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Geodesic tubes and normal flow space forms

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Abstract. We consider the shape operator of tubular hypersurfaces about geodesics in Riemannian manifolds which are equipped with a unit Killing vector field. We derive some characteristic properties for the special subclass of normal flow space forms.

1. Introduction

Up to local isometries, the two-point homogeneous spaces provide the simplest examples of locally homogeneous spaces. They may be considered as space forms in real, Kähler, quaternionic Kähler and Cayley geometry. In all these cases they can be defined by supposing that some particular sectional curvatures are locally or globally constant. The Sasakian space forms play a similar role in Sasakian geometry. One of the important features of all these space forms is that one can determine explicitly all the Jacobi vector fields along geodesics and use them to study a lot of geometric properties of these manifolds.

A key fact in Sasakian geometry is the existence of a unit Killing vector field on each Sasakian manifold. Several aspects of this geometry may also be considered on each Riemannian manifold equipped with such a vector field. This vector field determines an isometric flow on the manifold and for that reason the corresponding geometry has been called flow geometry or also generalized Sasakian geometry. We refer to [5], [6] for more details and further references. In [7] the authors studied the notion of a normal flow

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space form. It turns out that although there are many similarities with Sasakian geometry, there are also remarkable differences. For example, there exist normal flow space forms which do not have an analogue in Sasakian geometry.

Extrinsic and intrinsic geometric properties of geodesic spheres and tubes about curves or geodesics have been used to characterize all these space forms. Here, we continue this study and use Jacobi vector fields to derive characteristic properties of the shape operator of tubes about geodesics in normal flow space forms.

In Section 2 we collect some preliminaries about flow geometry and in Section 3 we recall some useful facts about the extrinsic geometry of tubes about geodesics. The characteristic properties are then derived in Section 4 and Section 5, based on the explicit expressions for the shape operator. These expressions are derived by means of the formulas for the Jacobi vector fields derived for normal flow space forms in [8].

2. Preliminaries about flow geometry

Let (M,g) be an n-dimensional, smooth, connected Riemannian manifold with $n \geq 2$. Further, let ∇ denote the Levi Civita connection of (M,g) and R the Riemannian curvature tensor with sign convention $R_{UV} = \nabla_{[U,V]} - [\nabla_U, \nabla_V]$ for $U, V \in \mathfrak{X}(M)$, the Lie algebra of smooth vector fields on M.

Further, let $\mathfrak{F}_\xi$ denote the isometric flow [12] on (M,g) generated by a unit Killing vector field ξ. The flow lines of $\mathfrak{F}_\xi$ are geodesics. A vector which is orthogonal to ξ is called a *transversal* or *horizontal* vector and a geodesic which is orthogonal to ξ is called a transversal or horizontal geodesic. $\mathfrak{F}_\xi$ determines locally a Riemannian submersion. In fact, for each $m \in M$, let $\mathcal{U}$ be a small open neighborhood of m such that ξ is regular on $\mathcal{U}$. Then the mapping $\pi : \mathcal{U} \to \widetilde{\mathcal{U}} = \mathcal{U}/\xi$ is a submersion. Further, let $\widetilde{g}$ denote the induced metric on $\widetilde{\mathcal{U}}$ given by $\left(\widetilde{g}(\widetilde{X}, \widetilde{Y})\right)^* = g(\widetilde{X}^*, \widetilde{Y}^*)$ for $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(\widetilde{\mathcal{U}})$ and where $\widetilde{X}^*$, $\widetilde{Y}^*$ denote the horizontal lifts of $\widetilde{X}$, $\widetilde{Y}$ with respect to the $(n-1)$-dimensional horizontal distribution on $\mathcal{U}$ determined by $\eta = 0$, η being the dual one-form of ξ with respect to g. Then $\pi : (\mathcal{U}, g_{|\mathcal{U}}) \to (\widetilde{\mathcal{U}}, \widetilde{g})$ is a Riemannian submersion.

Next, put $HU = -\nabla_U \xi$ and $h(U,V) = g(HU,V)$ for all $U, V \in \mathfrak{X}(M)$. Since ξ is a Killing vector field, h is skew-symmetric and moreover, we have $h = -d\eta$. The Levi Civita connection $\widetilde{\nabla}$ on $(\widetilde{\mathcal{U}}, \widetilde{g})$ is determined by

$$\nabla_{\widetilde{X}^*} \widetilde{Y}^* = (\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y})^* + h(\widetilde{X}^*, \widetilde{Y}^*)\xi \tag{2.1}$$

for $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(\widetilde{\mathcal{U}})$ and the curvature tensor R of (M,g) satisfies

$$R(X,\xi,Y,\xi) = g(HX,HY) = -g(H^2X,Y)$$

for all horizontal vector fields X, Y. Here we use, as usual, the notation $R(X,Y,Z,W) = g(R_{XY}Z,W)$. From this it follows that the ξ-*sectional curvature* $K(X,\xi)$ of the two-plane determined by X and ξ is non-negative for all horizontal X. Moreover, $K(X,\xi)$ is strictly positive for all X if and only if H is of maximal rank $n-1$. In this case, n is necessarily odd and η is a contact form. Then the flow $\mathfrak{F}_\xi$ is called a *contact flow*.

$\mathfrak{F}_\xi$ is called a *normal flow* [5] if, for all horizontal X, Y, the curvature transformations R_{XY} leave the horizontal subspaces invariant. This is equivalent to $R(X,Y,X,\xi) = 0$. It is worthwhile to note here that a Sasakian manifold is a Riemannian manifold equipped with a normal flow $\mathfrak{F}_\xi$ such that the ξ-sectional curvature equals 1 (see [1] for more details). Further, $\mathfrak{F}_\xi$ is normal if and only if

$$(\nabla_U H)V = g(HU,HV)\xi + \eta(V)H^2U \tag{2.2}$$

for all $U, V \in \mathfrak{X}(M)$. Then the curvature tensor satisfies

$$\begin{cases} R_{UV}\xi = \eta(V)H^2U - \eta(U)H^2V, \\ R_{U\xi}V = g(HU,HV)\xi + \eta(V)H^2U, \end{cases} \tag{2.3}$$

$U, V \in \mathfrak{X}(M)$. Using this and (2.1), we deduce that the curvature tensors of ∇ and $\widetilde{\nabla}$ are related by the formula

$$\begin{aligned} (\widetilde{R}_{\widetilde{X}\widetilde{Y}}\widetilde{Z})^* = R_{\widetilde{X}^*\widetilde{Y}^*}\widetilde{Z}^* &- g(H\widetilde{Y}^*, \widetilde{Z}^*)H\widetilde{X}^* \\ &+ g(H\widetilde{X}^*, \widetilde{Z}^*)H\widetilde{Y}^* + 2g(H\widetilde{X}^*, \widetilde{Y}^*)H\widetilde{Z}^* \end{aligned} \tag{2.4}$$

for all $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \mathfrak{X}(\widetilde{\mathcal{U}})$. Finally, put $\widetilde{H}\widetilde{X} = \pi_* H\widetilde{X}^*$ for $\widetilde{X} \in \mathfrak{X}(\widetilde{\mathcal{U}})$. Then $\mathfrak{F}_\xi$ is normal if and only if $\widetilde{\nabla}\widetilde{H} = 0$.

