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Weak topology and Markov–Kakutani theorem on hyperspace

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Abstract. Let K be a weakly compact convex subset of a Banach space X. One version of the Markov–Kakutani Theorem states that if $\mathcal{F} : (K, \tau_w) \to (K, \tau_w)$ is a commutative family of continuous linear transformations, then $\mathcal{F}$ has a common fixed point in K. Suppose now $CC(X)$ is the collection of all non-empty compact convex subsets of X. We shall define a certain weak topology $\mathcal{J}_w$ on $CC(X)$ and get the above-mentioned version of the Markov–Kakutani Theorem extended to the hyperspace $(CC(X), \mathcal{J}_w)$.

1. Introduction

The classical Markov–Kakutani Theorem states that if K is a compact, convex subset of a topological linear space E, then every commutative family $\mathcal{F}$ of continuous linear transformations of K into K must have a common fixed point in K. Since a linear transformation between Banach spaces is continuous if and only if it is weakly continuous, it follows that if K is a weakly compact convex subset of a Banach space X, then every commutative family $\mathcal{F}$ of weakly continuous linear transformations of K into K must have a common fixed point in K. In this paper, we shall have the above-mentioned version of the Markov–Kakutani theorem extended to the hyperspace $CC(X)$, where $CC(X)$ is the collection of all non-empty compact convex subsets of X.

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2. Notations and preliminaries

Let X be a Banach space, X^* its topological dual, and $CC(X)$ the collection of all non-empty compact convex subsets of X. For A, $B \in CC(X)$, define $N(A;\varepsilon) = \{x \in X : \|x-a\| < \varepsilon$ for some $a \in A\}$ and $h(A,B) = \inf\{\varepsilon > 0 : A \subset N(B;\varepsilon)$ and $B \subset N(A;\varepsilon)\}$ where h is known as the Hausdorff metric induced by the norm on X. Suppose A, $B \in CC(X)$ and α is a scalar, then it is known that both $\lambda A = \{\lambda a : a \in A\}$ and $A + B = \{a + b : a \in A, b \in B\}$ belong to $CC(X)$. Thus $CC(X)$ carries an "affine" structure on it in a natural way. $(CC(X), h)$ is known as the hyperspace over X and it has been investigated by several mathematicians from different view-points ([1], [2], [4]). We now define a subset $\mathcal{K} \subset CC(X)$ to be *convex* if for $A_1, A_2, \ldots, A_n \in \mathcal{K}$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in [0,1]$ with $\sum_{i=1}^n \alpha_i = 1$, we have $\sum_{i=1}^n \alpha_i A_i \in \mathcal{K}$; also a mapping $T : CC(X) \to CC(X)$ is said to be *affine* if $T(\sum_{i=1}^n \alpha_i A_i) = \sum_{i=1}^n \alpha_i T(A_i)$.

Lemma 1. *Let A, B, C, $D \in CC(X)$. We have*

(a) $h(\alpha A, \alpha B) = |\alpha| h(A,B)$, *where α is a scalar;*

(b) $h(A + C, B + D) \le h(A,B) + h(C,D)$;

(c) for each $x^* \in X^*$ and $A_1, A_2, \ldots, A_n \in CC(X)$,
$x^*(\sum_{i=1}^n \alpha_i A_i) = \sum_{i=1}^n \alpha_i x^*(A_i)$,
where $\alpha_1, \alpha_2, \ldots, \alpha_n \in [0,1]$ with $\sum_{i=1}^n \alpha_i = 1$;

(d) $A = B$ *if and only if* $x^*(A) = x^*(B)$ *for each* $x^* \in X^*$.

The proofs of (a), (b) and (c) follow immediately from the definitions and the proof of (d) is a simple application of the Hahn–Banach theorem and shall be omitted.

