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Convergence of semitypes

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Abstract. Convergence of types is a basic tool in limit theory for sums of random variables. In this paper we present the new concept of convergence of semitypes, which is the appropriate generalization for problems concerning domains of semistable attraction. We also show that the growth rate of the sampling sequence $\{k_n\}$ in $a_n^{-1}\mu^{k_n} * \delta(s_n) \Rightarrow \nu$ for a semistable but not stable limit law ν is uniquely determined by μ.

1. Introduction

Convergence of types is a basic tool in probability theory (see for example FELLER [6], VIII.2). Two distributions ν and ν_1 on $\mathbb{R}^1$ are of the same type if $\nu_1 = a\nu * \delta(s)$ for some $a > 0$ and $s \in \mathbb{R}^1$. Here $a\nu\{dx\} = \nu\{a^{-1}\,dx\}$, $*$ denotes convolution and $\delta(s)$ is the point mass at s. The main assertion of the convergence of types theorem is that if for some $b_n > 0$ we have $b_n^{-1}\mu_n * \delta(s_n) \Rightarrow \nu$ nondegenerate and there exists $a_n > 0$ such that $a_n^{-1}\mu_n * \delta(s'_n) \Rightarrow \nu_1$ nondegenerate then $b_n/a_n \to a > 0$ and $\nu_1 = a\nu * \delta(s)$. One important application of convergence of types is to domains of attraction (see for example FELLER [6], XVII.5). We say that μ belongs to the domain of attraction of a stable law ν, and we write $\mu \in \mathrm{DOA}(\nu)$, if there exist $b_n > 0$ and $s_n \in \mathbb{R}^1$ such that $b_n^{-1}\mu^n * \delta(s_n) \Rightarrow \nu$.

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We say that two different infinitely divisible laws ν, ν_1 are of the same semitype if

$$\nu_1 = a\nu^\lambda * \delta(s) \tag{1.1}$$

for some $a > 0$, $s \in \mathbb{R}^1$, and $\lambda > 0$, where ν^λ denotes the λ-fold convolution power of ν, defined in terms of characteristic functions. We say that μ belongs to the domain of semistable attraction of a nondegenerate probability distribution ν if for some sequence of positive integers $k_n \to \infty$ with $k_{n+1}/k_n \to c \geq 1$ there exist $b_n > 0$ and $s_n \in \mathbb{R}^1$ such that

$$b_n^{-1}\mu^{k_n} * \delta(s_n) \Rightarrow \nu. \tag{1.2}$$

In this case we write $\mu \in \mathrm{DOSA}(\nu, c)$. It follows from Kruglov [10] or Mejzler [16] that ν is (b, c) semistable. This means that

$$\nu^c = b\nu * \delta(s) \tag{1.3}$$

for some $b \geq \sqrt{c}$ and some shift $s \in \mathbb{R}$. It is interesting to note that infinitely divisible laws satisfying (1.3) with $s = 0$ were originally introduced by Lévy [11].

Semistable laws and their domains of semistable attraction were generalized to finite dimensional real vector spaces (see Jajte [8] and Łuczak [12]) and to simply connected nilpotent Lie groups (Nobel [17]). An application of Theorem 3.2 in Mejzler [16] or Proposition 7 in Nobel [17] yields that if (1.2) holds and additionally there exist $a_n > 0$, $s'_n \in \mathbb{R}^1$ and positive integers $k'_n \to \infty$ with $k'_{n+1}/k'_n \to c$ such that

$$a_n^{-1}\mu^{k'_n} * \delta(s'_n) \Rightarrow \nu_1 \tag{1.4}$$

for some nondegenerate distribution ν_1 then (1.1) holds, so ν and ν_1 are of the same semitype. This result is then used to show that domains of semistable attraction are either equal or disjoint (see Nobel [17] Corollary 5).

But in many situations one would also like to know a relationship between k_n and k'_n (resp. a_n and a'_n) as in the convergence of types theorem. This is done in this paper. We prove a convergence of semitype theorem (Theorem 2.7 below) which shows that in this situation if ν is semistable and not stable then $k'_n/k_n \to \lambda > 0$ and $b_n/a_n \to a > 0$ as in the classical convergence of types theorem. As a corollary we get that the growth rate of the sampling sequence $\{k_n\}$ in (1.2) is uniquely determined by μ when ν is semistable but not stable. If ν is stable then obviously nothing can be said about $\{k_n\}$.

