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On the Cartan connection for a class of generalized Lagrange spaces

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Abstract. In this paper we determine the Cartan connection of the generalized Lagrange space (M, a_{ij}) with metric tensor $a_{ij} = \theta(x,y)\overset{\circ}{h}_{ij} + \psi\big(F^2(x,y)\big)e^{2\alpha(x)}\overset{\circ}{l}_i\overset{\circ}{l}_j$, which includes particular metrics frequently encountered in applications in Physical and Biological problems. The results obtained are employed on the problem of equivalence of geodesics for certain metrics.

Introduction

The geometrical theory of spaces endowed with metrics having the forms

$$\text{(i)} \quad a_{ij} = e^{2\alpha} g_{ij} \quad \text{or (ii)} \quad \overline{g}_{ij} = g_{ij} + \beta \overset{\circ}{y}_i \overset{\circ}{y}_j$$

where g_{ij} is the metric tensor of a Finsler manifold and α, β are functions of position and direction has been studied in many papers [15], [19], [2], [4], [5], [12], [13], [14], [20], [21], [22]. Of course, these spaces used as geometrical models for gravitation and electromagnetism generally are not Lagrange manifolds and new geometrical ideas, initiated in [18] have been used. It is clear that such metrics are useful for a constructive axiomatic theory of general relativity based on conditions formulated by Ehlers, Pirani and Schild ([10], [19]), as well as for a geometrical model construction of relativistic optic [7], [8], [9], [15], [18], [19], [20], [21]. It seems that these metrics can be successfully used for biological models as well,

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as already given in the particular case of Finsler manifolds [3], or more general for some special Lagrangians [2], [4], [5].

The main result is given in Section 3 where the Cartan connection for a special GL-space is obtained. This is a substantial extension of similar results from Riemannian geometry (Levi-Civita connection) and Finsler geometry (Cartan connection).

The Cartan connection was obtained in [2] for homogeneous case and more general for φ-Lagrangians (different proof) [4], [5]. However, for these GL-spaces the obvious modifications of standard techniques (e.g. Finsler, m-homogeneous, φ-Lagrange) do not work.

The canonical d-connection used for the geometrical study of generalized Lagrange manifolds with metrics of type (i) or (ii) have in general nonvanishing deflection tensor. This is an unpleasant situation from at least two points of view: the energy functional is nonconservative along the paths of nonlinear connection, and the horizontal lift of these paths are not h-paths. In the present work, making a conformal change of metric (ii), we will find other metrics without the inconveniences alluded to above.

1. Preliminaries

Let M be a C^∞-real n-dimensional differentiable manifold and $\pi : TM \to M$ its tangent bundle.

If (x^i), $i = 1, \dots, n$ is a local system of coordinates on a domain U of a chart on M then (x^i, y^i) is the induced local system of coordinates on $\pi^{-1}(U)$ in TM.

We put $\partial_i := \partial/\partial x^i$ and $\dot{\partial}_i := \partial/\partial y^i$.

A change of coodinates on TM is given by

$$x^{i'} = x^{i'}(x^i), \quad y^{i'} = (\partial_i x^{i'})y^i, \quad \text{rank }(\partial_i x^{i'}) = n. \tag{1.1}$$

A *nonlinear connection* on M is defined by a distribution HTM over TM supplementary to the vertical distribution i.e. the kernel of the differential of π.

In HTM there exists a local frame defined by the local vector fields

$$\delta_i := \partial_i - N_i^k \dot{\partial}_k \tag{1.2}$$

(N_i^k) are called *the coefficients of the nonlinear connection*. We can introduce some special tensor fields, called *d-tensor field* as an object of algebra spanned by $\{1, \delta_i, \dot{\partial}_i\}$ over the ring $\mathcal{F}(TM)$ of smooth real valued functions on TM.

For a change of coordinates given by (1.1) the components of a d-tensor are transformed in exactly the same way as a tensor on M, in spite of y^i dependence.

