. \tag{15}$$

It follows from (14) and (15) that

$$0 < R < (Q_0 R + 1)/Q_0 = R_0 Q/Q_0 < Q/(q-1).$$

Thus, by using our assumption and the fact $R < R_0$, we have $(Q, R) \in \mathbf{J}$. On the other hand, by (15), we get

$$\begin{aligned} g(Q_0 n + R_0) - g(Q_0 n) =& \frac{1}{g(Q)}[g(Q_0 Q n + R_0 Q) - g(Q) g(Q_0 n)] = \\ =& \frac{g(Q_0)}{g(Q)}\{g(Qn + R) - g(Qn)\} + \\ &+ \frac{1}{g(Q)}\{g[Q_0(Qn + R) + 1] - g[Q_0(Qn + R)]\}, \end{aligned}$$

consequently $(Q_0, R_0) \in \mathbf{J}$, because $(Q, R) \in \mathbf{J}$ and $(Q_0, 1) \in \mathbf{J}$. Thus, we have proved (b).

Finally, we prove (c). Assume that $(h+1, 1) \in \mathbf{J}$ and $h \geq 2$. Let

$$T(x) := \sum_{n \leq x} |g(hn + 1) - g(hn)|.$$

For each positive integer d with $0 \leq d \leq h - 1$, we can choose positive integers $Q = Q(d)$ and $R = R(d)$ such that

$$(hd + 1)Q = h^2 R + 1. \tag{16}$$

We have

$$\begin{aligned} T(x) =& \sum_{d=0}^{h-1} \sum_{hm+d \leq x} |g[h^2 m + hd + 1] - g[h(hm + d)]| = \\ =& \sum_{d=0}^{h-1} \sum_{hm+d \leq x} \Bigg| \frac{1}{g(Q)}\big\{g[h^2(Qm + R) + 1] - g[h^2(Qm + R)]\big\} + \\ &+ \frac{g(h)}{g(Q)}\big\{g[Q(hm + d) + hR - Qd] - g[Q(hm + d)]\big\} \Bigg|, \end{aligned}$$

and so $T(x) = o(x)$ if $hR - Qd = 0$, because, by using (a), $(h+1, 1) \in \mathbf{J}$ and $h \geq 2$ imply that $(h^2, 1) \in \mathbf{J}$. If $hR - Qd \neq 0$, then we obtain from (16) that

$$0 < hR - Qd = (Q - 1)/h < Q/h,$$

which, by applying (b) with $q = h+1$, implies that $(Q, hR-Qd) \in \mathbf{J}$. This, with $(h^2, 1) \in \mathbf{J}$ shows that $T(x) = o(x)$, i.e. $(h, 1) \in \mathbf{J}$. This completes the proof of (c).

Now we prove Lemma 1.

As we have seen above, $(A, 1) \in \mathbf{J}$. If $A = 1$, then the assertion of Lemma 1 holds. If $A \geq 2$, then by using (c) one can deduce that $(2, 1) \in \mathbf{J}$, and so by applying (b) with $q = 2$, it follows that $(Q, R) \in \mathbf{J}$ for all integers Q and R which satisfy (12). This completes the proof of Lemma 1.

Lemma 2. *Assume that a completely multiplicative function $g(n)$ satisfies the conditions (3) and (4). Then for each positive integer κ, we have*

$$\sum_{n \leq x} |[g(n+1)]^\kappa - [g(n)]^\kappa| = o(x) \quad \text{as} \quad x \to \infty.$$

Proof. We first consider the case $\kappa = 1$.