Next, we collect some facts about locally Killing-transversally symmetric spaces. Let $m \in (M,g)$ and denote by $\sigma = \sigma_m : [-\delta, \delta] \to M$ a geodesic flow line through $m = \sigma(0)$ where δ is sufficiently small. A local diffeomorphism s_m of M defined in a neighborhood $\mathcal{U}$ of m is called a (*local*) *reflection with respect to* σ if for every transversal geodesic $\gamma(s)$ where $\gamma(0)$ lies in the intersection of $\mathcal{U}$ and σ, we have $(s_m \circ \gamma)(s) = \gamma(-s)$ for all s with $\gamma(\pm s) \in \mathcal{U}$, s being the arc length of γ. A Riemannian manifold (M,g) equipped with a flow $\mathfrak{F}_\xi$ such that each local reflection s_m is an isometry, is called a *locally Killing-transversally symmetric space* (briefly, a *locally* KTS-*space*) [5]. In this case, $\mathfrak{F}_\xi$ is a normal flow. These spaces may be characterized as follows.

Proposition 2.1 [5]**.** *The following statements are equivalent:*

(i) $(M,g,\mathfrak{F}_\xi)$ *is a locally* KTS-*space;*

(ii) $\mathfrak{F}_\xi$ *is normal and* $(\nabla_X R)(X,Y,X,Y) = 0$ *for all horizontal* X, Y.

Proposition 2.2 [5]**.** *Let* $\mathfrak{F}_\xi$ *be a normal flow on* (M,g). *Then* $(M,g,\mathfrak{F}_\xi)$ *is a locally* KTS-*space if and only if each base space* $\widetilde{\mathcal{U}}$ *of a local Riemannian submersion* $\pi : \mathcal{U} \to \widetilde{\mathcal{U}} = \mathcal{U}/\xi$ *is a locally symmetric space.*

Note that locally KTS-spaces are locally homogeneous spaces. Moreover, when η is a contact form, $(M,g,\mathfrak{F}_\xi)$ is called a *contact* locally KTS-space. In that case we mention the following useful result.

Proposition 2.3 [6]**.** *Any complete, contact locally* KTS-*space* $(M,g,\mathfrak{F}_\xi)$ *is transversally modelled on a Hermitian symmetric space* $\widetilde{M}$. *Moreover, if* $\widetilde{M} = \widetilde{M}_0 \times \widetilde{M}_1 \times \cdots \times \widetilde{M}_r$ *is its de Rham decomposition, where* $\widetilde{M}_0 = \mathbb{C}^p$, *then there exist* $r+p$ *real numbers* $c_1, \ldots, c_r$, $\mu_1, \ldots, \mu_p$ *such that on each distinguished chart* $\mathcal{U} \subset M$, *the smooth distributions* $\mathcal{H}_i$, $i = 0, 1, \ldots, r$, *obtained by the horizontal lifts of the tangent vectors of* $\widetilde{M}_i$, *verify*

(i) $\mathcal{H}(m) = \mathcal{H}_0(m) \oplus \mathcal{H}_1(m) \oplus \cdots \oplus \mathcal{H}_r(m)$ *is an* H*-invariant orthogonal decomposition of the horizontal subspace* $\mathcal{H}(m)$ *for each* $m \in \mathcal{U}$;

(ii) *each sectional curvature* $K(\mathcal{H}_j, \xi)$, $j = 1, \ldots, r$, *is a positive constant equal to* c_j^2;

(iii) *the* $(1,1)$*-tensor field*

$$J = J_0 \times \frac{1}{c_1}\widetilde{H}_1 \times \cdots \times \frac{1}{c_r}\widetilde{H}_r$$

is a Hermitian structure on $(\widetilde{\mathcal{U}} = \mathcal{U}/\xi, \widetilde{g})$*, where* J_0 *denotes the almost complex structure on* $\widetilde{M}_0 = \mathbb{C}^p = E^{2p}(x^1, \ldots, x^{2p})$ *and* $\widetilde{H}_j = \widetilde{H} \circ p_j$*,* $j = 1, \ldots, r$*, with* $p_j : \widetilde{M} \to \widetilde{M}_j$ *the projection of* $\widetilde{M}$ *on* $\widetilde{M}_j$*;*

(iv) $\widetilde{H}_0 = \widetilde{H} \circ p_0$ *on* $E^{2p}(x^1, \ldots, x^{2p})$ *is given by*

$$\widetilde{H}\frac{\partial}{\partial x^k} = \mu_k \frac{\partial}{\partial x^{p+k}}, \quad \widetilde{H}\frac{\partial}{\partial x^{p+k}} = -\mu_k \frac{\partial}{\partial x^k}, \qquad k = 1, \ldots, p.$$

Finally, we consider the preliminaries about normal flow space forms. A plane section in T_mM, $m \in M$, is called an *H-section* if there exists a horizontal X in T_mM such that $\{X, HX\}$ is a basis of the plane section. The sectional curvature $K(X, HX)$ of an H-section is called the *H-sectional curvature* corresponding to X. In [7] it is proved that the H- and ξ-sectional curvatures on a complete, contact locally KTS-space $(M, g, \mathfrak{F}_\xi)$ determine completely its curvature. A Riemannian manifold equipped with a contact flow $\mathfrak{F}_\xi$ is said to be a *flow space form* if the H-sectional curvature is pointwise constant, that is, $K(X, HX)$ is independent of X for each horizontal $X \in T_mM$ and all $m \in M$. Moreover, a *normal* flow space form has globally constant H-sectional curvature if and only if it is a locally KTS-space.

Normal flow space forms have been studied in detail in [7] where two special cases are considered according to whether the ξ-sectional curvature is constant or not. If $(M, g, \mathfrak{F}_\xi)$ is a normal flow space form with pointwise constant H-sectional curvature k and constant ξ-sectional curvature c^2, then $(M, c^2g, \varphi = c^{-1}H, c^{-1}\xi, c\eta)$ is a Sasakian manifold of constant φ-sectional curvature kc^{-2} and so, (M, g) is obtained by a homothetic change of metric from Sasakian space forms. See [7] for more information. Hence, as is proved in [7] and for $\dim M \geq 5$, the H-sectional curvature is a global constant k and the curvature tensor is given by

$$\begin{aligned}
R_{UV}W = {} & \frac{k+3c^2}{4}\Big\{g(U,W)V - g(V,W)U\Big\} \\
& + \frac{k-c^2}{4}\Big\{\eta(V)\eta(W)U - \eta(U)\eta(W)V \\
& + g(V,W)\eta(U)\xi - g(U,W)\eta(V)\xi\Big\} \\
& + \frac{k-c^2}{4c^2}\Big\{g(W,HU)HV - g(W,HV)HU - 2g(U,HV)HW\Big\}.
\end{aligned} \tag{2.5}$$

For $\dim M = 3$ we shall assume that k is also a global constant. Then we have the same formula for R as in (2.5). In this case $(\widetilde{\mathcal{U}}, \widetilde{g})$ is a Kähler manifold of constant holomorphic sectional curvature $h = k + 3c^2$. Here we have, based on [11, Theorem A]:

Theorem 2.1. *A Riemannian manifold (M, g) of dimension ≥ 5 equipped with a normal flow $\mathfrak{F}_\xi$ such that the ξ-sectional curvature is a non-vanishing constant, is a flow space form if and only if for any horizontal X, $R_{XHX}X$ is proportional to HX.*

This first kind of normal flow space forms will be denoted by $M^{2n+1}(c^2, k)$. We recall that we obtain Sasakian space forms for $c^2 = 1$.