As a corollary of our approach we derive a representation of the tail function of a measure attracted to a nonnormal semistable law which is the key to a series representation for these laws derived in MEERSCHAERT and SCHEFFLER [14].

2. Convergence of semitypes

Suppose there exist norming constants b_n, shifts s_n, and a sampling sequence $k_n \to \infty$ such that (1.2) holds for some nondegenerate distribution ν. Then the limit measure ν must be infinitely divisible. If one obtains the same limit in (1.2) for any choice of sampling sequence k_n then ν is stable, and we have the usual convergence of types theorem. On the other hand, if the limit depends on an arbitrary sampling sequence then we say that μ belongs to the domain of partial attraction of ν. For a modern approach to domains of partial attraction using quantile constructions see CSÖRGŐ [3] and CSÖRGŐ–DODUNEKOVA [5]. It is possible to construct so-called universal laws, which belong to the domain of partial attraction of every infinitely divisible law (see for example FELLER [6], XVII.9 and CSÖRGŐ–TOTIK [4]). Thus in order to obtain a useful extension of the convergence of types theorem, it is necessary to place some restriction on the sampling sequence. In this section we will restrict our attention to the situation where (1.2) holds for some sampling sequence k_n which satisfies $k_{n+1}/k_n \to c \geq 1$. It follows from Theorem 2.3 of MEJZLAR [16] or from JATJE [8] that if $c = 1$ then ν is stable, and otherwise ν is (b, c) semistable, that is, (1.3) holds. If $b = \sqrt{c}$ then ν is a normal law, and hence stable. In the following we will only consider the case where ν is not stable, so that necessarily $b > \sqrt{c}$.

Suppose then that (1.2) holds for some (b, c) semistable limit law ν which is not stable. In this section, we will prove that if additionally (1.4) holds for some $a_n > 0$ and some sequence of positive integers $k'_n \to \infty$ with $k'_{n+1}/k'_n \to c$ then $b_n/a_n \to a$ and $k'_n/k_n \to \lambda$. In particular ν, ν_1 are of the same semitype.

Theorem 1 of Kruglov [10] or Theorem 4.1 of Mejzlar [16] implies that the Lévy measure ϕ of the nonnormal (b, c) semistable law ν satisfies

$$\phi\{t : |t| > x\} = x^{-\alpha}\theta(\log x) \tag{2.1}$$

where $\alpha = \log c / \log b$ is necessarily in the interval $(0, 2)$, and θ is periodic with period $\log b$. In the case where θ is a constant function, ν is actually stable, but we have assumed that this is not the case. In view of the fact that the left hand side in (2.1) is monotone, we also have

$$\begin{aligned} \theta(y+\delta) &\le e^{\alpha\delta}\theta(y) \\ \theta(y-\delta) &\ge e^{-\alpha\delta}\theta(y) \end{aligned} \tag{2.2}$$

for all $y, \delta > 0$. Scheffler ([18]) shows that if $\mu \in \mathrm{DOSA}(\nu, c)$ and (1.2) holds the tail function $V_0(x) = \mu\{t : |t| > x\}$ satisfies

$$k_n V_0(b_n x) \to x^{-\alpha}\theta(\log x) \tag{2.3}$$

for all $x > 0$ such that x is a continuity point of the limit. Since V_0 is monotone, it follows immediately that if $x_n \to x > 0$ and x is a continuity point of the limit in (2.3) then $k_n V_0(b_n x_n) \to x^{-\alpha}\theta(\log x)$.

The following construction will be useful in the proof of the main result of this section. For all $x > b_1$ define $n(x) = \sup\{n : b_n \le x\}$, and let $b(x) = b_{n(x)}$, $g(x) = x/b(x)$. Convergence of types along with (2.1) yields $b_{n+1}/b_n \to b$, and so $\{g(x) : x > b_1\}$ is relatively compact in $(0, \infty)$ with every limit point lying in the interval $[1, b]$.