Definition 1.1 ([18]). A *generalized Lagrange space* (briefly GL-space) is a pair $\big(M, a_{ij}(x,y)\big)$ where a_{ij} is a symmetric and nondegenerate d-tensor field on $\overset{\circ}{T}M = TM \backslash \{0_x, x \in M\}$.

If there exists a smooth function $L : TM \to \mathbb{R}$ such that $a_{ij} = 1/2\ \dot{\partial}_i \dot{\partial}_j L$, then the pair (M, a_{ij}) or (M, L) is called a Lagrange space.

A Finsler space is a particular Lagrange space and of course, a particular GL-space. In this case L is smooth on $\overset{\circ}{T}M$, positively homogeneous and the quadratic form $a_{ij}(x,y)$ is positive definite.

The geometry of GL-space can be developed by the same methods as those employed in Lagrange or Finsler space [18], [6], [16].

In a GL-space endowed with the nonlinear connection (N_j^i) we can introduce so called *d-connection*.

$C\Gamma(N_j^i) = (L_{jk}^i, C_{jk}^i)$, which allows us to define the *h- and v-covariant derivative* like for Finsler spaces. For example, the h- and v-covariant derivative of the metric tensor a_{ij}, denoted by "$_|$" and respectively "$\big|$" are given by

$$a_{ij|k} = \delta_k a_{ij} - L_{ki}^s a_{sj} - L_{kj}^s a_{is}; \quad a_{ij}\,|_k = \dot{\partial}_k a_{ij} - C_{ki}^s a_{sj} - C_{kj}^s a_{is}.$$

The following result (see [18], page 186) will be used:

Theorem 1.2. *Let (M, a_{ij}) be a GL-Lagrange space and (N_j^i) a fixed nonlinear connection. Then there exist only one d-connectionq $C\Gamma(N_j^i) = (L_{jk}^i, C_{jk}^i)$ with the following properties*

1° $\quad a_{ij|k} = 0 \quad$ (*h-metrical*), $\qquad 2^\circ$ $\quad a_{ij}|_k = 0 \quad$ (*v-metrical*),

3° $\quad T_{jk}^i = L_{jk}^i - L_{kj}^i = 0, \qquad 4^\circ$ $\quad S_{jk}^i = C_{jk}^i - C_{kj}^i = 0.$

The coefficients of this d-connection are given by

$$L^i_{jk} = \tfrac{1}{2}\, a^{ip}(\delta_j a_{pk} + \delta_k a_{jp} - \delta_p a_{jk})$$
$$C^i_{jk} = \tfrac{1}{2}\, a^{ip}(\dot{\partial}_j a_{pk} + \dot{\partial}_k a_{jp} - \dot{\partial}_p a_{jk})$$

This connection will be called *the canonical d-connection* of the GL-space (M, a_{ij}).

A systematic presentation of the geometry of a GL-space is given in [18]. Also, throughout this paper we shall use the usual set up for general theory of Finsler manifolds. We follow closely the exposition of basic Finsler geometry in [6], [16].

2. Deflection free connection on GL-spaces

Let $C\overset{\circ}{\Gamma} = (\overset{\circ}{N}{}^i_j, F^i_{jk}, \overset{\circ}{C}{}^i_{jk})$ be the Cartan connection of the Finsler space $F^n = \big(M, F(x,y)\big)$ and $g_{ij} := 1/2\ \dot{\partial}_i \dot{\partial}_j F^2$ the metric tensor. In general, the geometrical objects related to the Finsler space will be indicated by "0" on the top.