Although there is in the contact case a similarity with the theory of Sasakian manifolds, there also differences. Indeed, the situation is quite different when we consider normal flow space forms with non-constant ξ-sectional curvature, a case which cannot occur in Sasakian geometry. We recall some facts for this case. When $(M, g, \mathfrak{F}_\xi)$ is a complete manifold with globally constant H-sectional curvature, it admits two smooth distributions $\mathcal{H}_1$ and $\mathcal{H}_2$ such that for each $m \in M$, $\mathcal{H}(m) = \mathcal{H}_1(m) \oplus \mathcal{H}_2(m)$ is an H-invariant decomposition of the horizontal subspace $\mathcal{H}(m)$ and each sectional curvature $K(\mathcal{H}_i, \xi)$, $i = 1, 2$, is a positive constant c_i^2 ($c_1^2 > c_2^2$). Further, such spaces are precisely the (M^{2n+1}, g) equipped with a normal contact flow $\mathfrak{F}_\xi$ which is transversally modelled on the Riemannian product $\mathbb{C}P^{n_1}(h_1) \times \mathbb{C}H^{n_2}(h_2)$ where $|h_2| < h_1$, $n_1 + n_2 = n$ and the ξ-sectional curvatures c_i^2, $i = 1, 2$, are given by

$$c_i^2 = (-1)^{i+1} h_i \frac{h_1 - h_2}{3(h_1 + h_2)}.$$

In what follows we shall denote these spaces by $M(n_1, n_2; h_1, h_2)$. The H-sectional curvature is the strictly negative constant $k = 2h_1h_2(h_1 + h_2)^{-1}$ and the curvature tensor is given by

$$
\begin{aligned}
R_{UV}W = \sum_{i=1}^{2} \Big\{ & \frac{h_i}{4}\Big(g(X_i, Z_i)Y_i - g(Y_i, Z_i)X_i\Big) \\
& +(-1)^i \frac{3(h_1+h_2)}{4(h_1-h_2)}\Big(g(HY_i, Z_i)HX_i - g(HX_i, Z_i)HY_i - 2g(HX_i, Y_i)HZ_i\Big) \\
& + (-1)^i \frac{h_i(h_1-h_2)}{3(h_1 + h_2)}\Big\{\Big(g(Y_i, Z_i)\eta(U) - g(X_i, Z_i)\eta(V)\Big)\xi \\
& + \eta(W)\Big(\eta(V)X_i - \eta(U)Y_i\Big)\Big\}\Big\} \\
& + g(HV, W)HU - g(HU, W)HV - 2g(HU, V)HW
\end{aligned}
\tag{2.6}
$$

for vector fields $U = \sum_{i=1}^{2} X_i + \eta(U)\xi$, $V = \sum_{i=1}^{2} Y_i + \eta(V)\xi$, $W = \sum_{i=1}^{2} Z_i + \eta(W)\xi$ on M.

In this context, we finish with the following characterization.

Theorem 2.2 [8]. *Let $(M, g, \mathfrak{F}_\xi)$ be a complete, contact locally* KTS-*space such that at each point m in M the self-adjoint operator $R_{\xi\cdot}\xi$ (or equivalently, $-H_m^2$) has two eigenspaces $\mathcal{V}_1(m)$ and $\mathcal{V}_2(m)$ with $\mathcal{H}(m) = \mathcal{V}_1(m) \oplus \mathcal{V}_2(m)$ and $\dim \mathcal{V}_i(m) = 2n_i \geq 4$, $i = 1, 2$. Then $(M, g, \mathfrak{F}_\xi)$ is a flow space form (with non-constant ξ-sectional curvature and globally constant H-sectional curvature) if and only if*

$$\|X_1\|^{-2} R_{X_1 H X_1} X_1 + \|X_2\|^{-2} R_{X_2 H X_2} X_2 \text{ and } R_{X H^\perp X} X$$

are proportional to HX for all $X = X_1 + X_2 \in \mathcal{V}_1(m) \oplus \mathcal{V}_2(m)$ and where, for non-vanishing X_1 and X_2, $H^\perp X$ denotes a vector orthogonal to HX in the plane spanned by HX_1 and HX_2.

In what follows we consider $H^\perp X$ given by

$$H^\perp X = \|HX_1\|^{-1}\|HX_2\|HX_1 - \|HX_2\|^{-1}\|HX_1\|HX_2$$

for each $X = X_1 + X_2 \in \mathcal{H}_1 \oplus \mathcal{H}_2$ where $\|X_1\|\,\|X_2\| \neq 0$.

3. The shape operator of geodesic tubes

In this section we briefly recall some facts about the treatment of tubular hypersurfaces about geodesics. We refer to [9], [10], [13] for more details and further references.

Let $\sigma : [a,b] \to M$ be a smooth embedded curve in (M,g) and denote by $\sigma^\perp$ the normal bundle of σ. Further, let $\exp_\sigma$ be the exponential map of this normal bundle, that is, $\exp_\sigma(\sigma(t),v) = \exp_{\sigma(t)} v$ for any $t \in [a,b]$ and all $v \in \sigma(t)^\perp$ where $\sigma(t)^\perp$ denotes the fiber of $\sigma^\perp$ over $\sigma(t)$. The set

$$\mathcal{U}_\sigma(r) = \{\exp_{\sigma(t)} v \mid v \in \sigma(t)^\perp,\ \|v\| < r, t \in [a,b]\}$$

is said to be the (open) tubular neighborhood or the (open) solid tube of radius r about σ. Since $[a,b]$ is compact and since $\sigma : [a,b] \to M$ is an embedding, we shall always assume that the radius r of this tube is so small that $\exp_\sigma$ is a diffeomorphism between $\mathcal{U}_\sigma(r)$ and the (open) solid tube $N_\sigma(r)$ of radius r about the zero section of $\sigma^\perp$. For each $s < r$, the set

$$\mathcal{P}_\sigma(s) = \{p \in \mathcal{U}_\sigma(r) \mid d(\sigma,p) = s\}$$

is a smooth hypersurface in M which is called the tubular hypersurface or just the tube of radius s about σ. These tubes determine a foliation of $\mathcal{U}_\sigma(r)$ by hypersurfaces $\mathcal{P}_\sigma(s)$ and the radial vector field $\frac{\partial}{\partial s}$ is a unit vector field orthogonal to the leaves of this foliation.

In what follows we shall suppose that σ is a unit speed geodesic. Then $\mathcal{P}_\sigma$ is called a *geodesic tube* about σ.

