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An analytic proof of the Lévy–Khinchin formula on $\mathbb{R}^n$

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Abstract. We give a proof of the Lévy–Khinchin formula using only some parts of the theory of distributions and Fourier analysis, but without using probability theory.

The Lévy–Khinchin formula says that every continuous and negative definite function $\psi : \mathbb{R}^n \to \mathbb{C}$ has the following representation

$$
\begin{aligned}
(1.1) \qquad \psi(\xi) &= c + id\xi + q(\xi) \\
&\quad + \int_{\mathbb{R}^n \setminus \{0\}} \left(1 - e^{-ix\xi} - \frac{ix\xi}{1+\|x\|^2}\right) \frac{1+\|x\|^2}{\|x\|^2}\, \mu(dx),
\end{aligned}
$$

with a positive constant $c \geq 0$, a vector $d \in \mathbb{R}^n$, a symmetric positive semidefinite quadratic form q, and a finite Borel measure μ on $\mathbb{R}^n \setminus \{0\}$. Every continuous negative definite ψ is uniquely determined by (c, d, q, μ) and any such quadruplet defines via (1.1) a continuous negative definite function.

The importance of continuous negative definite functions and the Lévy–Khinchin formula rests in their applications in the theory of limit theorems for independent and identically distributed random variables, cf. the monograph by B.W. Gnedenko and A.N. Kolmogorov [8] or the more recent book by V.V. Petrov [14]. Moreover, S. Bochner, cf. [2], showed that convolution semigroups of probability measures are characterized by negative definite functions. The Lévy–Khinchin formula can

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also be used to decompose a given Lévy process into simpler processes, see e.g. [2] or the textbook by L. Breiman [3].

Although there is many a textbook or monograph that gives a more or less elementary probabilistic proof of the Lévy–Khinchin formula in one space dimension, it is rather difficult to find a proper reference for higher dimensions; an exception is e.g. [6]. In the early sixties several purely analytic proofs of the Lévy–Khinchin representation for $\psi : \mathbb{R}^n \to \mathbb{C}$ were given. A proof by Ph. Courrège [4] used quite hard analytic results on certain integro-differential operators; in [16] M. Rogalski introduced the *radial limit method* which he attributes to A. Beurling. There are other proofs which employ extreme-point methods and Choquet theory, e.g. [13]. One should also mention the proof in M. Reed and B. Simon [15] which is but a bit sketchy and partly hidden in exercises. The growing interest in harmonic analysis and probability theory on (locally compact Abelian) groups led to further proofs, for example the one given by Kh. Harzallah [9] for real-valued continuous negative definite functions which was subsequently generalized to the complex case by G. Forst [7]. In the monograph [10] by H. Heyer these results are discussed rather comprehensively. In particular, he pointed out the fundamental strategy of all of these approaches: every continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{C}$ can be uniquely related with a convolution semigroup of sub-probability measures by Bochner's theorem. This, in turn, induces a semigroup of convolution operators on suitable function spaces. The Lévy–Khinchin formula is then derived using both certain representations for the infinitesimal generator of the operator semigroup and more or less complicated limiting procedures.

The proof we are going to present here follows exactly these lines. The new point of view, however, is to use some results of the theory of distributions (in the sense of L. Schwartz). The idea for this approach grew out of a lecture-course on *Pseudo-differential operators generating Feller processes* given by the first-named author, but it was the second author who brought some first attempts to a successful end. That such a proof was really necessary came up to us when lecturing an audience which was well-trained in analysis – especially Fourier analysis and the theory of partial differential equations – but did almost completely lack any background in probability or advanced potential theory. We hope that our approach is easily accessible to such an audience.

Due to a result of Ph. Courrège [5] an operator

$$-p(x,D)u(x) = -(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\xi}p(x,\xi)\widehat{u}(\xi)\,d\xi \tag{1.2}$$

with a continuous function $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ such that $p(x,\bullet) : \mathbb{R}^n \to \mathbb{C}$ is negative definite for all $x \in \mathbb{R}^n$, satisfies the positive maximum principle on $C_0^\infty(\mathbb{R}^n)$. By the Hille–Yosida–Ray theorem, $-p(x,D)$ is, therefore, a candidate for a pre-generator of a Feller semigroup, hence of a Feller process. In many cases it is indeed possible to construct a Feller semigroup by starting with (1.2), cf. [12] for a non-technical introduction. In order to treat these operators, it is necessary to develop some distribution theory and Fourier analysis well-known in the theory of partial differential equations, see L. Hörmander [11] as a good reference. On the other hand, the study of the semigroup and the associated stochastic process requires a good deal of the Fourier analysis used in probability theory, cf. C. Berg and G. Forst [1]. It is therefore natural, once we have both parts of Fourier analysis at our disposal, to combine them and try to prove the Lévy–Khinchin formula.

Our proof is given in Section 5. The other sections are added to fix some notations but also to make the paper self-contained for the readers' convenience. Let us finally remark that our proof is elementary in the sense that it only requires a small portion of the above mentioned distribution theory (only basic knowledge of tempered distributions) and Fourier analysis (here: Lévy's continuity theorem).

2. Negative definite functions and convolution semigroups

In this section we will collect some well-known facts on (continuous) negative definite functions. All results and their proofs can be found in the book by C. Berg and G. Forst [1].

Definition 2.1. A function $\psi : \mathbb{R}^n \to \mathbb{C}$ is called *negative definite* if for all $m \in \mathbb{N}$ and all $\xi^1, \dots, \xi^m \in \mathbb{R}^n$ the matrix $\big(\psi(\xi^k) + \overline{\psi(\xi^\ell)} - \psi(\xi^k - \xi^\ell)\big)_{k,\ell=1}^m$ is positive Hermitian.

It is easy to see that a function $\psi : \mathbb{R}^n \to \mathbb{C}$ is negative definite if and only if $\psi(0) \geq 0$, $\psi(\xi) = \overline{\psi(-\xi)}$, and if for all $m \in \mathbb{N}$, $\xi^1, \dots, \xi^m \in \mathbb{R}^n$, and $z_1, \ldots, z_m \in \mathbb{C}$ such that $\sum_{k=1}^m z_k = 0$

$$\sum_{k,\ell=1}^{m} \psi(\xi^k - \xi^\ell) z_k \bar{z}_\ell \leq 0 \tag{2.1}$$

holds. We will need some further properties of negative definite functions.

Lemma 2.2. *Let $\psi : \mathbb{R}^n \to \mathbb{C}$ be a negative definite function. Then $\psi(0) \geq 0$ and for all $\xi \in \mathbb{R}^n$ we have $\operatorname{Re}\psi(\xi) \geq 0$ and $\psi(\xi) = \overline{\psi(-\xi)}$. If ψ is continuous,*

$$|\psi(\xi)| \leq c_\psi (1 + |\xi|^2), \quad \xi \in \mathbb{R}^n, \tag{2.2}$$

holds with some constant $c_\psi > 0$.