Next, we let $\mathbb{Z}$ denote the complex plane, $CC(\mathbb{Z})$ denote the collection of all non-empty compact, convex subsets of $\mathbb{Z}$ and h the natural Hausdorff metric on $CC(\mathbb{Z})$. Note that for any $x^* \in X^*$ and $A \in CC(X)$, it follows from the linearity and continuity of x^* that $x^*(A)$ is a non-empty compact, convex subset of $\mathbb{Z}$, i.e. $x^*(A) \in CC(\mathbb{Z})$. We shall now prove the following

Lemma 2. *Suppose A, $B \in CC(X)$, then $h(x^*(A), x^*(B)) \le \|x^*\| h(A,B)$ for each $x^* \in X^*$. Thus $x^* : (CC(X), h) \to (CC(\mathbb{Z}), h)$ is continuous (for simplicity the same h is used to denote different Hausdorff metrics on $CC(X)$ and $CC(\mathbb{Z})$.).*

Proof. Let $r > h(A,B)$. Then $A \subset N(B;r)$ and $B \subset N(A;r)$. Hence for each $a \in A$, there exists $b \in B$ such that $\|a - b\| < r$ and consequently $\|x^*(a) - x^*(b)\| = \|x^*(a-b)\| \leq \|x^*\|\|a-b\| < \|x^*\|r$, which in turn implies that $x^*(A) \subset N(x^*(B); \|x^*\|r)$. Similarly $x^*(B) \subset N(x^*(A); \|x^*\|r)$. Hence $h(x^*(A), x^*(B)) \leq \|x^*\|r$, which implies that $h(x^*(A), x^*(B)) \leq \|x^*\|h(A,B)$ and the proof is complete.

Recall now that the weak topology τ_w on X is defined to be the weakest topology on X which makes each $x^* : (X, \tau_w) \to (\mathbb{Z}, |\ |)$ continuous. Now that we have, by Lemma 2, that each $x^* : (CC(X), h) \to (CC(\mathbb{Z}), h)$ is continuous, we may define $\mathcal{J}_w$ to be the weakest topology on the hyperspace $CC(X)$ such that each $x^* : (CC(X), \mathcal{J}_w) \to (CC(\mathbb{Z}), h)$ is continuous. The notion of weak convergence has been studied by some mathematicians ([3], [5], [6]) and this paper has been inspired by their work; in particular by the paper of F. S. De Blasi and J. Myjak [3]. However, our approach is somewhat different from theirs. We shall use the notation $\mathcal{W}(A; x_1^*, \dots, x_n^*; \varepsilon) = \{B \in CC(X) \mid h(x_i^*B, x_i^*A) < \varepsilon \text{ for } i = 1, 2, \dots, n\}$ to denote a $\mathcal{J}_w$-neighborhood of A in $CC(X)$.

Lemma 3. *Suppose S, $T : (CC(X), \mathcal{J}_w) \to (CC(X), \mathcal{J}_w)$ are continuous and α is a scalar. Then $S + T : (CC(X), \mathcal{J}_w) \to (CC(X), \mathcal{J}_w)$ and $\alpha S : (CC(X), \mathcal{J}_w) \to (CC(X), \mathcal{J}_w)$ are also continuous where $(S+T)(A) = SA + TA$ and $(\alpha S)A = \alpha SA$.*

Proof. To show that $S + T$ is continuous at A we let $\mathcal{W}(SA+TA; x_1^*, x_2^*, \dots, x_n^*; \varepsilon)$ be a $\mathcal{J}_w$-neighborhood of $SA+TA$. Since S is continuous at A, for $\mathcal{W}_1(SA; x_1^*, x_2^*, \dots, x_n^*; \frac{\varepsilon}{2})$ which is a $\mathcal{J}_w$-neighborhood of SA, there exists a $\mathcal{J}_w$-neighborhood $\mathcal{U}(A)$ such that $B \in \mathcal{U}(A)$ implies $SB \in \mathcal{W}_1$. Similarly, for $\mathcal{W}_2(TA; x_1^*, x_2^*, \dots, x_n^*; \frac{\varepsilon}{2})$, there exists a $\mathcal{J}_w$-neighborhood $\mathcal{V}(A)$ such that $B \in \mathcal{V}(A)$ implies $TB \in \mathcal{W}_2$. Consequently for $B \in \mathcal{U} \cap \mathcal{V}$ we have $SB \in \mathcal{W}_1$, $TB \in \mathcal{W}_2$ and it follows from Lemma 1 that

$$
\begin{gathered}
h(x_i^*(SB+TB),\ x_i^*(SA+TA)) = h(x_i^*(SB) + x_i^*(TB), \\
x_i^*(SA) + x_i^*(TA)) \leq h(x_i^*(SB), \\
x_i^*(SA)) + h(x_i^*(TB), x_i^*(TA)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon;
\end{gathered}
$$

i.e., $(S+T)(B) \in \mathcal{W}$ proving that $(S+T)$ is continuous at A. It can be proved in a similar fashion, that $\alpha A : (CC(X), \mathcal{J}_w) \to (CC(X), \mathcal{J}_w)$.