For example $\overset{\circ}{y}_i = g_{ij} y^j$, $\overset{\circ}{\ell}_i = \overset{\circ}{y}_i / F$, $\overset{\circ}{\ell}{}^i = y^i / F$, $\overset{\circ}{h}_{ij} = g_{ij} - \overset{\circ}{\ell}_i \overset{\circ}{\ell}_j$. We consider a GL-space whose metric tensor is of the form

$$a_{ij} = e^{2\alpha} \overline{g}_{ij}, \quad \overline{g}_{ij} = g_{ij} + \beta \overset{\circ}{y}_i \overset{\circ}{y}_j, \tag{2.1}$$

where $\alpha = \alpha(x,y)$, $\beta = \beta(x,y)$ are smooth scalar functions such that $1 + \beta F^2 > 0$. It is easy to see that the reciprocal components of a_{ij} are given by

$$a^{ij} = e^{-2\alpha} \overline{g}^{ij}, \quad \overline{g}^{ij} = g^{ij} - \frac{\beta}{1 + F^2 \beta} y^i y^j. \tag{2.2}$$

We put

$$U^i_{jk} := \delta^i_k \alpha_j + \delta^i_j \alpha_k - \overline{g}_{jk} \overline{g}^{ih} \alpha_h + \tfrac{1}{2}\ \overline{g}^{ih} (\beta_j \overset{\circ}{y}_h \overset{\circ}{y}_k + \beta_k \overset{\circ}{y}_j \overset{\circ}{y}_h - \beta_h \overset{\circ}{y}_j \overset{\circ}{y}_k) \tag{2.3}$$

$$\dot{U}^i_{jk} := \delta^i_k \dot{\alpha}_j + \delta^i_j \dot{\alpha}_k - \overline{g}_{jk} \overline{g}^{ih} \dot{\alpha}_h + \tfrac{1}{2}\ \overline{g}^{ih} (\dot{\beta}_j \overset{\circ}{y}_h \overset{\circ}{y}_k + \dot{\beta}_k \overset{\circ}{y}_j \overset{\circ}{y}_h - \dot{\beta}_h \overset{\circ}{y}_j \overset{\circ}{y}_k) \tag{2.4}$$

where $\alpha_i := \delta_i \alpha$, $\dot{\alpha}_i := \dot{\partial}_i \alpha$. We have

Proposition 2.1. *Let $C\overset{\circ}{\Gamma} = (\overset{\circ}{N}{}^i_j, F^i_{jk}, \overset{\circ}{C}{}^i_{jk})$ be the Cartan connection of the Finsler space $F^n = \big(M, F(x,y)\big)$. The coefficients of the canonical d-connection $C\Gamma(\overset{\circ}{N}{}^i_j) = (L^i_{jk}, C^i_{jk})$ of the GL-space (M, a_{ij}) are given by*

$$L^i_{jk} = F^i_{jk} + U^i_{jk} \tag{2.5}$$

$$C^i_{jk} = \overset{\circ}{C}{}^i_{jk} + \dot{U}^i_{jk} + \frac{\beta}{1+\beta F^2}\, g_{jk} y^i. \tag{2.6}$$

Proof. We can apply Theorem 1.2 or we look for L^i_{jk} in the following form:

$$L^i_{jk} = F^i_{jk} + A^i_{jk}$$

where A^i_{jk} is a symmetric d-tensor field.

From $a_{ij|k} = 0 \iff a_{\underset{\circ}{ij|k}} - a_{hj} A^h_{ik} - a_{ih} A^h_{jk} = 0$ using the Christoffel process, after a tedious calculation we obtain (2.5). Similarly, we get (2.6).

For the d-connection $C\Gamma(\overset{\circ}{N}{}^i_j) = (L^i_{jk}, C^i_{jk})$ we put the following questions:

a) When the horizontal lift of the paths of the nonlinear connection $(\overset{\circ}{N}{}^i_j)$, (in fact the geodesics of the Finsler spaces, parametrized by arc length) are h-paths? (This is equivalent with $D^i_0 = L^i_{00}$.)

b) For which α and β the deflection tensor, D^i_j, vanishes?