The following characterization of negative definite functions is due to J.L. Schoenberg:

Theorem 2.3. *A function $\psi : \mathbb{R}^n \to \mathbb{C}$ is negative definite if and only if $\psi(0) \geq 0$ and if for all $t > 0$ the function*

$$\mathbb{R}^n \ni \xi \mapsto (2\pi)^{-n/2}\, e^{-t\psi(\xi)} \tag{2.3}$$

is positive definite.

Remark 2.4. *(A)* Recall that a function $f : \mathbb{R}^n \to \mathbb{C}$ is *positive definite* if for all $m \in \mathbb{N}$ and $\xi^1, \ldots, \xi^m \in \mathbb{R}^n$ the matrix $\big(f(\xi^k - \xi^\ell)\big)_{k,\ell=1}^m$ is positive definite.

(B) The normalizing factor $(2\pi)^{-n/2}$ in (2.3) is added just for convenience and is of no further relevance.

It is an important consequence of Schoenberg's theorem that we can establish a relation between continuous negative definite functions and convolution semigroups on $\mathbb{R}^n$.

Definition 2.5. Let $\{\mu_t\}_{t\geq 0}$ be a family of sub-probability measures on $\mathbb{R}^n$, i.e. μ_t is a Borel measure on $\mathbb{R}^n$ with total mass $\mu_t(R^n) \leq 1$. We call $\{\mu_t\}_{t\geq 0}$ a *convolution semigroup* on $\mathbb{R}^n$ if $\mu_t \star \mu_s = \mu_{t+s}$ for all $s, t \geq 0$ and if $\lim_{t\to 0} \mu_t = \mu_0 = \epsilon_0$ (in the sense of vague convergence of measures) are satisfied.

Here, ϵ_0 denotes the Dirac measure (unit mass) at the origin. The *Fourier transform* of any finite Borel measure μ is given by

$$\widehat{\mu}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi}\, \mu(dx), \qquad \xi \in \mathbb{R}^n. \tag{2.4}$$

Theorem 2.6. *Let $\psi : \mathbb{R}^n \to \mathbb{C}$ be a continuous negative definite function. Then there exists a unique convolution semigroup $\{\mu_t\}_{t\geq 0}$ on $\mathbb{R}^n$ such that*

$$\widehat{\mu}_t(\xi) = (2\pi)^{-n/2}\, e^{-t\psi(\xi)}, \quad t \geq 0,\ \xi \in \mathbb{R}^n \tag{2.5}$$

holds. Conversely, for any convolution semigroup $\{\mu_t\}_{t\geq 0}$ on $\mathbb{R}^n$ there is a unique continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{C}$ satisfying (2.5).

In the definition of convolution semigroups we used already the notion of vague convergence of measures. We say that a net $\{\mu_i\}_{i\in I}$ of Borel measures on $\mathbb{R}^n$ *converges vaguely* to a Borel measure μ if for all $\phi \in C_0(\mathbb{R}^n)$, i.e. all continuous and compactly supported function,

$$\lim_{i\in I} \int_{\mathbb{R}^n} \phi(x)\, \mu_i(dx) = \int_{\mathbb{R}^n} \phi(x)\, \mu(dx). \tag{2.6}$$

A net of bounded Borel measures $\{\mu_i\}_{i\in I}$ on $\mathbb{R}^n$ is said to *converge weakly* to a bounded Borel measure μ if (2.6) holds for all continuous and bounded functions $\phi \in C_b(\mathbb{R}^n)$. It is well-known that for any convolution semigroup $\{\mu_t\}_{t\geq 0}$

$$\lim_{t\to 0} \mu_t = \epsilon_0 \quad \text{weakly}$$

and not only vaguely.

We shall also need some results on how the Fourier transform acts on bounded measures. First of all, let us recall *Bochner's theorem*.

Theorem 2.7. *A continuous function $u : \mathbb{R}^n \to \mathbb{C}$ is positive definite if and only if there exists a bounded measure μ on $\mathbb{R}^n$ such that*

$$u(\xi) = \widehat{\mu}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi}\, \mu(dx). \tag{2.8}$$

Theorem 2.8. *The Fourier transform is a homeomorphism between the space of bounded measures (equipped with the weak topology) and the set of continuous positive definite functions (equipped with the topology of uniform convergence on compact sets). In particular, weakly converging nets of measures are mapped onto locally uniformly converging nets of continuous positive definite functions and vice versa.*

3. Tempered distributions

Let us recall some results from the theory of distributions which we shall need later on. We shall follow the presentation of L. Hörmander [11].

The *Schwartz space* $\mathcal{S}(\mathbb{R}^n)$ consists of all arbitrarily often differentiable functions $\phi : \mathbb{R}^n \to \mathbb{C}$ such that

$$p_{\alpha,\beta}(\phi) := \sup_{x\in\mathbb{R}^n} |x^\beta \partial^\alpha \phi(x)| \tag{3.1}$$

is finite for all $\alpha, \beta \in \mathbb{N}_0^n$. The family $\{p_{\alpha,\beta}\}_{\alpha,\beta\in\mathbb{N}_0^n}$ is a family of seminorms that turns $\mathcal{S}(\mathbb{R}^n)$ into a Fréchet space. On $\mathcal{S}(\mathbb{R}^n)$ we define the Fourier transform by

$$\mathcal{F}\phi(\xi) = \widehat{\phi}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} \phi(x)\, dx. \tag{3.2}$$

It is well-known that the Fourier transform is a bijective and bicontinuous mapping of $\mathcal{S}(\mathbb{R}^n)$ onto itself and that its inverse is given by

$$\mathcal{F}^{-1}\phi(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi} \phi(x)\, dx. \tag{3.3}$$

In particular, we have $\mathcal{F}^2\phi(x) = \phi(-x)$ and $\mathcal{F}^4 = \mathrm{id}$, thus $\mathcal{F}^{-1} = \mathcal{F}^3$. For $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$ we have Parseval's identity

$$\int_{\mathbb{R}^n} \phi(x)\overline{\psi(x)}\, dx = \int_{\mathbb{R}^n} \widehat{\phi}(\xi)\overline{\widehat{\psi}(\xi)}\, d\xi \tag{3.4}$$

and the convolution theorem holds

$$(\phi \star \psi)\widehat{\ }(\xi) = (2\pi)^{n/2}\widehat{\phi}(\xi)\widehat{\psi}(\xi). \tag{3.5}$$

In addition, we have the formula

$$\xi^\beta(-i\partial_\xi)^\alpha\widehat{\phi}(\xi) = (-1)^{|\alpha|}\big((-i\partial_x)^\beta(x^\alpha\phi(\bullet))\big)\widehat{\ }(\xi). \tag{3.6}$$

Definition 3.1. The space $\mathcal{S}'(\mathbb{R}^n)$ of all *tempered distributions* is the space of all continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$.