Fermi coordinate systems are well-adapted to treat the geometry of tubes. They are defined as follows. Let $\sigma : [a,b] \to M$ be such a geodesic and let $\{e_1 = \sigma'(a), e_2, \dots, e_n\}$ be an orthonormal basis of $T_{\sigma(a)}M$. Further, let $E_1 = \sigma'$ and let $E_2, \dots, E_n$ be the normal vector fields along σ which are parallel with respect to the Levi Civita connection ∇ and such that $E_i(a) = e_i$, $i = 2, \dots, n$. Then the Fermi coordinates $(x_1, \dots, x_n)$, with respect to $\sigma(a)$ and the frame field $\{E_1, \dots, E_n\}$, are defined by

$$x_1\Big(\exp_{\sigma(t)} \sum_{j=2}^n t_j E_j\Big) = t - a, \quad x_i\Big(\exp_{\sigma(t)} \sum_{j=2}^n t_j E_j\Big) = t_i, \quad i = 2, \dots, n.$$

Next, let $p = \exp_{\sigma(t)}(sv)$, $v \in \sigma(t)^\perp$, $\|v\| = 1$. Then $p \in \mathcal{P}_\sigma(s)$. Further, let γ be the unit speed geodesic connecting $\sigma(t)$ and p and adapt the frame field $\{E_1, \dots, E_n\}$ along σ such that $E_2(t) = \gamma'(0)$. Moreover, let $\{F_1, \dots, F_n\}$ be the frame field along γ obtained by parallel translation of $\{E_1(t), \dots, E_n(t)\}$. Now, let $X_1, X_3, \dots, X_n$ be the Jacobi vector fields along γ which satisfy the following set of initial conditions:

$$\begin{cases} X_1(0) = E_1(t), \quad X_1'(0) = \Big(\nabla_{\gamma'} \dfrac{\partial}{\partial x_1}\Big)(\sigma(t)) = 0, \\ X_i(0) = 0, \qquad\quad X_i'(0) = E_i(t) \quad i = 3, \dots, n. \end{cases} \tag{3.1}$$

Then we have

$$X_1(s) = \frac{\partial}{\partial x_1}(\gamma(s)), \quad X_i(s) = s\frac{\partial}{\partial x_i}(\gamma(s)), \quad i = 3, \dots, n.$$

Now, define the endomorphism-valued function $B : s \mapsto B(s)$ by

$$X_i(s) = BF_i(s), \quad i = 1, 3, \dots, n. \tag{3.2}$$

Then this function satisfies the Jacobi equation

$$B'' + R_\gamma \circ B = 0$$

where $R_\gamma = R_{\gamma' .}\gamma'$. The initial values are

$$B(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B'(0) = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

Next, since $\frac{\partial}{\partial s}$ is a unit normal vector of $\mathcal{P}_\sigma(s)$ at p, the *shape operator* S_σ of $\mathcal{P}_\sigma(r)$ at p is given by

$$(S_\sigma X)(p) = \left(\nabla_X \frac{\partial}{\partial s}\right)(p)$$

for any X tangent to $\mathcal{P}_\sigma(s)$ at p and then it follows from (3.2) that $S_\sigma(p)$ is given by

$$S_\sigma(p) = (B'B^{-1})(s). \tag{3.3}$$

Moreover, by differentiation along γ, we have that S_σ satisfies the following Riccati type differential equation:

$$S_\sigma' + S_\sigma^2 + R_\gamma = 0. \tag{3.4}$$

4. Quasi-umbilicity and ξ-sectional curvature

A hypersurface N is said to be *quasi-umbilical* at $p \in N$ if its shape operator $S(p)$ has at least $d-1$ equal eigenvalues where $d = \dim N$. When there are exactly $d-1$ equal eigenvalues we may write $S(p) = \alpha_1 I + \alpha_2 \mu \otimes u$ where μ is the dual one-form of u and in this case we say that N is quasi-umbilical with respect to u. We now extend this concept of quasi-umbilicity and say that N is k*-quasi-umbilical* at p with respect to u if $S(p)$ has exactly k different eigenvalues with corresponding eigenspaces orthogonal to u. Clearly, 1-quasi-umbilical is equivalent to quasi-umbilical.

Now we return to a Riemannian manifold (M, g) equipped with a flow $\mathfrak{F}_\xi$ and derive some results about the ξ-sectional curvature which are related to the k-quasi-umbilicity of geodesic tubes. So, let $(M, g, \mathfrak{F}_\xi)$ be a complete, contact locally KTS-space and let $\mathcal{P}_\sigma(s)$ be a tube of radius s about a transversal geodesic σ. Then $p = \gamma(s) = \exp_m s\xi$, $m = \sigma(t)$, is a point of $\mathcal{P}_\sigma(s)$. We put $u = \sigma'$ and we denote by u also its parallel translate along γ. According to Proposition 2.3, M is transversally modelled on a Hermitian symmetric space $\widetilde{M} = \widetilde{M}_0 \times \widetilde{M}_1 \times \cdots \times \widetilde{M}_r$. Let $\dim \widetilde{M}_i = 2n_i$. Further, choose an orthonormal basis $\{v_{ij} \mid j = 1, \ldots, 2n_i\}$ of $\mathcal{H}_i(m)$, $i = 1, \ldots, r$, and denote by E_l, F_{ij} the vector fields along γ obtained by parallel translating $\left(\frac{\partial}{\partial x^l}\right)^*$, $l = 1, \ldots, 2p$, and v_{ij}. Then it follows from [8, Theorem 4.1] that every Jacobi vector field, orthogonal to γ, is given by

$$
\begin{aligned}
X = \sum_{k=1}^{p} \Big\{ &(A_k \sin \mu_k s + B_k \cos \mu_k s) E_k \\
&+ (A_{p+k} \sin \mu_k s + B_{p+k} \cos \mu_k s) E_{p+k} \Big\} \\
&+ \sum_{i=1}^{r} \sum_{j=1}^{2n_i} (C_{ij} \sin c_i s + D_{ij} \cos c_i s) F_{ij}
\end{aligned}
$$

where A_l, B_l, C_{ij} and D_{ij} are constants.

In particular, if $u \in \mathcal{H}_{i_0}$ for some $i_0 \in \{1, \ldots, r\}$, we take $\{v_{ij}$, $j = 1, \ldots, 2n_i\}$ with $v_{i_0 1} = u$. Then, using (3.3), we get by a straightforward calculation

$$
(4.1)\qquad
\begin{aligned}
S_\sigma(p)E_k &= \mu_k \cot \mu_k s E_k, && k=1,\dots,p,\\
S_\sigma(p)E_{p+k} &= \mu_k \cot \mu_k s E_{p+k}, && k=1,\dots,p,\\
S_\sigma(p)F_{i_0 1} &= -c_{i_0} \tan c_{i_0} s F_{i_0 1},\\
S_\sigma(p)F_{ij} &= c_i \cot c_i s F_{ij}, && i=1,\dots,r;\\
& \qquad j=1,\dots,2n_i \quad \text{and} \quad (i,j)\neq(i_0,1).
\end{aligned}
$$

A similar expression holds when $u = \left(\frac{\partial}{\partial x^1}\right)^*(m) \in \mathcal{H}_0(m)$.

We have

Theorem 4.1. *A complete, contact locally* KTS-*space* $(M, g, \mathfrak{F}_\xi)$ *has constant ξ-sectional curvature if and only if there exists a horizontal geodesic σ such that $\mathcal{P}_\sigma(s)$ is quasi-umbilical with respect to u at $p = \exp_m s\xi$ for all small s.*

Proof. First, suppose that $(M, g, \mathfrak{F}_\xi)$ has constant ξ-sectional curvature c^2. Let $u(m) \in \mathcal{H}_{i_0}(m)$ for some $i_0 = 1,\dots,r$. With the same conventions as in (4.1) we then have

$$
\begin{aligned}
S_\sigma(p)E_l &= c \cot cs E_l, && l=1,\dots,2p,\\
S_\sigma(p)F_{i_0 1} &= -c \tan cs F_{i_0 1},\\
S_\sigma(p)F_{ij} &= c \cot cs F_{ij}, && i=1,\dots,r;\\
& \qquad j=1,\dots,2n_i \quad \text{and} \quad (i,j)\neq(i_0,1).
\end{aligned}
$$

A similar expression holds when $u(m) = \left(\frac{\partial}{\partial x^1}\right)^*(m) \in \mathcal{H}_0(m)$. In both cases we have that $\mathcal{P}_\sigma(s)$ is quasi-umbilical with respect to u at p.