Lemma 2.1. *Let $1 \le \lambda < b$. Then for some $x_0 > b_1$, for all $x \ge x_0$ we have*

$$\begin{aligned} g(x/\lambda) &= g(x)/\lambda && \text{if } g(x) \ge \lambda \\ g(x/\lambda) &= g(x)/\lambda \cdot \frac{b_{n(x)}}{b_{n(x)-1}} && \text{if } g(x) < \lambda \\ g(\lambda x) &= \lambda g(x) && \text{if } \lambda g(x) \le b \\ g(\lambda x) &= \lambda g(x) \cdot \frac{b_{n(x)}}{b_{n(x)+1}} && \text{if } \lambda g(x) > b. \end{aligned} \tag{2.4}$$

Proof. Note that $x = g(x) b_n$ for all $x > b_1$, where $n = n(x)$. If $\lambda \le g(x)$ then $n(x/\lambda) = n(x)$ and so $g(x/\lambda) = (x/\lambda)/b_n = (x/b_n)/\lambda = g(x)/\lambda$.

Similarly, if $\lambda g(x) \le b$ then $n(\lambda x) = n(x)$ and so $g(\lambda x) = \lambda x/b_n = \lambda g(x)$. Choose n_0 so that $b_n/b_{n-1} > \lambda$ for all $n \ge n_0$. Since $n(x) \to \infty$ as $x \to \infty$ we may choose x_0 so that $n(x) \ge n_0$ for all $x \ge x_0$. If $\lambda > g(x)$ then $n(x/\lambda) < n(x)$. For all such x, setting $n = n(x)$ as before, we have $x/\lambda \ge b_n/\lambda > b_{n-1}$ and so $n(x/\lambda) = n(x) - 1$. Then $g(x/\lambda) = (x/\lambda)/b_{n-1} = (g(x)b_n)/\lambda b_{n-1} = (g(x)/\lambda)\cdot(b_n/b_{n-1})$. Similarly, if $\lambda g(x) > b$ then $n(\lambda x) > n(x)$. For all such x we have $\lambda x < \lambda b_{n+1} < b_{n+2}$ and so $n(\lambda x) = n(x) + 1$. Then $g(\lambda x) = (\lambda x)/b_{n+1} = \lambda(g(x)b_n)/b_{n+1} = (\lambda g(x)) \cdot (b_n/b_{n+1})$.

Since $\mu \in \mathrm{DOSA}(\nu, c)$ and ν is a nonnormal (b, c) semistable law, a result of Scheffler ([19]) shows that $\int |x|^\rho d\mu(x)$ is finite if $0 < \rho < \alpha$ and infinite if $\rho > \alpha$. Since by (1.4) we also have $\mu \in \mathrm{DOSA}(\nu_1, c)$ where ν_1 is (b', c) semistable, for some $b' \ge \sqrt{c}$, in view of this moment result we see that $\log b'/\log c = \alpha$. Then $b' = b$ and hence $\nu_1^c = b\nu_1 * \delta(s_1)$ for some $s_1 \in \mathbb{R}$. Suppose that ν_1 is stable. Then Theorem 2.1 of Meerschaert and Scheffler [15] implies that $\mu \in \mathrm{DOA}(\nu_1)$ and hence we also have $d_n^{-1}\mu^{k_n} * \delta(s_n') \Rightarrow \nu_1$ for some $d_n > 0$ and some shifts $s_n' \in \mathbb{R}$. Then convergence of types together with (1.2) implies that ν is also stable, which is a contradiction. Hence ν_1 is also a (b, c) semistable law which is not stable. Let ϕ_1 be the Lévy measure of ν_1. Another application of Theorem 1 of Kruglov [10] yields $\phi_1\{t : |t| > x\} = x^{-\alpha}\theta_1(\log x)$ where θ_1 is $\log b$ periodic. As in (2.3) we obtain

$$k_n' V_0(a_n x) \to x^{-\alpha}\theta_1(\log x) \tag{2.5}$$

at continuity points x of the limit. By a simple change of scale, we may assume without loss of generality that $x = 1$ is a continuity point of both limits, and that $\theta(0) = 1$. Now write $V_0(x) = x^{-\alpha}L(x)f(x)$ where

$$\begin{aligned} L(x) &= x^\alpha g(x)^{-\alpha} V_0(b_{n(x)}) \\ f(x) &= g(x)^\alpha V_0(x)/V_0(b_{n(x)}). \end{aligned} \tag{2.6}$$