It is easy to see that $L^p(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ for all $1 \le p \le \infty$ and that all bounded measures and all measurable functions which grow at most polynomially define elements in $\mathcal{S}'(\mathbb{R}^n)$. For a bounded measure μ we have

$$\langle \mu, \phi \rangle := \int_{\mathbb{R}^n} \phi(x)\, \mu(dx), \quad \phi \in \mathcal{S}(\mathbb{R}^n). \tag{3.7}$$

By duality we can extend several operations onto $\mathcal{S}'(\mathbb{R}^n)$:

$$\langle \partial^\alpha u, \phi \rangle = (-1)^{|\alpha|} \langle u, \partial^\alpha \phi \rangle\,; \tag{3.8}$$

$$\langle \widehat{u}, \phi \rangle = \left\langle u, \widehat{\phi} \right\rangle; \tag{3.9}$$

$$\langle g u, \phi \rangle = \langle u, g\phi \rangle; \tag{3.10}$$

where $u \in \mathcal{S}'(\mathbb{R}^n)$, $\phi \in \mathcal{S}(\mathbb{R}^n)$, and g is a C^∞-function which is together with all of its partial derivatives polynomially bounded. Moreover, we set

$$(u \star \phi)(x) = \langle u, \phi(x - \bullet) \rangle, \quad x \in \mathbb{R}^n. \tag{3.1}$$

Clearly, these definitions extend the earlier definitions. In particular, the Fourier transform of a bounded measure μ as in (2.4) coincides with (3.9). For the Dirac measure ϵ_0 we find

$$\widehat{\epsilon_0} = (2\pi)^{-n/2} \quad \text{and} \quad \widehat{1} = (2\pi)^{n/2} \epsilon_0. \tag{3.12}$$

Moreover, (3.5), (3.6) remain true in $\mathcal{S}'(\mathbb{R}^n)$ with an appropriate interpretation, and (3.4) extends to $\phi, \psi \in L^2(\mathbb{R}^n)$ as well as to $f \in \left\{ g \in L^1(\mathbb{R}^n) : \widehat{g} \in L^1(\mathbb{R}^n) \right\}$ and bounded measures μ, i.e. we have

$$\int_{\mathbb{R}^n} f(x)\, \mu(dx) = \int_{\mathbb{R}^n} \widehat{f}(\xi) \widehat{\mu}(-\xi)\, d\xi = \int_{\mathbb{R}^n} \widehat{f}(\xi) \overline{\widehat{\mu}(\xi)}\, d\xi \tag{3.13}$$

since $\widehat{\mu}(-\xi) = \overline{\widehat{\mu}(\xi)}$.

The *support* of $u \in \mathcal{S}'(\mathbb{R}^n)$ is the complement of all open sets $\Omega \subset \mathbb{R}^n$ where $u\big|_{C_0^\infty(\Omega)} \equiv 0$ holds. Here, $C_0^\infty(\Omega)$ are the test functions, that are the arbitrarily often differentiable, compactly supported (in Ω) functions.

Theorem 3.2. *Suppose that* $u \in \mathcal{S}'(\mathbb{R}^n)$ *and* $\operatorname{supp} u = \{0\}$. *Then there exists a number* $m \in \mathbb{N}_0$ *such that*

$$u = \sum_{|\alpha| \le m} c_\alpha \partial^\alpha \epsilon_0 \tag{3.14}$$

is valid with some constants $c_\alpha \in \mathbb{C}$.

Let $u \in \mathcal{S}'(\mathbb{R}^n)$ be such that $\widehat{u} = c_0\epsilon_0 + \sum_{j=1}^n c_j \frac{\partial}{\partial x_j}\epsilon_0$. Then (3.6) and (3.12) imply

$$u(x) = \tilde{c}_0 + \sum_{j=1}^n \tilde{c}_j x_j. \tag{3.15}$$

Later on, we will consider the set

$$\begin{aligned} \mathcal{S}_0(\mathbb{R}^n) &:= \Big\{ v \in \mathcal{S}(\mathbb{R}^n) : \widehat{v}(0) = \frac{\partial}{\partial \xi_j}\widehat{v}(0) = 0,\ 1 \le j \le n \Big\} \\ &= \Big\{ \widehat{v} \in \mathcal{S}(\mathbb{R}^n) : v(0) = \frac{\partial}{\partial \xi_j} v(0) = 0,\ 1 \le j \le n \Big\}. \end{aligned} \tag{3.16}$$

Using (3.6) we find for any $v \in \mathcal{S}_0(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} v(x)\,dx = \int_{\mathbb{R}^n} x_j v(x)\,dx = 0, \quad 1 \le j \le n. \tag{3.17}$$

Lemma 3.3. *Let $f, g : \mathbb{R}^n \to \mathbb{C}$ be two measurable functions which are polynomially bounded. If for all $\phi \in \mathcal{S}_0(\mathbb{R}^n)$*

$$\int_{\mathbb{R}^n} f(x)\phi(x)\,dx = \int_{\mathbb{R}^n} g(x)\phi(x)\,dx, \tag{3.18}$$

then there exist a constant $c \in \mathbb{C}$ and a vector $d \in \mathbb{C}^n$ such that

$$f(x) = c + d \cdot x + g(x). \tag{3.19}$$

Proof. Clearly, both f and g induce elements of $\mathcal{S}'(\mathbb{R}^n)$, thus (3.18) implies

$$\langle f, \mathcal{F}^2\phi\rangle = \langle g, \mathcal{F}^2\phi\rangle \tag{3.20}$$

and

$$\langle \widehat{f}, \widehat{\phi}\rangle = \langle \widehat{g}, \widehat{\phi}\rangle \tag{3.21}$$

for all $\phi \in S_0(\mathbb{R}^n)$. Since $\mathcal{S}_0(\mathbb{R}^n)$ is rich enough we find $\operatorname{supp}(\widehat{f}-\widehat{g}) \subset \{0\}$, implying

$$\widehat{f} - \widehat{g} = \sum_{|\alpha|\leq m} c_\alpha \partial^\alpha \epsilon_0. \tag{3.22}$$

Since for $|\alpha| \geq 2$ there are $\phi \in \mathcal{S}_0(\mathbb{R}^n)$ with $\partial^\alpha\phi(0) \neq 0$, we have necessarily $m < 2$ in (3.22) and the assertion follows from (3.15). $\square$

4. Convolution semigroups and Feller semigroups

Let $\{\mu_t\}_{t\geq 0}$ be a convolution semigroup on $\mathbb{R}^n$. For any measurable and bounded function u, that is $u \in B_b(\mathbb{R}^n)$, we can define the operator

$$T_t u(x) = \int_{\mathbb{R}^n} u(x-y)\,\mu_t(dy), \quad x \in \mathbb{R}^n. \tag{4.1}$$