Conversely, let $\mathcal{P}_\sigma(s)$ be a tube about a horizontal geodesic σ and write $u = u_0^1 + \cdots + u_0^p + u_1 + \cdots + u_r$ where

$$
u_0^k \in \left\{ \left(\frac{\partial}{\partial x^k}\right)^*(m), \left(\frac{\partial}{\partial x^{p+k}}\right)^*(m) \right\}, \quad u_i \in \mathcal{H}_i(m), \quad i=1,\dots,r.
$$

We suppose that all u_0^k and u_i are different from zero. In the other cases, we may proceed in a similar way. We take the orthonormal bases

$\{\|u_0^1\|^{-1}u_0^1, \dots, \|u_0^p\|^{-1}u_0^p, u_0^{1\perp}, \dots, u_0^{p\perp}\}$ of $\mathcal{H}_0(m)$ where $u_0^{k\perp}$ is a unit vector in

$$\left\{\left(\frac{\partial}{\partial x^k}\right)^*(m), \left(\frac{\partial}{\partial x^{p+k}}\right)^*(m)\right\}$$

orthogonal to u_0^k, and $\{v_{ij},\ j = 1, \dots, 2n_i\}$ of $\mathcal{H}_i(m)$, $i = 1, \dots, r$, where $v_{i1} = \|u_i\|^{-1}u_i$. Denote by E_l, $l = 1, \dots, 2p$, and F_{ij} the parallel translates along γ. Then we have, for $p = \exp_m s\xi$,

$$(4.2) \qquad \begin{cases} S_\sigma(p)E_{p+k} = \mu_k \cot \mu_k s E_{p+k}, & k = 1, \dots, p, \\ S_\sigma(p)F_{ij} = c_i \cot c_i s F_{ij} \text{ for } (i,j) \neq (i,1), & i = 1, \dots, r. \end{cases}$$

Hence, if $\mathcal{P}_\sigma(s)$ is quasi-umbilical at p, we must have $\mu_1^2 = \cdots = \mu_p^2 = c_1^2 = \cdots = c_r^2$ and this implies that the ξ-sectional curvature is constant. $\square$

Theorem 4.2. *Let $(M, g, \mathfrak{F}_\xi)$ be a $(2n+1)$-dimensional, complete, contact locally* KTS*-space with non-constant ξ-sectional curvature. Then the self-adjoint operator $R_{\xi\cdot}\xi$ (or equivalently, $-H_m^2$) has at a point $m \in M$ two eigenspaces $\mathcal{V}_1(m)$ and $\mathcal{V}_2(m)$ with $\mathcal{H}(m) = \mathcal{V}_1(m) \oplus \mathcal{V}_2(m)$ if and only if there exists a horizontal geodesic σ through m such that $\mathcal{P}_\sigma(s)$ is 2-quasi-umbilical with respect to u at $p = \exp_m s\xi$.*

Proof. First, let λ_1^2, λ_2^2 be the eigenvalues of $R_{\xi\cdot}\xi$ corresponding to $\mathcal{V}_1(m)$ and $\mathcal{V}_2(m)$ where λ_1, λ_2 are taken as positive numbers. Let u be a unit vector in $\mathcal{V}_1(m)$ and take an orthonormal basis $\{e_1, \dots, e_{2n+1}\}$ of T_mM such that $e_{2n+1} = \xi$ and where $\{e_1 = u, \dots, e_d\}$, $\{e_{d+1}, \dots, e_{2n}\}$ are orthonormal bases of $\mathcal{V}_1(m)$ and $\mathcal{V}_2(m)$, respectively. Further, let $\{E_1, \dots, E_{2n+1}\}$ be the parallel translated frame along the flow line through m. Following the notation given in Proposition 2.3, $\mathcal{H}(m) = \mathcal{H}_0(m) \oplus \mathcal{H}_1(m) \oplus \cdots \oplus \mathcal{H}_r(m)$ and $\mathcal{H}_i(m)$, $i = 1, \dots, r$, are eigenspaces of $R_{\xi\cdot}\xi$ with corresponding eigenvalues c_i^2 at m. Moreover, each plane $\left\{\left(\frac{\partial}{\partial x^k}\right)^*, \left(\frac{\partial}{\partial x^{p+k}}\right)^*\right\}$ is an eigenspace with eigenvalue μ_k^2. Hence, using (4.1), we have

$$\begin{aligned} &S_\sigma(p)E_1 = -\lambda_1 \tan \lambda_1 s E_1, \\ &S_\sigma(p)E_j = \lambda_1 \cot \lambda_1 s E_j, \quad j = 2, \dots, d, \\ &S_\sigma(p)E_k = \lambda_2 \cot \lambda_2 s E_k, \quad k = d+1, \dots, 2n. \end{aligned}$$

So, $\mathcal{P}_\sigma(s)$ is 2-quasi-umbilical with respect to u at p.

To prove the converse, we use (4.2). It then follows that $\mathcal{P}_\sigma(s)$ is 2-quasi-umbilical at p if and only if there are at most two different values for μ_i^2 and c_j^2. But since the ξ-sectional curvature is non-constant, we must have exactly two different values. This proves the required result. □

5. Geodesic tubes and normal flow space forms

In this final section we derive some characterizations of normal flow space forms by considering the shape operator of tubes about flow lines or tubes about horizontal geodesics. We start with

5.1 Tubes about flow lines

Let $(M, g, \mathfrak{F}_\xi)$ be a Riemannian manifold equipped with a normal flow $\mathfrak{F}_\xi$ and let σ be a segment of the flow line through $m \in M$. Further, let $\mathcal{P}_\sigma(s)$ be the tube of radius s about σ. Let $p = \exp_m su$ where u is a unit horizontal vector and denote by $\kappa_\sigma(p)$ the normal curvature of the geodesic of $\mathcal{P}_\sigma(s)$ tangent to $v = \|Hu\|^{-1}Hu$ at p, that is,

$$\kappa_\sigma(p) = g(S_\sigma(p)v, v).$$

Here, κ_σ is a real-valued function on each $\mathcal{P}_\sigma(s)$.

Our first result generalizes that for Sasakian space forms given in [4].