Lemma 2.2. *The function $L(x)$ defined in (2.6) above is slowly varying.*

Proof. Apply Lemma 2.1. It suffices to show that $L(\lambda x)/L(x) \to 1$ for all $1 \le \lambda < b$ (see for example Seneta [20, p. 8]). When $\lambda g(x) \le b$ we have $L(\lambda x)/L(x) = 1$ and in the remaining case when $\lambda g(x) > b$, setting $n = n(x)$ as before, we have

$$\frac{L(\lambda x)}{L(x)} = (b_n/b_{n+1})^{-\alpha} \frac{k_{n+1}V_0(b_{n+1})}{k_n V_0(b_n)} \frac{k_n}{k_{n+1}}$$

where $b_n/b_{n+1} \to b^{-1}$ and $k_n/k_{n+1} \to c^{-1} = b^{-\alpha}$. Using this along with (2.3) the result follows easily.

Lemma 2.3. *The function $f(x)$ defined in (2.6) above satisfies*

$$\text{(2.7)} \qquad \text{(a)} \quad f(\lambda x) \le \lambda^{\alpha} f(x) E(x) \qquad \text{with } E(x) \to 1 \text{ as } x \to \infty$$

$$\text{(2.8)} \qquad \text{(b)} \quad f(x/\lambda) \ge \lambda^{-\alpha} f(x)\tilde{E}(x) \quad \text{with } \tilde{E}(x) \to 1 \text{ as } x \to \infty$$

for all $1 \le \lambda < b$.

Proof. The proof of (a) and (b) is similar, so we will only prove part (b). Apply Lemma 2.1. In the case $g(x) \ge \lambda$ we have $f(x/\lambda)/f(x) = \lambda^{-\alpha}V_0(x/\lambda)/V_0(x) \ge \lambda^{-\alpha}$ in view of the fact that V_0 is monotone. In the case $g(x) < \lambda$ we have $f(x/\lambda)/f(x) = \lambda^{-\alpha}\tilde{E}(x)V_0(x/\lambda)/V_0(x) \ge \lambda^{-\alpha}\tilde{E}(x)$ by monotonicity of V_0 where, setting $n = n(x)$ as before, we have

$$\tilde{E}(x) = \frac{b_n^{\alpha} V_0(b_n)}{b_{n-1}^{\alpha} V_0(b_{n-1})} \to 1$$

since $b_n/b_{n-1} \to b$ and $V_0(b_n)/V_0(b_{n-1}) \to b^{-\alpha}$ as $x \to \infty$.

Lemma 2.4. *For all $x > 0$, every limit point of $\{f(a_n x)\}$ lies between m, M where*

$$\text{(2.9)} \qquad \begin{aligned} m &= \inf\{\theta(\log y) : 1 \le y \le b\} \\ M &= \sup\{\theta(\log y) : 1 \le y \le b\} \end{aligned}$$

Proof. If $f(a_l x) \to t$ along a subsequence, then we can choose a further subsequence along which $g(a_l x) \to y$ as well. If y is a continuity point of the limit in (2.3) then along this subsequence, writing $n_l = n(a_l x)$, we have

$$\text{(2.10)} \qquad f(a_l x) = g(a_l x)^{\alpha} \frac{V_0(g(a_l x) b_{n_l})}{V_0(b_{n_l})} \to y^{\alpha} \cdot y^{-\alpha}\theta(\log y)$$

and so $t = \theta(\log y)$. This certainly gives $m \le t \le M$. On the other hand, if y is not a continuity point then we will apply Lemma 2.3 to show that $\theta(\log y+) \le t \le \theta(\log y-)$. Note that in view of the definition (2.1) we will always have $\theta(\log y-) \ge \theta(\log y+)$. Choose $\lambda > 1$ arbitrarily close to 1 such that y/λ is a continuity point, with $g(a_l x/\lambda) = g(a_l x)/\lambda$ for all large l. Then $g(a_l x/\lambda) \to y/\lambda$ and it follows as above that $f(a_l x/\lambda) \to \theta(\log t/\lambda)$ along the chosen subsequence. Then Lemma 2.3 implies that $f(a_l x) \le \lambda^\alpha f(a_l x/\lambda)/\tilde{E}(a_l x)$ and so $t \le \lambda^\alpha \theta(\log y/\lambda)$ for $\lambda > 1$ arbitrarily close to 1, and it follows that $t \le \theta(\log y-)$. Next choose $\lambda > 1$ arbitrarily close to 1 such that λy is a continuity point, with $g(\lambda a_l x) = \lambda g(a_l x)$ for all large l. Essentially the same argument as before yields $t \ge \theta(\log y+)$.