Clearly, $T_t u$ is again in $B_b(\mathbb{R}^n)$ and the family $\{T_t\}_{t\geq 0}$ is a one-parameter semigroup of linear operators on $B_b(\mathbb{R}^n)$, i.e.

$$T_{t+s} = T_t \circ T_s \quad \text{and} \quad T_0 = \mathrm{id}. \tag{4.2}$$

Moreover, $\{T_t\}_{t\geq 0}$ is a Feller semigroup in the following sense:

Definition 4.1. Let $\{T_t\}_{t\geq 0}$ be a family of operators acting on the continuous functions vanishing at infinity, $T_t : C_\infty(\mathbb{R}^n) \to C_\infty(\mathbb{R}^n)$, such that $T_{t+s} = T_t \circ T_s$, and $T_0 = \mathrm{id}$, $\lim_{t\to 0} \|T_t u - u\|_\infty = 0$, and $0 \leq T_t u \leq 1$ whenever $0 \leq u \leq 1$ are fulfilled. Then $\{T_t\}_{t\geq 0}$ is said to be a *Feller semigroup*.

Since $\mathcal{S}(\mathbb{R}^n) \subset C_\infty(\mathbb{R}^n)$ we find for $u \in \mathcal{S}(\mathbb{R}^n)$

$$T_t u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi} e^{-t\psi(\xi)} \widehat{u}(\xi)\, d\xi, \quad x \in \mathbb{R}^n, \tag{4.3}$$

where $\psi : \mathbb{R}^n \to \mathbb{C}$ is the continuous negative definite function associated with the convolution semigroup $\{\mu_t\}_{t\geq 0}$

$$\widehat{\mu}_t(\xi) = (2\pi)^{-n/2} e^{-t\psi(\xi)}, \quad \xi \in \mathbb{R}^n. \tag{4.4}$$

Let $(A, D(A))$ denote the infinitesimal generator of the semigroup $\{T_t\}_{t\geq 0}$,

$$Au = \lim_{t\to 0} \frac{T_t u - u}{t}, \tag{4.5}$$

$$D(A) = \big\{u \in C_\infty(\mathbb{R}^n) : \text{the limit (4.5) exists in } C_\infty(\mathbb{R}^n)\big\}. \tag{4.6}$$

It is a straightforward calculation that for $u \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned} &\lim_{t\to 0} \left| \frac{T_t u(x) - u(x)}{t} + (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi} \psi(\xi) \widehat{u}(\xi)\, d\xi \right| \\ &= \lim_{t\to 0} (2\pi)^{-n/2} \left| \int_{\mathbb{R}^n} e^{ix\xi} \left(\frac{e^{-t\psi(\xi)} - 1}{t} + \psi(\xi) \right) \widehat{u}(\xi)\, d\xi \right| = 0 \end{aligned} \tag{4.7}$$

uniformly for all $x \in \mathbb{R}^n$. This shows that $\mathcal{S}(\mathbb{R}^n) \subset D(A)$ and that for those u the generator admits the representation

$$Au(x) = -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi} \psi(\xi) \widehat{u}(\xi)\, d\xi, \quad x \in \mathbb{R}^n. \tag{4.8}$$

On the other hand, we have also

$$\begin{aligned} Au(x) &= \lim_{t\to 0} \frac{T_t u(x) - u(x)}{t} \\ &= \lim_{t\to 0} \frac{1}{t} \left(\int_{\mathbb{R}^n} u(x-y)\, \mu_t(dy) - \int_{\mathbb{R}^n} u(x-y)\, \epsilon_0(dy) \right) \end{aligned}$$

and comparing this formula with (4.8) we get

$$\begin{aligned} &-(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi} \psi(\xi) \widehat{u}(\xi)\, d\xi \\ &= \lim_{t\to 0} \frac{1}{t} \left(\int_{\mathbb{R}^n} u(x-y)\, \mu_t(dy) - \int_{\mathbb{R}^n} u(x-y)\, \epsilon_0(dy) \right) \end{aligned} \tag{4.9}$$

for all $x \in \mathbb{R}^n$ and $u \in \mathcal{S}(\mathbb{R}^n)$.

Evaluating carefully the right-hand side of (4.9) one should be able to derive a representation formula for the negative definite function ψ.

5. The Lévy–Khinchin formula

Going along the lines laid out in the preceding paragraph we will now prove the following theorem, known as the *Lévy–Khinchin formula*.

Theorem 5.1. *Let $\psi : \mathbb{R}^n \to \mathbb{C}$ be a continuous negative definite function. Then there exist a constant $c \geq 0$, a vector $d \in \mathbb{R}^n$, a symmetric positive semidefinite quadratic form $q(\xi)$ on $\mathbb{R}^n$ and a finite measure μ on $\mathbb{R}^n \setminus \{0\}$ such that*

$$\psi(\xi) = c + id\xi + q(\xi) + \int_{\mathbb{R}^n\setminus\{0\}} \left(1 - e^{-ix\xi} - \frac{ix\xi}{1+\|x\|^2}\right) \frac{1+\|x\|^2}{\|x\|^2}\, \mu(dx) \tag{5.1}$$

holds.

In order to prove Theorem 5.1 we need some preparations. Recall that there is a one-to-one correspondence between negative definite functions ψ and convolution semigroups $\{\mu_t\}_{t\geq 0}$ which is expressed by (2.5),

$$\widehat{\mu}_t(\xi) = (2\pi)^{-n/2}\, e^{-t\psi(\xi)}, \quad t \geq 0,\ \xi \in \mathbb{R}^n. \tag{5.2}$$

Lemma 5.2. *There exists a finite measure μ on $\mathbb{R}^n$ such that we have in the sense of weak convergence of measures*

$$\frac{1}{t}\,\frac{\|x\|^2}{1+\|x\|^2}\, \mu_t(dx) \to \mu(dx) \quad \textit{as} \quad t \to 0. \tag{5.3}$$

In particular, the measures in (5.3) have uniformly (in $t > 0$) bounded total mass.