Theorem 5.1. *A Riemannian manifold (M, g) with $\dim M = 2n+1 \geq 5$ and equipped with a normal flow $\mathfrak{F}_\xi$ such that the ξ-sectional curvature is a non-vanishing constant, is a flow space form if and only if for any horizontal u, $S_\sigma(p)Hu$ belongs to the plane $\{\xi, Hu\}$ for all m and all small tubes $\mathcal{P}_\sigma(s)$ about the flow line through m. Moreover, $h = -(\kappa_\sigma^2 + \kappa_\sigma')$ is constant on every tube $\mathcal{P}_\sigma(s)$ and $k = h - 3c^2$ is the constant H-sectional curvature.*

Proof. Let $(M, g, \mathfrak{F}_\xi) = M(c^2, k)$ and consider an orthonormal basis $\{e_1, \ldots, e_{2n+1}\}$ at a point m of the flow line σ such that $e_1 = \xi$, $e_2 = u$, $e_3 = \|Hu\|^{-1}Hu$. Further, let $\{F_1, \ldots, F_{2n+1}\}$ be the frame field along $\gamma(s) = \exp_m su$ obtained by parallel translating $\{e_1, \ldots, e_{2n+1}\}$. Then it follows from (3.3) and the explicit expressions of the Jacobi vector fields on Sasakian space forms given in [2] and adapted to our case, that the shape operator S_σ has the form

$$S_\sigma = \begin{pmatrix} A & 0 \\ 0 & fI_{2n-2} \end{pmatrix}$$

with respect to $\{F_1, F_3, \ldots, F_{2n+1}\}$. Here, $f = \frac{\sqrt{h}}{2}\cot\frac{\sqrt{h}}{2}s$ for $h > 0$, $f = \frac{1}{s}$ for $h = 0$ and $f = \frac{\sqrt{-h}}{2}\coth\frac{\sqrt{-h}}{2}s$ for $h < 0$, $h = k + 3c^2$. Further, A is a symmetric 2×2 matrix with entries which depend only on s. Since the field of planes spanned by ξ and Hu is parallel along γ (as follows from (2.2)), we obtain that $S_\sigma Hu \in \operatorname{span}\{\xi, Hu\}$.

Conversely, suppose $S_\sigma Hu \in \operatorname{span}\{\xi, Hu\}$. Then we have at p:

$$S_\sigma Hu = \nabla_{\gamma'}(Hu) = -c^2\xi + \kappa_\sigma Hu.$$

Hence, we obtain

$$S_\sigma^2 Hu = -c^2\kappa_\sigma\xi + (c^2 + \kappa_\sigma^2)Hu$$

and by using (2.2), we also have

$$S'_\sigma Hu = (2c^2 + \kappa'_\sigma)Hu + c^2\kappa_\sigma\xi.$$

So, we obtain from this and the Riccati equation (3.4):

$$R_{uHu}u = -(3c^2 + \kappa_\sigma^2 + \kappa'_\sigma)Hu.$$

Since any point $p \in M$ belongs to a $\mathcal{P}_\sigma(s)$, we get that $R_{uHu}u$ is proportional to Hu at each point of M and for each horizontal vector u. Then it follows from Theorem 2.1 that $(M, g, \mathfrak{F}_\xi)$ is a flow space form with constant H-sectional curvature $k = -(3c^2 + \kappa_\sigma^2 + \kappa'_\sigma)$. □

Next, let $(M, g, \mathfrak{F}_\xi) = M(n_1, n_2; h_1, h_2)$ be a complete normal flow space form with non-constant ξ-sectional curvature and globally constant H-sectional curvature. Using the solutions of the Jacobi equation on normal flow space forms given in [8], we obtain the shape operator of $\mathcal{P}_\sigma(s)$ by applying (3.3). Here we have, when σ is a flow line:

A. $u = u_1 \in \mathcal{H}_1$

Let $\{e_1, \ldots, e_{2n+1}\}$ be an orthonormal basis at m such that $e_1 = \xi$, $e_2 = u_1$, $e_3 = \|Hu_1\|^{-1}Hu_1$ and where $\{e_2, \ldots, e_{2n_1+1}\}$ and $\{e_{2n_1+2}, \ldots, e_{2n+1}\}$ are orthonormal bases of $\mathcal{H}_1(m)$ and $\mathcal{H}_2(m)$, respectively. With respect to the parallel basis $\{F_1, F_3, \ldots, F_{2n+1}\}$, $S_\sigma(\exp_m su_1)$ has the matrix form

$$S_\sigma(\exp_m su_1) = \begin{pmatrix} A & 0 & 0 \\ 0 & \dfrac{\sqrt{h_1}}{2}\cot\dfrac{\sqrt{h_1}}{2}sI_{2n_1-2} & 0 \\ 0 & 0 & \dfrac{1}{s}I_{2n_2} \end{pmatrix};$$

B. $u = u_2 \in \mathcal{H}_2$

Taking now $\{e_1, \dots, e_{2n+1}\}$ assuming that $e_1 = \xi$, $e_2 = u_2$, $e_3 = \|Hu_2\|^{-1} Hu_2$ and with $\{e_4, \dots, e_{2n_1+3}\}$, $\{e_2, e_3, e_{2n_1+4}, \dots, e_{2n+1}\}$ as bases of $\mathcal{H}_1(m)$ and $\mathcal{H}_2(m)$, respectively, we have

$$S_\sigma(\exp_m su_2) = \begin{pmatrix} A & 0 & 0 \\ 0 & \frac{1}{s} I_{2n_1} & 0 \\ 0 & 0 & \frac{\sqrt{-h_2}}{2} \coth \frac{\sqrt{-h_2}}{2} s I_{2n_2-2} \end{pmatrix}$$

where in both cases A is a 2×2 matrix.

C. $u = u_1 + u_2 \in \mathcal{H}_1 \oplus \mathcal{H}_2$

In this case, we put

$$e_1 = \xi, \quad e_2 = u, \quad e_3 = \|Hu\|^{-1} Hu,$$
$$e_4 = \|Hu\|^{-1} H^\perp u, \quad e_5 = \|u_1\|^{-1}\|u_2\| u_1 - \|u_2\|^{-1}\|u_1\| u_2$$

and we consider the orthonormal basis $\{e_1, \dots, e_{2n+1}\}$ at m such that $\{e_6, \dots, e_{2n_1+3}\}$ and $\{e_{2n_1+4}, \dots, e_{2n+1}\}$ are bases of $\mathcal{E}_1(m)$ and $\mathcal{E}_2(m)$, respectively, where $\mathcal{E}_i$, $i = 1, 2$, denotes the field of $2(n_i - 1)$-planes on $\mathcal{H}_i$ along γ and orthogonal to span $\{u_i, Hu_i\}$. With respect to the parallel frame field $\{F_1, F_3, \dots, F_{2n+1}\}$, $S_\sigma(\exp_m su)$ is of the form

$$S_\sigma(\exp_m su) = \begin{pmatrix} B & 0 & 0 & 0 \\ 0 & \frac{1}{s} & 0 & 0 \\ 0 & 0 & f_1 I_{2n_1-2} & 0 \\ 0 & 0 & 0 & f_2 I_{2n_2-2} \end{pmatrix}$$

where $f_1 = \frac{\|u_1\|\sqrt{h_1}}{2} \cot \frac{\|u_1\|\sqrt{h_1}}{2} s$, $f_2 = \frac{\|u_2\|\sqrt{-h_2}}{2} \coth \frac{\|u_2\|\sqrt{-h_2}}{2} s$ and B is a symmetric 3×3 matrix.