Lemma 2.5. *Suppose that for all $x > 0$, either 1 or b is a limit point of $\{g(a_n x)\}$. Then for every $x > 0$, every $\lambda \in [1, b]$ is a limit point of $\{g(a_n x)\}$.*

Proof. Suppose that $g(a_l y) \to 1$ along a subsequence. Given $x > 0$ and $\lambda \in (1, b)$, let $y = x/\lambda$. Then along this same subsequence we have for all large l that $g(a_l x) = g(a_l \lambda y) = \lambda g(a_l y) \to \lambda$. On the other hand, suppose that $g(a_l y) \to b$ along a subsequence. Given $x > 0$ and $\lambda \in (1, b)$, let $y = \lambda x$. Then along this same subsequence we have for all large l that $g(a_l x) = g(a_l y/\lambda) = g(a_l y)/\lambda \to b/\lambda$. Then every $\lambda \in (1, b)$ is a limit point of $\{g(a_n x)\}$, and it follows easily that both $1, b$ are also limit points.

Lemma 2.6. *Both $\{b_n/a_n\}$ and $\{k_n/k'_n\}$ are relatively compact in $(0, \infty)$.*

Proof. First suppose that for some $x > 0$ we have $1 < a \le g(a_n x) \le d < b$ for all $n \ge n_0$. Choose $\varepsilon < \min\{ba/d - 1, 1 - d/ba\}$ and enlarge n_0 if necessary to ensure that $(1-\varepsilon)b \le a_{n+1}/a_n \le (1+\varepsilon)b$ for all $n \ge n_0$. Let $n_l = n(a_l x)$ so that $a_l x = g(a_l x) b_{n_l}$ and observe that for all l sufficiently large to make $n_l \ge n_0$ we have

$$\frac{b_{n_{l+1}}}{b_{n_l}} = \frac{a_{l+1}}{a_l} \frac{g(a_l x)}{g(a_{l+1} x)} \le (1+\varepsilon) b \frac{d}{a} < b^2$$

as well as $b_{n_{l+1}}/b_{n_l} \ge (1-\varepsilon)ba/d > 1$. This second inequality implies that $n_{l+1} \ge n_l + 1$ for all large l. But if $n_{l+1} \ge n_l + 2$ infinitely often then we contradict the first inequality, since then $b_{n_{l+1}}/b_{n_l} \ge b_{n_l+2}/b_{n_l}$ infinitely often, and this last term tends to b^2 in the limit. Hence for all large l

we have $n_{l+1} = n_l + 1$, and this implies that for some integer k we have $n_l = l + k$ for all large l. Then we may write $a_l = b_{l+k} g(a_l x)/x$ for all large l, and since g is bounded and $b_{l+k}/b_l \to b^k$ we see that $\{b_n/a_n\}$ is relatively compact.

Otherwise Lemma 2.5 holds, and in this case we will derive a contradiction. If $x > 0$ is a continuity point of the limit in (2.5') then

$$k_n' a_n^{-\alpha} L(a_n) f(a_n x) \to \theta_1(\log x)$$

in view of Lemma 2.2. Then every limit point of $k_n' a_n^{-\alpha} L(a_n)$ is of the form $\theta_1(\log x)/t$ where t is a limit point of $\{f(a_n x)\}$. Given $\varepsilon > 0$ we can choose $y_1, y_2 \in [1, b]$ such that both are continuity points of $\theta(\log y)$ and $t_1 = \theta(\log y_1) < m + \varepsilon$ while $t_2 = \theta(\log y_2) > M - \varepsilon$. Then (2.10) implies that $f(a_n x)$ has both t_1, t_2 as limit points, and so in view of Lemma 2.4 and the fact that $\varepsilon > 0$ is arbitrarily small we have

$$\limsup_{n\to\infty} k_n' a_n^{-\alpha} L(a_n) = \frac{\theta_1(\log x)}{m}$$
$$\liminf_{n\to\infty} k_n' a_n^{-\alpha} L(a_n) = \frac{\theta_1(\log x)}{M}$$

for any continuity point x. But the left hand side of the above expressions do not depend on x, hence $\theta_1(\log x)$ is a constant. This is a contradiction, since we have assumed that ν_1 is not stable.