Let $Q \geq 2$ be a fixed positive integer. For each integer $\gamma \geq 0$ let

$$\mathcal{B}_\gamma = \{n \in \mathbf{N} \mid Q^\gamma \parallel (n+1)\}$$

and

$$S_\gamma(x) := x^{-1} \sum_{\substack{n \leq x \\ n \in \mathcal{B}_\gamma}} |g(n+1) - g(n)|.$$

By using the conditions (3) and (4), one can get from Lemma 1 that

$$S_0(x) = x^{-1} \sum_{\substack{n \leq x \\ n \in \mathcal{B}_0}} |g(n+1) - g(n)| = o(1) \quad \text{as} \quad x \to \infty. \tag{17}$$

Thus, by using (17) and Lemma 1, it follows that

$$\begin{aligned} S_\gamma(x) := x^{-1} &\sum_{\substack{n \leq x \\ n \in \mathcal{B}_\gamma}} |g(n+1) - g(n)| = \\ &= x^{-1} \sum_{\substack{m+1 \leq (x+1)/Q^\gamma \\ m \in \mathcal{B}_0}} \left| g(Q^\gamma) g(m+1) - g(Q^\gamma m + Q^\gamma - 1) \right| = \\ &= x^{-1} \sum_{\substack{m+1 \leq (x+1)/Q^\gamma \\ m \in \mathcal{B}_0}} \Big| g(Q^\gamma) \left[g(m+1) - g(m) \right] - \\ &\quad - \left[g(Q^\gamma m + Q^\gamma - 1) - g(Q^\gamma m) \right] \Big| \\ &= o(1). \end{aligned} \tag{18}$$

The relations (17) and (18) together with (3) imply that for each positive integer M, we have

$$x^{-1}\sum_{n\le x}|g(n+1)-g(n)|\le S_0(x)+\sum_{1\le j\le M}S_j(x)+x^{-1}\sum_{\substack{n\le x\\ Q^M|(n+1)}}2\le$$

$$\le o(M+1)+\frac{2}{Q^M},$$

and so

$$\limsup_{x\to\infty}x^{-1}\sum_{n\le x}|g(n+1)-g(n)|<<Q^{-M}.$$

This with $M\to\infty$ shows that

$$x^{-1}\sum_{n\le x}|g(n+1)-g(n)|=o(1),$$

which proves Lemma 2 in the case $\kappa=1$.

Now let $\kappa>1$ be an integer. By using the relation

$$x^{\kappa+1}-y^{\kappa+1}=x(x^{\kappa}-y^{\kappa})+y^{\kappa}(x-y),$$

it is easily shown that

$$\mathbf{J}(g)\subseteq\mathbf{J}(g^{\kappa})\quad(\kappa=1,2,\dots).$$

Thus, Lemma 2 is a consequence of the above relation and the fact $(1,1)\in\mathbf{J}(g)$. Lemma 2 is proved.

Lemma 3. *Let $f(n)$ be a multiplicative function which satisfies $|f(n)|\le 1$. Let $1\le w_0\le x$. Then there is a real number t, $|t|<(\log x)^{1/19}$, so that*

$$\sum_{n\le x/w}f(n)=w^{-1-it}\sum_{n\le x}f(n)+O\left[\frac{x}{w}\left(\frac{\log 2w_0}{\log x}\right)^{1/19}\right]$$

uniformly for $1\le w\le w_0$. If f is real-valued, then we may set $t=0$. The implied constant is absolute.

Proof. This is Theorem 1 of Elliott [2].

Proof of the theorem. Assume that a completely multiplicative function $g(n)$ satisfies the conditions (3), (4) and (5) for some positive integers A, B, k and a non-zero complex number C. Let

$$f(n):=[g(n)]^k\quad(n=1,2,\dots).$$

It is obvious that $f(n)$ is a completely multiplicative function, $f(n)$ satisfies (3), furthermore by applying Lemma 2 with $\kappa = k$ it follows that $(1,1) \in \mathbf{J}(f)$. This shows that $(Q,R) \in \mathbf{J}(f)$ holds for all positive integers Q and R. Let

$$S(x) := \sum_{n \le x} f(n).$$

Let w_0 be a sufficiently large real number. For each $x > w_0$, by applying Lemma 3, there is a real number $t(x)$ satisfying $|t(x)| \le (\log x)^{1/19}$ such that for $1 \le Q \le w_0$ we have

$$\sum_{m \le x/Q} f(m) = Q^{-1-it(x)} S(x) + O\left[\frac{x}{Q}\left(\frac{\log 2w_0}{\log x}\right)^{1/19}\right].$$