Now, we define the real-valued function k on each tube $\mathcal{P}_\sigma(s)$ by

$$k(p) = -\Big(3\|Hu\|^2 + \kappa_\sigma^2 + \kappa'_\sigma\Big)(p), \quad p = \exp_m su.$$

Then we have

Theorem 5.2. *Let $(M, g, \mathfrak{F}_\xi)$ be a complete, contact locally* KTS-*space such that at each point m in M the self-adjoint operator $R_{\xi \cdot}\xi$ (or equivalently, $-H_m^2$) has two eigenspaces $\mathcal{V}_1(m)$ and $\mathcal{V}_2(m)$ with $\mathcal{H}(m) = \mathcal{V}_1(m) \oplus \mathcal{V}_2(m)$ and $\dim \mathcal{V}_i(m) = 2n_i \geq 4$, $i = 1, 2$. Then $(M, g, \mathfrak{F}_\xi)$ is a flow space form (with non-constant ξ-sectional curvature and globally constant H-sectional curvature) if and only if, for all unit vectors $u_i \in \mathcal{V}_i$ and $u \in \mathcal{V}_1 \oplus \mathcal{V}_2$, the shape operator S_σ of any small tube $\mathcal{P}_\sigma$ about the flow lines σ satisfies:*

(i) *$S_\sigma(\exp_m su_i)Hu_i \in \{\xi, Hu_i\}$ and $S_\sigma(\exp_m su)$ preserves the vector space spanned by $\{\xi, Hu, H^\perp u\}$;*

(ii) *$k(\exp_m su_1) = k(\exp_m su_2)$ and $N'_{33} + N_{23}^2 + N_{33}^2 = 0$ where N_{ij}, $1 \leq i, j \leq 3$, are the entries of the matrix associated to $S_\sigma(\exp_m su)$ with respect to the orthonormal basis $\{\xi, \|Hu\|^{-1}Hu, \|Hu\|^{-1}H^\perp u\}$.*

Proof. Using (i) we derive by a straightforward computation for $i = 1, 2$

$$
\begin{aligned}
S_\sigma(\exp_m su_i)Hu_i &= -\|Hu_i\|^2\xi + \kappa_\sigma(\exp_m su_i)Hu_i,\\
S_\sigma(\exp_m su)H^\perp u &= N_{23}Hu + N_{33}H^\perp u,\\
S_\sigma^2(\exp_m su_i)Hu_i &= -\|Hu_i\|^2\kappa_\sigma(\exp_m su_i)\xi\\
&\quad + \Big(\|Hu_i\|^2 + \kappa_\sigma^2(\exp_m su_i)\Big)Hu_i,\\
S_\sigma^2(\exp_m su)H^\perp u &= -\|Hu\|^2 N_{23}\xi + N_{23}(N_{22} + N_{33})Hu\\
&\quad + (N_{23}^2 + N_{33}^2)H^\perp u.
\end{aligned}
$$

Moreover, from (2.2) and since $H^\perp u$ is parallel along γ, we obtain

$$
\begin{aligned}
S'_\sigma Hu_i &= \|Hu_i\|^2\kappa_\sigma(\exp_m su_i)\xi + \Big(2\|Hu_i\|^2 + \kappa'_\sigma(\exp_m su_i)\Big)Hu_i,\\
S'_\sigma H^\perp u &= \|Hu\|^2 N_{23}\xi + N'_{23}Hu + N'_{33}H^\perp u.
\end{aligned}
$$

So, using (3.4), we get

$$
\begin{aligned}
\|u_i\|^{-2}R_{u_i Hu_i}u_i &= k(\exp_m su_i)Hu_i,\\
R_{uH^\perp u}u &= -\Big\{\Big(N_{23}(N_{22} + N_{33}) + N'_{23}\Big)Hu + \Big(N'_{33} + N_{23}^2 + N_{33}^2\Big)H^\perp u\Big\}
\end{aligned}
$$

So, (ii) implies that $\|u_1\|^{-2}R_{u_1Hu_1}u_1+\|u_2\|^{-2}R_{u_2Hu_2}u_2$ and $R_{uH^\perp u}u$ are proportional to Hu_1+Hu_2 and to Hu, respectively. By a same argument as in the proof of Theorem 5.1, it follows from Theorem 2.2 that $(M,g,\mathfrak{F}_\xi)$ is a normal flow space form.

The converse follows from the direct computations given above. □

5.2 Tubes about horizontal geodesics

Now, we consider small tubes $\mathcal{P}_\sigma(s)$ about a horizontal geodesic σ and put $\sigma'=u$ where u is a unit vector. Next, let γ be the unit speed geodesic of M meeting σ orthogonally at $\gamma(0)=m$ and tangent to the horizontal vector v such that $v(m)=\|Hu\|^{-1}Hu(m)$.

First we prove the following theorem which generalizes the one for Sasakian space forms [3].

Theorem 5.3. *Let (M,g) be a Riemannian manifold with* $\dim M=2n+1\geq 5$ *and equipped with a normal flow $\mathfrak{F}_\xi$ such that the ξ-sectional curvature is a non-vanishing constant. Then, with the conventions made above, M is a flow space form if and only if for all horizontal geodesic σ, the shape operator S_σ along γ preserves the plane spanned by $\{\xi,Hv\}$.*

Proof. First, let $(M,g,\mathfrak{F}_\xi)=M(c^2,k)$ and let $\{e_1,\dots,e_{2n+1}\}$ be an orthonormal basis at m with initial conditions $e_1=u$, $e_2=v$, $e_3=\xi$. Again, denote by $\{F_1,\dots,F_{2n+1}\}$ the basis obtained by parallel translation of $\{e_1,\dots,e_{2n+1}\}$ along γ. Taking into account that $Hv=-\|Hu\|u$ at m, it follows from (3.3) and the results in [2] (with slight modifications as mentioned in Theorem 5.1) that, with respect to the basis $\{F_1,F_3,\dots,F_{2n+1}\}$, S_σ has the following form along γ:

$$S_\sigma=\begin{pmatrix} A & 0\\ 0 & fI_{2n-2}\end{pmatrix}$$

where $f=\frac{\sqrt{h}}{2}\cot\frac{\sqrt{h}}{2}s$ for $h>0$, $f=\frac{1}{s}$ for $h=0$ and $f=\frac{\sqrt{-h}}{2}\coth\frac{\sqrt{-h}}{2}s$ for $h<0$, $h=k+3c^2$. Here, A is a 2×2 matrix with entries depending on s. Hence, S_σ preserves the field of planes spanned by ξ and Hv because these planes are parallel.

Conversely, the hypothesis implies that also S_σ^2 and S_σ' preserve the plane $\{\xi,Hv\}$. It then follows as in Theorem 5.1 that $R_{vHv}v\in\{\xi,Hv\}$ and because of the normality of the flow, $R_{vHv}v$ is proportional to Hv. So, the result follows from Theorem 2.1. □

Next, we consider complete normal flow space forms with non-constant ξ-sectional curvature and globally constant H-sectional curvature. Let σ_i, $i = 1, 2$, or σ be the horizontal geodesics through $m \in M$ tangent to the unit vector $u_i \in \mathcal{H}_i(m)$ or $u = u_1 + u_2 \in \mathcal{H}_1(m) \oplus \mathcal{H}_2(m)$, respectively. Further, let γ_i, $i = 1, 2$, or γ denote the unit speed geodesics of M meeting σ_i or σ orthogonally at $m = \sigma(t)$ and tangent to v_i or v, respectively, where $v_i(m) = \|Hu_i\|^{-1} Hu_i(m)$ and

$$v(m) = \Big(\|Hu_1\|^{-1} \cos\theta Hu_1 + \|Hu_2\|^{-1} \sin\theta Hu_2\Big)(m)$$

with $\theta = \arctan(-\|Hu_1\|/\|Hu_2\|)$. Then $H^{\perp}v$ is proportional to u at m. Using again the explicit formulas given in [8], we may obtain the shape operator of the tubes $\mathcal{P}_\sigma(s)$ at the points $\gamma_i(s)$ or $\gamma(s)$. We have