Then $\{b_n/a_n\}$ is relatively compact. Suppose $b_n/a_n \to a$ along a subsequence, and choose $x > 0$ such that x is a continuity point of $\theta(\log x)$ and ax is a continuity point of $\theta_1(\log x)$. From (2.3) and (2.5') we have along this same subsequence that

$$\frac{k_n}{k_n'} k_n' V_0\left(a_n \frac{b_n}{a_n} x\right) \to x^{-\alpha}\theta(\log x)$$
$$k_n' V_0\left(a_n \frac{b_n}{a_n} x\right) \to (ax)^{-\alpha}\theta_1(\log ax)$$

which implies that

$$\frac{k_n}{k_n'} \to \frac{x^{-\alpha}\theta(\log x)}{(ax)^{-\alpha}\theta_1(\log ax)} \tag{2.11}$$

along this subsequence. It follows that $\{k_n/k_n'\}$ is relatively compact.

Theorem 2.7 (Convergence of semitypes). *Suppose that for some sequence of positive integers $k_n \to \infty$ with $k_{n+1}/k_n \to c > 1$ there exist $b_n > 0$ and $s_n \in \mathbb{R}^1$ such that $b_n^{-1}\mu^{k_n} * \delta(s_n) \Rightarrow \nu$, a semistable law which is not stable. If there exist $a_n > 0$, $s'_n \in \mathbb{R}^1$, and a sequence of positive integers satisfying $k'_{n+1}/k'_n \to c$ such that $a_n^{-1}\mu^{k'_n} * \delta(s'_n) \Rightarrow \nu_1$ nondegenerate then $b_n/a_n \to a$ and $k'_n/k_n \to \lambda$.*

Proof. Apply Lemma 2.6. If along a subsequence we have $(b_n/a_n, k'_n/k_n) \to (a, \lambda)$ then we have both

$$\begin{aligned} a_n^{-1}\mu^{k'_n} * \delta(s'_n) &\Rightarrow \nu_1 \\ a_n^{-1}b_n b_n^{-1}\mu^{k_n k'_n/k_n} * \delta(\bar{s}_n) &\Rightarrow a\nu^\lambda \end{aligned} \tag{2.12}$$

for some sequence of shifts $\bar{s}_n$, so that $\nu_1 = a\nu^\lambda * \delta(s)$ for some $s \in \mathbb{R}^1$. We know that ν is (b, c) semistable, and although (b, c) are not unique, Theorem 3.2 of Łuczak [12] implies that we can always choose the unique smallest $b > 1$. If (a_0, λ_0) is another limit point of the sequence $(b_n/a_n, k_n/k'_n)$ then $a\nu^\lambda = a_0\nu^{\lambda_0}$, and then it follows from (2.1) that $a_0 = ab^k$ for some integer k. Then all limit points of b_n/a_n are of the form ab^k, and by relative compactness there are only a finite number of these. To show that in fact there is only one limit point, we will argue by contradiction. If there are more than one limit point then there is a subsequence along which $b_n/a_n \to a$ and $b_{n+1}/a_{n+1} \to ab^k$ for some $k \neq 0$. But since $a_{n+1}/a_n \to b$ and $b_{n+1}/b_n \to b$ we must have $b_{n+1}/a_{n+1} \sim b_n/a_n$, which is a contradiction. So $b_n/a_n \to a$, and then it follows by the same argument as for (2.11) that k'_n/k_n converges, and so we must have $k'_n/k_n \to \lambda$. This proves the direct half of the theorem, and the converse follows immediately from (2.12).