From this, we have

$$\tag{19} \begin{aligned} \sum_{\substack{n \le x \\ n \equiv 0 (\mathrm{mod}\, Q)}} f(n) = f(Q) \sum_{m \le x/Q} f(m) = \\ = Q^{-1-it(x)} f(Q) S(x) + O\left[\frac{x}{Q} f(Q)\left(\frac{\log 2w_0}{\log x}\right)^{1/19}\right]. \end{aligned}$$

By using (19) and the fact $(Q,R) \in \mathbf{J}(f)$ for all integers Q and R, we deduce that

$$\begin{aligned} \sum_{\substack{n \le x \\ n \equiv R (\mathrm{mod}\, Q)}} f(n) = \sum_{\substack{n \le x \\ n \equiv R (\mathrm{mod}\, Q)}} [f(n) - f(n-R)] + \sum_{\substack{m \le x-R \\ m \equiv 0 (\mathrm{mod}\, Q)}} f(m) = \\ = Q^{-1-it(x)} f(Q) S(x) + O\left[\frac{x}{Q} f(Q)\left(\frac{\log 2w_0}{\log x}\right)^{1/19}\right] + o(x) \end{aligned}$$

holds for each $R = 0, \dots, Q-1$. Thus by adding the above relations, we get

$$\tag{20} S(x) = Q^{-it(x)} f(Q) S(x) + O\left[x f(Q)\left(\frac{\log 2w_0}{\log x}\right)^{1/19}\right] + o(Qx).$$

By the condition (5), we can choose $D > 0$ and a sequence $\{x_i\}_{i=1}^{\infty}$, $x_i \to \infty$ such that

$$\left|\frac{S(x_i)}{x_i}\right| \ge D > 0 \quad \text{as} \quad x_i \to \infty.$$

Then (20) gives

$$D\left|1-\frac{f(Q)}{Q^{it(x_i)}}\right| \le \left|1-\frac{f(Q)}{Q^{it(x_i)}}\right| \cdot \left|\frac{S(x_i)}{x_i}\right| = o(1)$$

and so

$$Q^{it(x_i)} \to f(Q) \quad \text{as} \quad x_i \to \infty. \tag{21}$$

Since (21) holds for all integers Q for which $1 \le Q \le w_0$, and for each Q we get from (21) that

$$t(x_i) \to t \quad \text{as} \quad x_i \to \infty, \tag{22}$$

thus (21) and (22) imply

$$f(Q) = Q^{it} \tag{23}$$

for all $1 \le Q \le w_0$. This with $w_0 \to \infty$ shows that (23) holds for all positive integers Q.

Since

$$f(n) = [g(n)]^k \quad \text{and} \quad f(n) = n^{it} \quad (n = 1, 2, \dots),$$

it follows that for each positive integer n there exists a complex number $G(n)$ such that

$$g(n) = n^{it/k}.G(n). \tag{24}$$

It is obvious that $G(n)$ is a completely multiplicative function and

$$[G(n)]^k = 1 \quad (n = 1, 2, \dots).$$

Let $\tau := t/k$. By (24) we have

$$\begin{aligned} G(An+B) - G(An) = &\frac{g(An+B) - g(An)}{(An)^{i\tau}} - \\ &- G(An+B)\frac{(An+B)^{i\tau} - (An)^{i\tau}}{(An)^{i\tau}} \end{aligned}$$

which with (4) implies that

$$\sum_{n \le x} |G(An+B) - G(An)| = o(x) \quad \text{as} \quad x \to \infty.$$

By using Lemma 2 with $g(n)$ replaced by $G(n)$, the last relation implies (8). This completes the proof of our theorem.

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