Proof. We want to apply Theorem 2.8. Therefore we have to calculate the Fourier transforms of the measures $\frac{1}{t}\frac{\|x\|^2}{1+\|x\|^2}\, \mu_t(dx)$. Note that

$$\begin{aligned}
\frac{\|x\|^2}{1+\|x\|^2} &= \frac{1}{2}\int_0^\infty \left(1 - e^{-\lambda\|x\|^2/2}\right) e^{-\lambda/2}\, d\lambda \\
&= \frac{1}{2}\int_0^\infty \int_{\mathbb{R}^n} (2\pi\lambda)^{-n/2} \left(1 - e^{-ix\xi}\right) e^{-\|\xi\|^2/(2\lambda)}\, e^{-\lambda/2}\, d\xi\, d\lambda \\
&= \int_{\mathbb{R}^n} \left(1 - e^{-ix\xi}\right) g(\xi)\, d\xi,
\end{aligned}$$

where

$$g(\xi) = \frac{1}{2}\int_0^\infty (2\pi\lambda)^{-n/2} e^{-\|\xi\|^2/(2\lambda)}\, e^{-\lambda/2}\, d\lambda. \tag{5.4}$$

It is easy to see that

$$\int_{\mathbb{R}^n} g(\xi)\,d\xi < \infty \quad \text{and} \quad \int_{\mathbb{R}^n} \|\xi\|^4 g(\xi)\,d\xi < \infty. \tag{5.5}$$

Hence, we find for the Fourier transform of $\frac{1}{t}\,\frac{\|x\|^2}{1+\|x\|^2}\,\mu_t(dx)$

$$\begin{aligned}\frac{1}{t}\int_{\mathbb{R}^n} e^{-ix\eta}\frac{\|x\|^2}{1+\|x\|^2}\,\mu_t(dx) &= \frac{1}{t}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\eta}\left(1-e^{-ix\xi}\right) g(\xi)\,d\xi\,\mu_t(dx)\\ &= \frac{1}{t}\int_{\mathbb{R}^n}\left(e^{-t\psi(\eta)}-e^{-t\psi(\xi+\eta)}\right) g(\xi)\,d\xi.\end{aligned}$$

Using a Taylor expansion we obtain

$$\frac{1}{t}\int_{\mathbb{R}^n} e^{-ix\eta}\frac{\|x\|^2}{1+\|x\|^2}\,\mu_t(dx) = \int_{\mathbb{R}^n}(\psi(\xi+\eta)-\psi(\eta))g(\xi)\,d\xi + \frac{1}{2}t\,I(t,\eta)$$

where

$$I(t,\eta) = \int_{\mathbb{R}^n}(R(t,\eta)-R(t,\eta+\xi))g(\xi)\,d\xi,$$

with $\frac{t^2}{2}R(t,\eta)$ being the remainder of the Taylor expansion of $t\psi(\eta)\mapsto e^{-t\psi(\eta)}$ up to order two. We shall prove that for any $h>0$

$$\sup_{t>0}\,\sup_{\|\eta\|<h} |I(t,\eta)| < \infty. \tag{5.6}$$

Indeed,

$$\begin{aligned}|I(t,\eta)| &\le \int_{\mathbb{R}^n}\big(|R(t,\eta)|+|R(t,\xi+\eta)|\big)g(\xi)\,d\xi\\ &\le \int_{\mathbb{R}^n}\big(|\psi(\eta)|^2+|\psi(\xi+\eta)|^2\big)g(\xi)\,d\xi\\ &\le c_\psi\int_{\mathbb{R}^n}\big(1+\|\eta\|^4+\|\xi\|^4\big)g(\xi)\,d\xi,\end{aligned}$$

and with (5.5) the estimate (5.6) follows. Thus we find that the Fourier transforms of $\frac{1}{t}\,\frac{\|x\|^2}{1+\|x\|^2}\,\mu_t(dx)$ converge uniformly on compact sets to a (continuous) positive definite function, and by Theorem 2.8 there exists a finite measure μ such that (5.3) holds. The uniform boundedness of the total masses follows from the above calculations if we set $\eta = 0$. $\square$

Lemma 5.3. *Denote by $1_{B_1(0)}$, resp. $1_{B_2(0)}$ the indicator functions of the open balls with radius 1, resp., 2 centered at the origin. Fix a function $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $1_{B_1(0)} \le \chi \le 1_{B_2(0)}$. For $k, \ell = 1, \ldots, n$ there exist (signed) measures $\nu_{k\ell}$ on $\mathbb{R}^n$ with finite total mass such that*

$$\frac{1}{t}\,\frac{x_k x_\ell}{1+\|x\|^2}\,\chi(x)\,\mu_t(dx) \to \nu_{k\ell} \quad \text{as} \quad t \to 0 \tag{5.7}$$

holds in the sense of weak convergence of measures.

Proof. Using the identity $2x_k x_\ell = (x_k + x_\ell)^2 - x_k^2 - x_\ell^2$ it is sufficient to show that $\frac{1}{t}\,\frac{(x_k+x_\ell)^2}{1+\|x\|^2}\,\chi(x)\,\mu_t(dx) \to \rho_{k\ell}$ as $t \to 0$ for all $k, \ell = 1, 2, \ldots, n$ in the sense of weak convergence of measures. (Clearly, $\nu_{k\ell}$ is given by $\frac{1}{2}\rho_{k\ell} - \frac{1}{8}\rho_{kk} - \frac{1}{8}\rho_{\ell\ell}$ and has finite total mass.)

Since

$$\frac{(x_k + x_\ell)^2}{1+\|x\|^2}\,\chi(x) \le \frac{2\,\|x\|^2}{1+\|x\|^2}, \qquad 1 \le k, \ell \le n,\ x \in \mathbb{R}^n,$$

Lemma 5.2 shows that the measures $\frac{1}{t}\,\frac{(x_k+x_\ell)^2}{1+\|x\|^2}\,\chi(x)\,\mu_t(dx)$, $t > 0$, have uniformly bounded total mass with respect to t, k, ℓ, and χ. Again we want to apply Theorem 2.8, and for this reason we compute the Fourier transforms of the these measures. Note that

$$\begin{aligned} &\frac{1}{t}\int_{\mathbb{R}^n} e^{-ix\eta}\,\frac{(x_k + x_\ell)^2}{1+\|x\|^2}\,\chi(x)\,\mu_t(dx) \\ &= \frac{1}{t}\int_{\mathbb{R}^n} e^{-ix\eta}\,\frac{(x_k + x_\ell)^2}{1+\|x\|^2}\,\chi(x)\,(\mu_t - \epsilon_0)(dx). \end{aligned}$$

If Φ_η denotes the function

$$\Phi_\eta(x) = e^{-ix\eta}\frac{(x_k + x_\ell)^2}{1+\|x\|^2}\,\chi(x), \tag{5.8}$$

we have $\Phi_\eta \in \mathcal{S}(\mathbb{R}^n)$ and, using (3.13), we find

$$\begin{aligned}\frac{1}{t}\int_{\mathbb{R}^n} e^{-ix\eta}\,\frac{(x_k+x_\ell)^2}{1+\|x\|^2}\,\chi(x)\,\mu_t(dx) &= \frac{1}{t}\int_{\mathbb{R}^n} \widehat{\Phi_\eta}(\xi)\big(\overline{\widehat{\mu}_t(\xi)}-\overline{\widehat{\epsilon}_0(\xi)}\big)\,d\xi\\ &= (2\pi)^{-n/2}\,\frac{1}{t}\int_{\mathbb{R}^n}\widehat{\Phi_\eta}(\xi)\big(e^{-t\overline{\psi(\xi)}}-1\big)\,d\xi.\end{aligned}$$