In the theorem above, the assumption that $b_n^{-1}\mu^{k_n} * \delta(s_n) \Rightarrow \nu$ for some $k_{n+1}/k_n \to c > 1$ already implies that ν is either stable or semistable. If ν is not stable then neither is ν_1. If ν is stable, then so is ν_1, and in this case we can say nothing about the behavior of the sequence $\{k_n/k'_n\}$ or the sequence $\{b_n/a_n\}$. But if ν is semistable and not stable we have shown:

Corollary 2.8. *Suppose that $\mu \in \mathrm{DOSA}(\nu, c)$ for some nondegenerate semistable law ν which is not stable. Then the growth rate of the sampling sequence $\{k_n\}$ is uniquely determined by μ, meaning that if*

$$a_n^{-1}\mu^{k_n} * \delta(s_n) \Rightarrow \nu \quad \text{for some } \{k_n\} \text{ with } \frac{k_{n+1}}{k_n} \to c$$

and

$$b_n^{-1}\mu^{k'_n} * \delta(s'_n) \Rightarrow \nu \quad \text{for some } \{k'_n\} \text{ with } \frac{k'_{n+1}}{k'_n} \to c$$

then there exists an integer j such that $k_{n+j}/k'_n \to 1$ as $n \to \infty$.

Proof. Convergence of semitypes yields that $k'_n/k_n \to \lambda$ where $\nu = a\nu^\lambda * \delta(s)$ for some $\lambda, a > 0$ and some $s \in \mathbb{R}$. Since ν is (b, c) semistable and not stable we must have $\lambda = c^j$ for some integer j. The result follows easily.

3. Remarks

In order for a probability measure μ to belong to some stable domain of attraction, the tails of this measure must satisfy a regular variation condition. In particular, in order that $b_n^{-1}\mu^n * \delta(s_n) \to \nu$ nondegenerate nonnormal we must have $V_0(t) = \mu\{x : |x| > t\}$ regularly varying with some index $-\alpha \in (0, 2)$. The number α is the index of the stable limit law ν. The norming constants can be constructed from the tail function by letting $b_n = \sup\{t : nV_0(t) \geq 1\}$. Our results illuminate the tail behavior of a probability measure μ which belongs to some domain of semistable attraction, as well as the relation between the tails and the norming constants. Suppose that (1.2) holds. Then arguing as in the proof of Lemma 2.6 it is easy to see that $k_n b_n^{-\alpha} L(b_n) f(b_n x) \to \theta(\log x)$ for continuity points $x > 0$ of the limit. In particular, since $f(b_n) = 1$ by definition, we have $k_n b_n^{-\alpha} L(b_n) \to 1$. Define $R(t) = t^{-\alpha}L(t)$ regularly varying. Then we can always take $b_n = \sup\{t : k_n R(t) \geq 1\}$. The representation

$$V_0(x) = x^{-\alpha} L(x) f(x) \tag{3.1}$$

expresses the tail of μ as the product of a regularly varying function and a bounded, asymptotically log periodic function f. This representation is one of the key ingrediences of a series representation for nonnormal semistable laws considered in MEERSCHAERT and SCHEFFLER [14]. In fact we have $f(b_n x) \to \theta(\log x)$ at continuity points, together with the inequalities of Lemma 2.3 at discontinuity points. Since continuity points are dense, this gives sharp bounds on the behavior of f. If the Lévy measure ϕ of the limit ν is continuous, then we can write

$$V_0(x) = x^{-\alpha} L(x)[\theta(\log g(x)) + h(x)] \tag{3.2}$$

where L is slowly varying and $h(x) \to 0$ as $x \to \infty$. This extends a recent result of GRINEVICH and KHOKHLOV [7]. In the case where ϕ has jumps, it seems that no such representation is possible.

A stable measure ν satisfies $\nu\{x : |x| > t\} = Cx^{-\alpha}$ for some $C > 0$. For a (b, c) semistable measure we obtain from (2.6) that $\nu\{x : |x| > t\} = x^{-\alpha} f(t)$ where f is a bounded, asymptotically log periodic function. In this case we may take $b_n = b^n$ and $k_n = c^n$, so that $g(x) = x/b^{n(x)}$. If the Lévy measure ϕ of ν is continuous then $\nu\{x : |x| > t\} = x^{-\alpha}[\theta(\log x) + h(x)]$ where $h(x) \to 0$. If $\nu_1 = a\nu^\lambda * \delta(s)$, so that both measures are of the same semitype, then it follows from (2.3) and Theorem 2.7 that $\theta_1(\log ax) = \lambda a^\alpha \theta(\log x)$. Alternatively, this follows from (2.1) and the Lévy representation. Then two different semistable laws which are of the same semitype, but not the same type, are related by a phase shift in the log periodic portion of the tail.