Theorem. *Let X be a Banach space and $\mathcal{K}$ $\mathcal{J}_w$-compact, convex subset of $(CC(X), \mathcal{J}_w)$. Suppose $\mathcal{F}$ is a commutative family of continuous affine mappings of $(\mathcal{K}, \mathcal{J}_w)$ into itself. Then $\mathcal{F}$ has a common fixed point in $\mathcal{K}$.*

Proof. For each $T \in \mathcal{F}$ and each integer n, let $T_n = (\frac{1}{n}) \sum_{k=0}^{n-1} T^k$, where $T^0 = I$ is the identity mapping. It follows that $T_n(\mathcal{K}) \subset \mathcal{K}$. If follows from Lemma 3 that T_n is $\mathcal{J}_w$-continuous and consequently $T_n(\mathcal{K})$ is $\mathcal{J}_w$-compact. The commutativity of $\mathcal{F}$ implies that $T_n(S_m(\mathcal{K}))=S_m(T_n(\mathcal{K})) \subset T_n(\mathcal{K}) \cap S_m(\mathcal{K})$. Consequently, $\{T_n(\mathcal{K}) : n = 1, 2, \dots , T \in \mathcal{F}\}$ is a family of $\mathcal{J}_w$-compact subsets of $\mathcal{K}$ with finite intersection property and hence has non-empty intersection, i.e., $\bigcap T_n(\mathcal{K}) \neq \phi$ where the intersection is taken over $n = 1, 2, \dots$, and $T \in \mathcal{F}$. Let $A_0 \in \bigcap T_n(\mathcal{K})$. We claim that $TA_0 = A_0$ for all $T \in \mathcal{F}$. Assume the contrary, then there exists $T \in \mathcal{F}$ with $TA_0 \neq A_0$ which implies that $h(x^*(TA_0), x^*(A_0)) > 0$ for some $x^* \in X^*$ by Lemma 1. For each n, $A_0 \in T_n(\mathcal{K})$ implies the existence of some $B_n \in \mathcal{K}$ with $A_0 = \frac{1}{n} \sum_{k=0}^{n-1} T^k(B_n)$. T is affine implies that $TA_0 = (\frac{1}{n}) \sum_{k=1}^{n} T^k(B_n)$. Since $x^* : (CC(X), \mathcal{J}_w) \to (CC(\mathbb{Z}), h)$ is continuous and $\mathcal{K}$ is $\mathcal{J}_w$-compact, it follows that $x^*(\mathcal{K})$ is a compact subset of the metric space $(CC(\mathbb{Z}), h)$ and hence totally bounded which in turn implies that diam $(\mathcal{K}) = \sup\{h(x^*(A), x^*(B)) : A, B \in \mathcal{K}\} < \infty$. It follows now from the lemmas that

$$h(x^*(TA_0), x^*(A_0)) = h\left(x^*\left(\frac{1}{n}\sum_{k=1}^{n} T^k B_n\right), x^*\left(\frac{1}{n}\sum_{k=0}^{n-1} T^k B_n\right)\right) =$$
$$\frac{1}{n} h\big(x^*(TB_n) + x^*(T^2 B_n) + \cdots + x^*(T^n B_n),$$
$$x^*(B_n) + x^*(TB_n) + \cdots + x^*(T^{n-1} B_n)\big) \leq \frac{1}{n} h\big(x^*(B_n), x^*(T^n B_n)\big)$$
$$\leq \frac{1}{n}\,\mathrm{diam}(\mathcal{K}).$$

Diam $(\mathcal{K}) < \infty$ and that n is arbitrary implies that $h(x^*(TA_0), x^*(A_0))$=0. This is a contradiction. Thus $TA_0 = A_0$ for all $T \in \mathcal{F}$ and the proof is complete.

Remark. Suppose $\mathcal{K}$ consists of singletons. Then we obtain the version of the Markov–Kakutani Theorem mentioned in the introduction of this paper.

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