As in the proof of Lemma 5.2 a Taylor expansion yields

$$\begin{aligned}&\frac{1}{t}\int_{\mathbb{R}^n} e^{-ix\eta}\,\frac{(x_k+x_\ell)^2}{1+\|x\|^2}\,\chi(x)\,\mu_t(dx)\\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n}\widehat{\Phi_\eta}(\xi)\overline{\psi(\xi)}\,d\xi+\frac{1}{2}t\,\tilde{I}(t,\eta)\end{aligned}$$

where

$$\tilde{I}(t,\eta) = (2\pi)^{-n/2}\int_{\mathbb{R}^n}\widehat{\Phi_\eta}(\xi)R(t,\xi)\,d\xi$$

with the remainder term $R(t,\xi)$ of the Taylor expansion up to order two. Now

$$\begin{aligned}(2\pi)^{n/2}|\tilde{I}(t,\eta)| &= \Big|\int_{\mathbb{R}^n}\widehat{\Phi_\eta}(\xi)R(t,\xi)\,d\xi\Big| = \Big|\int_{\mathbb{R}^n}\widehat{\Phi_0}(\xi)R(t,\xi-\eta)\,d\xi\Big|\\ &\le \int_{\mathbb{R}^n}\big|\widehat{\Phi_0}(\xi)\big|\big|\psi(\xi-\eta)\big|^2\,d\xi \le c_\psi\int_{\mathbb{R}^n}\big|\widehat{\Phi_0}(\xi)\big|(1+\|\eta\|^4+\|\xi\|^4)\,d\xi\end{aligned}$$

which implies that

$$\sup_{t>0}\sup_{\|\eta\|\le h}|\tilde{I}(t,\eta)|<\infty \tag{5.9}$$

for all $h>0$. Hence, the Fourier transforms of $\frac{1}{t}\,\frac{(x_k+x_\ell)^2}{1+\|x\|^2}\,\chi(x)\,\mu_t(dx)$ converge as $t\to 0$ uniformly on compact sets, and by Theorem 2.8 the assertion follows. $\square$

Corollary 5.4. *Let U be an open neighborhood of the origin $0\in\mathbb{R}^n$ such that $2U\subset B_1(0)$ and let $\phi_U\in C_0^\infty(\mathbb{R}^n)$ be a function satisfying $1_U\le\phi_U\le 1_{2U}$. Then*

$$\lim_{t\to 0}\frac{1}{t}\int_{\mathbb{R}^n}x_kx_\ell\phi_U(x)\mu_t(dx)=\int_{\mathbb{R}^n}\phi_U(x)\nu_{k\ell}(dx)+O((\operatorname{diam}2U)^2) \tag{5.10}$$

is valid.

PROOF. Let χ be the cut-off function introduced in Lemma 5.3. Since $\operatorname{supp}\phi_U \subset \{x \in \mathbb{R}^n : \chi(x) = 1\}$, we find

$$\frac{1}{t}\int_{\mathbb{R}^n} x_k x_\ell \phi_U(x)\,\mu_t(dx) = \frac{1}{t}\int_{\mathbb{R}^n} (1+\|x\|^2)\phi_U(x)\,\frac{x_k x_\ell}{1+\|x\|^2}\,\chi(x)\,\mu_t(dx)$$

and since $(1+\|\bullet\|^2)\phi_U \in C_0(\mathbb{R}^n)$, it follows from Lemma 5.3 that

$$\lim_{t\to 0}\frac{1}{t}\int_{\mathbb{R}^n} x_k x_\ell \phi_U(x)\,\mu_t(dx) = \int_{\mathbb{R}^n}(1+\|x\|^2)\phi_U(x)\,\nu_{k\ell}(dx).$$

Let $|\nu_{k,\ell}|(dx)$ denote the modulus of the signed measure $\nu_{k\ell}(dx)$. Then

$$\begin{aligned}\Big|\int_{\mathbb{R}^n}\|x\|^2\phi_U(x)\,\nu_{k\ell}(dx)\Big| &\le \int_{2U}\|x\|^2\,|\nu_{k\ell}|(dx)\\ &\le (\operatorname{diam}2U)^2\max_{1\le k,\ell\le n}|\nu_{k\ell}|(\mathbb{R}^n)\end{aligned}$$

and the assertion follows since the $\nu_{k\ell}$ are of bounded total mass. □

Let us recall the definition of the set $\mathcal{S}_0(\mathbb{R}^n)$,

$$\mathcal{S}_0(\mathbb{R}^n) = \left\{v \in \mathcal{S}(\mathbb{R}^n) : \widehat{v}(0) = \frac{\partial}{\partial\xi_j}\widehat{v}(0) = 0,\ 1\le j\le n\right\}.$$

Define

$$\begin{aligned}\mathcal{F} := \Big\{f : \mathbb{R}^n\setminus\{0\}\to\mathbb{R} : f(x) = \frac{1+\|x\|^2}{\|x\|^2}\,\widehat{v}(x)\\ \text{for some } v\in\mathcal{S}_0(\mathbb{R}^n)\Big\}.\end{aligned} \tag{5.12}$$

Lemma 5.5. *Let $f\in\mathcal{F}$, $f(x) = \frac{1+\|x\|^2}{\|x\|^2}\,\widehat{v}(x)$ with $v\in\mathcal{S}_0(\mathbb{R}^n)$. Then we have*

$$\frac{\|x\|^2}{1+\|x\|^2}\,f(x) = \frac{1}{2}\sum_{k,\ell=1}^n x_k x_\ell\,\frac{\partial^2}{\partial x_k\partial x_\ell}\,\widehat{v}(0)\;+\;O(\|x\|^3).$$

In particular, the left-hand side of (5.13) *has an extension onto $\mathbb{R}^n$.*

PROOF. By definition we have for $x\in\mathbb{R}^n\setminus\{0\}$

$$\frac{\|x\|^2}{1+\|x\|^2}\,f(x) = \widehat{v}(x)\in\mathcal{S}_0(\mathbb{R}^n)$$

which is a function on $\mathbb{R}^n$. The Taylor expansion of $\widehat{v}$ at $0 \in \mathbb{R}^n$ yields

$$\widehat{v}(x) = \frac{1}{2} \sum_{k,\ell=1}^{n} x_k x_\ell \, \frac{\partial^2}{\partial x_k \partial x_\ell} \, \widehat{v}(0) + R(x)$$

where we can estimate the remainder by

$$|R(x)| \leq \frac{n^3}{6} \sup_{1 \leq k,\ell,m \leq n} \left\| \frac{\partial^3}{\partial x_k \partial x_\ell \partial x_m} \, \widehat{v} \right\|_\infty \|x\|^3,$$

which implies (5.13). $\square$

Lemma 5.6. *For all $f \in \mathcal{F}$, $f(x) = \frac{1+\|x\|^2}{\|x\|^2} \, \widehat{v}(x)$ and $v \in \mathcal{S}_0(\mathbb{R}^n)$, we have*

$$-(2\pi)^{-n/2} \int_{\mathbb{R}^n} \psi(\xi) v(\xi) \, d\xi = \sum_{k,\ell=1}^{n} q_{k\ell} \, \frac{\partial^2}{\partial x_k \partial x_\ell} \, \widehat{v}(0) + \int_{\mathbb{R}^n \setminus \{0\}} f(x) \, \mu(dx),$$

where $(q_{k\ell})_{k,\ell=1}^n$ is a symmetric, positive semidefinite matrix and μ the measure constructed in Lemma 5.2.