From a "statistical point of view", domains of semistable attraction are interesting because they place a weaker restriction on the tails of a measure μ than domains of attraction. In the latter case, one can construct norming constants from the empirical version of the tail function $V_0(t)$, using the fact that $V_0(b_n) \sim n$. For domains of semistable attraction, it is necessary to first construct the norming constants b_n, which capture the log periodic behavior of the tail. The functions L and f can then be obtained using an empirical estimate of the tail $V_0(x)$, and the sampling constants can be chosen to satisfy $k_n b_n^{-\alpha} L(b_n) \sim 1$. The problem of constructing a sequence b_n from the data is still open.

Another interesting open problem is the multivariable analogue of domains of semistable attraction. Suppose that (1.2) holds where μ, ν are full probability measure on $\mathbb{R}^d$ and b_n are linear operators. Then we say that ν is operator semistable and that μ belongs to its generalized domain of semistable attraction. The structure of generalized domain of semistable attraction is discussed in MEERSCHAERT and SCHEFFLER [15]. The problem of convergence of semitypes, especially the assertion on the growth rate of the sampling sequence $\{k_n\}$, in the operator semistable case is still an open and challenging problem.

References

[1] A. Araujo and E. Giné, The Central Limit Theorem for Real and Banach Valued Random Variables, *Wiley*, *New York*, 1980.

[2] N. Bingham, C. Goldie and J. Teugels, Regular Variation, Encyclopedia of Mathematics and its Applications, vol. 27, *Cambridge University Press*, 1987.

[3] S. Csörgő, A probabilistic approach to domains of partial attraction, *Advances in Applied Math.* **11** (1990), 282–327.

[4] S. Csörgő and V. Totik, Quantile constructions for Khinchin's and Pruitt's theorems and for Doeblin's universal laws, *Advances in Applied Math* **14** (1993), 226–246.

[5] S. Csörgő and R. Dodunekova, The domain of partial attraction of an infinitely divisible law without a normal component, *Advances in Applied Math* **16** (1995), 184–205.

[6] W. Feller, An Introduction to Probability Theory and its Applications, vol. II (2nd edn), *Wiley*, *New York*, 1971.

[7] I. Grinevich and Y. Khokhlov, The domains of attraction of semistable laws, preprint, 1994.

[8] R. Jajte, Semistable probability measures on $\mathbb{R}^N$, *Studia Math.* **61** (1977), 29–39.

[9] Z. J. Jurek and J. D. Mason, Operator-limit distributions in probability theory, *Wiley* & *Sons*, *New York*, 1993.

[10] V. M. Kruglov, On the extension of the class of stable distributions, *Theory Probab. Appl.* **17** (1972), 685–694.

[11] P. Lévy, Théorie l'addition des variables aléatoires, *Gauthier-Villars*, *Paris*, 1937.

[12] A. Łuczak, Elliptical symmetry and characterization of operator-stable and operator-semistable measures, *Ann. Probab.* **12** (1984), 1217–1223.

[13] M. Meerschaert, Regular variation and generalized domains of attraction in $\mathbb{R}^k$, *Stat. Probab. Lett.* **18** (1993), 233–239.

[14] M. Meerschaert and H. P. Scheffler, Series Representation for semistable laws and their domains of semistable attraction, *J. Theoretical Probab.* **9** (1996), 931–959.

[15] M. Meerschaert and H. P. Scheffler, The structure of generalized domains of semistable attraction, *Stat. Probab. Lett.* **33** (1997), 367–372.

[16] D. Mejzler, On a certain class of infinitely divisible distributions, *Israel Journal of Math.* (1973), 1–19.

[17] S. Nobel, Limit theorems for probability measures on simply connected nilpotent Lie groups, *J. Theoretical Probab.* **4** (1991), 261–284.

[18] H. P. Scheffler, Domains of semi-stable attraction of nonnormal semi-stable laws, *J. Multivariate Anal.* **51** (1994), 432–444.

[19] H. P. Scheffler, Moments of distributions attracted to operator semi-stable laws, *Stat. Prob. Lett.* **24** (1995), 187–192.

[20] E. Seneta, Regularly Varying Functions, Lecture Notes in Mathematics, vol. 508, *Springer*, *Berlin*, 1976.

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