A. $u = u_1 \in \mathcal{H}_1$ Let $\{e_1, \ldots, e_{2n+1}\}$ be an orthonormal basis at m such that $e_1 = u_1$, $e_2 = v_1$, $e_3 = \xi$ and where $\{e_1, e_2, e_4, \ldots, e_{2n_1+1}\}$ and $\{e_{2n_1+2}, \ldots, e_{2n+1}\}$ are bases of $\mathcal{H}_1(m)$ and $\mathcal{H}_2(m)$, respectively. With respect to the corresponding parallel frame field $\{F_1, F_3, \ldots, F_{2n+1}\}$, we have:

$$S_\sigma(\exp_m sv_1) = \begin{pmatrix} A & 0 & 0 \\ 0 & \dfrac{\sqrt{h_1}}{2} \cot \dfrac{\sqrt{h_1}}{2} sI_{2n_1-2} & 0 \\ 0 & 0 & \dfrac{1}{s} I_{2n_2} \end{pmatrix};$$

B. $u = u_2 \in \mathcal{H}_2$ Here we take a basis with $e_1 = u_2$, $e_2 = v_2$, $e_3 = \xi$ and such that $\{e_4, \ldots, e_{2n_1+3}\}$ and $\{e_1, e_2, e_{2n_1+4}, \ldots, e_{2n+1}\}$ are bases of $\mathcal{H}_1(m)$ and $\mathcal{H}_2(m)$, respectively. Then we have with respect to the corresponding parallel frame field:

$$S_\sigma(\exp_m sv_2) = \begin{pmatrix} A & 0 & 0 \\ 0 & \dfrac{1}{s} I_{2n_1} & 0 \\ 0 & 0 & \dfrac{\sqrt{-h_2}}{2} \coth \dfrac{\sqrt{-h_2}}{2} sI_{2n_2-2} \end{pmatrix}$$

and where in both cases A is a symmetric 2×2 matrix.

C. $u = u_1 + u_2 \in \mathcal{H}_1 \oplus \mathcal{H}_2$

Let $\{e_1, \dots, e_{2n+1}\}$ be an orthonormal basis at m such that

$$e_1 = u, \quad e_2 = v, \quad e_3 = \|Hv\|^{-1} Hv,$$
$$e_4 = \xi, \quad e_5 = \|u_1\|^{-1}\|u_2\|u_1 - \|u_2\|^{-1}\|u_1\|u_2$$

and where $\{e_6, \dots, e_{2n_1+3}\}$, $\{e_{2n_1+4}, \dots, e_{2n+1}\}$ are bases of $\mathcal{E}_1(m)$ and $\mathcal{E}_2(m)$, respectively. $\mathcal{E}_1$ and $\mathcal{E}_2$ are as in 5.1. With respect to the corresponding parallel frame field, S_σ has the form

$$S_\sigma(\exp_m sv) = \begin{pmatrix} B & 0 & 0 & 0 \\ 0 & \frac{1}{s} & 0 & 0 \\ 0 & 0 & f_1 I_{2n_1-2} & 0 \\ 0 & 0 & 0 & f_2 I_{2n_2-2} \end{pmatrix}$$

where $f_1 = \frac{\|u_1\|\sqrt{h_1}}{2} \cot \frac{\|u_1\|\sqrt{h_1}}{2} s$, $f_2 = \frac{\|u_2\|\sqrt{-h_2}}{2} \coth \frac{\|u_2\|\sqrt{-h_2}}{2} s$ and B is a symmetric 3×3 matrix.

Then we have

Theorem 5.4. *Let $(M, g, \mathfrak{F}_\xi)$ be a complete, contact locally* KTS*-space such that at each point $m \in M$ the self-adjoint operator $R_{\xi.}\xi$ (or equivalently, $-H_m^2$) has two eigenspaces $\mathcal{V}_1(m)$ and $\mathcal{V}_2(m)$ with $\mathcal{H}(m) = \mathcal{V}_1(m) \oplus \mathcal{V}_2(m)$ and $\dim \mathcal{V}_i(m) = 2n_i \geq 4$, $i = 1, 2$. Then $(M, g, \mathfrak{F}_\xi)$ is a flow space form (with non-constant ξ-sectional curvature and globally constant H-sectional curvature) if and only if for all unit speed geodesics σ_i, σ such that $\sigma_i'(0) = u_i \in \mathcal{V}_i(m)$ and $\sigma'(0) = u \in \mathcal{V}_1(m) \oplus \mathcal{V}_2(m)$ and with the conventions as above, the following conditions are satisfied by the shape operators S_{σ_i} and S_σ:*

(i) *$S_{\sigma_i}(\exp_m sv_i)$ and $S_\sigma(\exp_m sv)$ preserve the vector spaces spanned by $\{\xi, Hv_i\}$ and $\{\xi, Hv, H^\perp v\}$, respectively;*

(ii) *if $M^i = (M^i_{jk})_{1 \leq j,k \leq 2}$ and $N = (N_{lm})_{1 \leq l,m \leq 3}$ denote the corresponding matrices associated to these endomorphisms with respect to the orthonormal bases $\{\xi, \|Hv_i\|^{-1}Hv_i\}$ and $\{\xi, \|Hv\|^{-1}Hv, \|Hv\|^{-1}H^\perp v\}$, then we have*

$$(M^1_{12})^2 + (M^1_{22})^2 + {M^1_{22}}' - 2\|Hv_1\| M^1_{12} = (M^2_{12})^2 + (M^2_{22})^2 + {M^2_{22}}' - 2\|Hv_2\| M^2_{12},$$

$$\sum_{l=1}^{3} (N_{l3})^2 + N_{33}' = 0.$$

Proof. When $(Mg, \mathfrak{F}_\xi) = M(n_1, n_2; h_1, h_2)$, then the result follows from the formulas given above.

So, conversely, suppose that (i) is satisfied. Then, we get

$$S^2_{\sigma_i}(\exp_m sv_i)Hv_i = \|Hv_i\| M^i_{12}\Big(M^i_{11} + M^i_{22}\Big)\xi + \Big((M^i_{12})^2 + (M^i_{22})^2\Big)Hv_i,$$

$$S^2_\sigma(\exp_m sv)H^\perp v = \sum_{l=1}^{3}\Big\{\|Hv\|N_{1l}N_{l3}\xi + N_{2l}N_{l3}Hv + (N_{l3})^2 H^\perp v\Big\}$$

and by using (2.2), we also have

$$S'_{\sigma_i}Hv_i = \|Hv_i\|\Big({M^i_{12}}' + \|Hv_i\|(M^i_{22} - M^i_{11})\Big)\xi$$
$$+ \Big({M^i_{22}}' - 2\|Hv_i\|M^i_{12}\Big)Hv_i,$$

$$S'_\sigma H^\perp v = \|Hv\|\Big(N'_{13} + \|Hv\|N_{33}\Big)\xi + \Big(N'_{23} - \|Hv\|N_{13}\Big)Hv + H'_{33}H^\perp v.$$

So, taking into account the normality of the flow and using (3.4) together with the conditions (ii), we obtain that $\|v_1\|^{-2}R_{v_1Hv_1}v_1 + \|v_2\|^{-2}R_{v_2Hv_2}v_2$ and $R_{vH^\perp v}v$ are proportional to $Hv_1 + Hv_2$ and Hv, respectively. Then it follows from Theorem 2.2 that $(M, g, \mathfrak{F}_\xi)$ is a normal flow space form. $\square$

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