Proof. Let f and v be as above. Then

$$\begin{aligned}
\frac{1}{t} \int_{\mathbb{R}^n} f(x) \, \frac{\|x\|^2}{1+\|x\|^2} \, \mu_t(dx) &= \frac{1}{t} \int_{\mathbb{R}^n} \widehat{v}(x) \, \mu_t(dx) \\
&= \frac{1}{t} \int_{\mathbb{R}^n} \widehat{v}(x) \, (\mu_t - \epsilon_0)(dx) = \frac{1}{t} \int_{\mathbb{R}^n} v(\xi) (\widehat{\mu}_t(\xi) - (2\pi)^{-n/2}) \, d\xi,
\end{aligned}$$

where we used (3.13), the fact that $\mathcal{F}^2 v(\xi) = v(-\xi)$, and that $\overline{\psi(\xi)} = \psi(-\xi)$. Therefore,

$$\frac{1}{t} \int_{\mathbb{R}^n} f(x) \, \frac{\|x\|^2}{1+\|x\|^2} \, \mu_t(dx) = (2\pi)^{-n/2} \, \frac{1}{t} \int_{\mathbb{R}^n} v(\xi) \big(e^{-t\psi(\xi)} - 1\big) \, d\xi.$$

Since $\left|e^{-t\psi(\xi)} - 1\right| \leq t|\psi(\xi)| \leq c_\psi t(1 + \|\xi\|^2)$ and $v \in \mathcal{S}(\mathbb{R}^n)$ we can pass to the limit under the integral sign and get

$$\lim_{t \to 0} \frac{1}{t} \int_{\mathbb{R}^n} f(x) \frac{\|x\|^2}{1+\|x\|^2} \mu_t(dx) = -(2\pi)^{-n/2} \int_{\mathbb{R}^n} v(\xi) \psi(\xi) d\xi. \tag{5.14}$$

With ϕ_U as in Corollary 5.4 it follows that

$$\frac{1}{t}\int_{\mathbb{R}^n} f(x)\,\frac{\|x\|^2}{1+\|x\|^2}\,\mu_t(dx) = \frac{1}{t}\int_{\mathbb{R}^n} \phi_U(x) f(x)\,\frac{\|x\|^2}{1+\|x\|^2}\,\mu_t(dx)$$
$$+\frac{1}{t}\int_{\mathbb{R}^n} (1-\phi_U(x)) f(x)\,\frac{\|x\|^2}{1+\|x\|^2}\,\mu_t(dx) = I_1(t) + I_2(t).$$

Since $(1-\phi_U)f \in C_b(\mathbb{R}^n)$, Lemma 5.2 gives

$$I_2(t) \to \int_{\mathbb{R}^n} (1-\phi_U(x)) f(x)\,\mu(dx) \quad \text{as} \quad t \to 0.$$

As in the proof of Lemma 5.5. we see

$$I_1(t) = \frac{1}{t}\int_{\mathbb{R}^n} \phi_U(x) \sum_{k,\ell=1}^{n} \frac{1}{2}\, x_k x_\ell\, \frac{\partial^2}{\partial x_k \partial x_\ell}\,\widehat{v}(0)\,\mu_t(dx)$$
$$+\frac{1}{t}\int_{\mathbb{R}^n} \phi_U(x) R(x)\,\mu_t(dx)$$

where R is a continuous function satisfying $|R(x)| \le c\|x\|^3$. Therefore,

$$\tilde{R}(x) := \begin{cases} \frac{R(x)(1+\|x\|^2)}{\|x\|^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is also continuous on $\mathbb{R}^n$ and we get

$$\frac{1}{t}\int_{\mathbb{R}^n} \phi_U(x) R(x)\,\mu_t(dx) = \frac{1}{t}\int_{\mathbb{R}^n} \phi_U(x)\tilde{R}(x)\,\frac{\|x\|^2}{1+\|x\|^2}\,\mu_t(dx).$$

Since $\phi_U(\bullet)\tilde{R}(\bullet)$ is bounded and continuous we find that the limit

$$\lim_{t\to 0} \frac{1}{t}\int_{\mathbb{R}^n} \phi_U(x) R(x)\,\mu_t(dx) = h_U \in \mathbb{R} \tag{5.16}$$

exists. But $\operatorname{supp}\phi_U \subset 2\bar{U} \subset \overline{B_1(0)}$, thus

$$\Big|\frac{1}{t}\int_{\mathbb{R}^n} \phi_U(x) R(x)\,\mu_t(dx)\Big| \le \frac{c}{t}\int_{2U} \|x\|^3\,\mu_t(dx)$$
$$\le \tilde{c}\,\operatorname{diam}(2U)\,\frac{1}{t}\int_{\mathbb{R}^n} \frac{\|x\|^2}{1+\|x\|^2}\,\mu_t(dx).$$

The last term is, however, uniformly bounded in $t > 0$. This gives

$$|h_U| \le c' \operatorname{diam}(2U) \tag{5.17}$$

and by Corollary 5.4

$$\lim_{t\to 0} I_1(t) = \frac{1}{2}\sum_{k,\ell=1}^{n} \int_{\mathbb{R}^n} \phi_U(x)\,\nu_{k\ell}(dx)\,\frac{\partial^2}{\partial x_k \partial x_\ell}\,\widehat{v}(0) + h_U.$$

So far we have proved

$$\begin{aligned}&\lim_{t\to 0}\frac{1}{t}\int_{\mathbb{R}^n} f(x)\,\frac{\|x\|^2}{1+\|x\|^2}\,\mu_t(dx)\\ =&\frac{1}{2}\sum_{k,\ell=1}^{n}\int_{\mathbb{R}^n}\phi_U(x)\nu_{k\ell}(dx)\frac{\partial^2}{\partial x_k\partial x_\ell}\,\widehat{v}(0)+\int_{\mathbb{R}^n}(1-\phi_U(x))f(x)\,\mu(dx)+h_U.\end{aligned}$$

Since the measures $\nu_{k\ell}$, $1 \le k, \ell \le n$, are finite we can pass to the limit $U \downarrow \{0\}$ to obtain

$$\begin{aligned}&\lim_{t\to 0}\frac{1}{t}\int_{\mathbb{R}^n} f(x)\,\frac{\|x\|^2}{1+\|x\|^2}\,\mu_t(dx)\\ &= \frac{1}{2}\sum_{k,\ell=1}^{n}\nu_{k\ell}(\{0\})\,\frac{\partial^2}{\partial x_k\partial x_\ell}\,\widehat{v}(0) + \int_{\mathbb{R}^n\setminus\{0\}} f(x)\,\mu(dx)\end{aligned}$$

or in view of (5.14)

$$\begin{aligned}&-(2\pi)^{-n/2}\int_{\mathbb{R}^n}\psi(\xi)v(\xi)\,d\xi\\ &= \frac{1}{2}\sum_{k,\ell=1}^{n}\nu_{k\ell}(\{0\})\,\frac{\partial^2}{\partial x_k\partial x_\ell}\,\widehat{v}(0) + \int_{\mathbb{R}^n\setminus\{0\}} f(x)\,\mu(dx).\end{aligned} \tag{5.18}$$

It remains to show that the matrix $(q_{k\ell})_{k,\ell=1}^n = \big((1/2)\nu_{k\ell}(\{0\})\big)_{k,\ell=1}^n$ is symmetric and positive semidefinite. But for any choice of $\xi_k \in \mathbb{R}$,

$1 \leq k \leq n$,

$$\begin{aligned}\sum_{k,\ell=1}^{n} q_{k\ell}\xi_k\xi_\ell &= \sum_{k,\ell=1}^{n} \lim_{U\downarrow\{0\}} \frac{1}{2}\int_{\mathbb{R}^n} \phi_U(x)\,\nu_{k,\ell}(dx)\,\xi_k\xi_\ell \\ &= \frac{1}{2}\sum_{k,\ell=1}^{n} \lim_{U\downarrow\{0\}}\lim_{t\to 0}\frac{1}{t}\int_{\mathbb{R}^n}\phi_U(x)\,x_k x_\ell\xi_k\xi_\ell\,\mu_t(dx) \\ &= \frac{1}{2}\lim_{U\downarrow\{0\}}\lim_{t\to 0}\frac{1}{t}\int_{\mathbb{R}^n}\phi_U(x)\Big(\sum_{k=1}^{n} x_k\xi_k\Big)^2\mu_t(dx) \geq 0,\end{aligned}$$

and the symmetry is obvious. □

Proof of Theorem 5.1. Let $f \in \mathcal{F}$, $f(x) = \frac{1+\|x\|^2}{\|x\|^2}\,\widehat{v}(x)$ and $v \in \mathcal{S}_0(\mathbb{R}^n)$. With μ as in Lemma 5.2 we find

$$\begin{aligned}\int_{\mathbb{R}^n\setminus\{0\}} f(x)\,\mu(dx) &= \int_{\mathbb{R}^n\setminus\{0\}}\frac{1+\|x\|^2}{\|x\|^2}\,\widehat{v}(x)\,\mu(dx) \\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n\setminus\{0\}}\int_{\mathbb{R}^n} e^{-ix\xi}v(\xi)\,d\xi\,\frac{1+\|x\|^2}{\|x\|^2}\,\mu(dx) \\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n\setminus\{0\}}\int_{\mathbb{R}^n}\left(e^{-ix\xi}-1+\frac{ix\xi}{1+\|x\|^2}\right) \\ &\qquad\times v(\xi)\,d\xi\,\frac{1+\|x\|^2}{\|x\|^2}\,\mu(dx),\end{aligned} \tag{5.19}$$

because $v \in \mathcal{S}_0(\mathbb{R}^n)$ implies $\int_{\mathbb{R}^n} v(\xi)\,d\xi = \int_{\mathbb{R}^n}\xi_j v(\xi)\,d\xi = 0$ for all $1 \leq j \leq n$. Since

$$\begin{aligned}\left|e^{-ix\xi}-1+\frac{ix\xi}{1+\|x\|^2}\right| &\leq \left|e^{-ix\xi}-1+ix\xi\right| + \left|ix\xi-\frac{ix\xi}{1+\|x\|^2}\right| \\ &\leq \frac{1}{2}\,\|x\|^2\|\xi\|^2 + \frac{\|x\|^2}{1+\|x\|^2}\,\|x\|\|\xi\|,\end{aligned}$$

we find for $\|x\| \leq 1$ that

$$\left|\left(e^{-ix\xi}-1+\frac{ix\xi}{1+\|x\|^2}\right)\frac{1+\|x\|^2}{\|x\|^2}\right| \leq \frac{1}{2}(1+\|x\|^2)\|\xi\|^2+\|x\|\|\xi\| \leq \|\xi\|^2+\|\xi\|,$$

and for $\|x\| > 1$ we have

$$\left|\left(e^{-ix\xi} - 1 + \frac{ix\xi}{1+\|x\|^2}\right)\frac{1+\|x\|^2}{\|x\|^2}\right| \le 4 + \|\xi\|.$$

We may, therefore, change the order of integration in (5.19) and obtain

$$\begin{aligned}&\int_{\mathbb{R}^n\setminus\{0\}} f(x)\,\mu(dx)\\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\setminus\{0\}}\left(e^{-ix\xi} - 1 + \frac{ix\xi}{1+\|x\|^2}\right)\frac{1+\|x\|^2}{\|x\|^2}\,\mu(dx)\,v(\xi)\,d\xi.\end{aligned}$$

By Lemma 5.6

$$\int_{\mathbb{R}^n}\psi(\xi)v(\xi)d\xi = -(2\pi)^{n/2}\sum_{k,\ell=1}^{n} q_{k\ell}\frac{\partial^2}{\partial x_k\partial x_\ell}\widehat{v}(0) - (2\pi)^{n/2}\int_{\mathbb{R}^n\setminus\{0\}} f(x)\mu(dx)$$

$$=\int_{\mathbb{R}^n}\left\{\sum_{k,\ell=1}^{n} q_{k\ell}\xi_k\xi_\ell + \int_{\mathbb{R}^n\setminus\{0\}}\left(1-e^{-ix\xi}-\frac{ix\xi}{1+\|x\|^2}\right)\frac{1+\|x\|^2}{\|x\|^2}\mu(dx)\right\}v(\xi)d\xi.$$

Applying Lemma 3.3 we find

$$\begin{aligned}\psi(\xi) = c + id\xi + \sum_{k,\ell=1}^{n} q_{k\ell}\xi_k\xi_\ell\\ + \int_{\mathbb{R}^n\setminus\{0\}}\left(1 - e^{-ix\xi} - \frac{ix\xi}{1+\|x\|^2}\right)\frac{1+\|x\|^2}{\|x\|^2}\,\mu(dx),\end{aligned}$$

with some $c \in \mathbb{C}$ and $d \in \mathbb{C}^n$. For $\xi = 0$ we know $\psi(0) \ge 0$, hence $c \in \mathbb{R}$ and $c \ge 0$, and, since $\psi(\xi) = \overline{\psi(-\xi)}$, we see that $d \in \mathbb{R}^n$.

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