Theorem 2.2. *Let $C\Gamma(\overset{\circ}{N}{}^i_j) = (L^i_{jk}, C^i_{jk})$ the canonical d-connection given by* (2.5), (2.6) *and $\mathcal{E} = a_{ij} y^i y^j$ the energy function. Then*

(i) $\mathcal{E}_{|k} = 0 \iff 2\alpha = \sigma - \log\,(1 + F^2\beta)$ *with* $\sigma_k = 0 \iff D^i_0 = 0$

(ii) $D^i_k = 0 \iff 2\alpha = \sigma - \log\,(1 + F^2\beta)$ *with* $\sigma_k = 0$ *and* $\beta_0 := y^i \beta_i = 0$.

Proof. (i) The h-covariant derivatie of $\mathcal{E} = e^{2\alpha}(1+\beta F^2)F^2$ is

$$\mathcal{E}_{|k} = e^{2\alpha} F^2 \big(2(1+\beta F^2)\alpha_k + F^2 \beta_k\big). \tag{2.7}$$

Thus

$$\mathcal{E}_{|k} = 0 \iff 2\alpha_k = -\frac{\beta_k F^2}{1+\beta F^2} \iff 2\alpha = \sigma - \log\,(1+\beta F^2)$$

where $\sigma = \sigma(x,y)$ and $\sigma_k = 0$.

On the other hand

$$D^i_j = L^i_{j0} - \overset{\circ}{N}{}^i_j = U^i_{j0}.$$

That is

$$\overline{g}_{ih} D^h_j = (\overline{g}_{ij}\alpha_k + \overline{g}_{ik}\alpha_j - \overline{g}_{jk}\alpha_i)y^k + \tfrac{1}{2}\,(\beta_k \overset{\circ}{y}_i \overset{\circ}{y}_j + \beta_j \overset{\circ}{y}_i \overset{\circ}{y}_k - \beta_i \overset{\circ}{y}_j \overset{\circ}{y}_k)y^k.$$

Taking (2.1) into account we get

$$\begin{aligned} \overline{g}_{ih} D^h_j = \big((\beta\alpha_k + \tfrac{1}{2}\,\beta_k)\overset{\circ}{y}_i\overset{\circ}{y}_j + (\beta\alpha_j + \tfrac{1}{2}\,\beta_j)\overset{\circ}{y}_i\overset{\circ}{y}_k \\ - (\beta\alpha_i + \tfrac{1}{2}\,\beta_i)\overset{\circ}{y}_j\overset{\circ}{y}_k\big)y^k \\ + (g_{ij}\alpha_k + g_{ik}\alpha_j - g_{jk}\alpha_i)y^k. \end{aligned} \tag{2.8}$$

Using $\alpha_k = -\frac{F^2\beta_k}{2(1+\beta F^2)}$ we obtain after same computations

$$\overline{g}_{ih} D^h_j = \frac{\beta_0}{2(1+\beta F^2)}\,(\overset{\circ}{y}_i\overset{\circ}{y}_j - F^2 g_{ij}). \tag{2.9}$$

Transvecting by y^j in (2.8) we get $D^i_0 = 0$. Now, suppose that $D^i_0 = 0$. From (2.8) we have

$$2\alpha_o\overset{\circ}{y}_i - \alpha_i F^2 + (2\beta\alpha_0 + \beta_0)F^2\overset{\circ}{y}_i - (\beta\alpha_i + \tfrac{1}{2}\,\beta_i)F^4 = 0 \tag{2.10}$$

Transvecting by y^i we get

$$2\alpha_0 = -\frac{\beta_0 F^2}{1+\beta F^2}.$$

Using again (2.10) we obtain

$$2\alpha_k = -\frac{\beta_k F^2}{1+\beta F^2} \tag{2.11}$$

and (i) follows.

(ii) If $D^i_k = 0$ then $D^i_0 = 0$ and using (i) we get $2\alpha = \sigma - \log(1 + F^2\beta)$ with $\sigma_k = 0$. On the other hand making use of (2.9) we obtain $\beta_0 = 0$. The converse implication follows from (2.8) and (2.9).

Corollary 2.3. *In the* GL*-space with metric tensor* (2.1) *horizontal lift of the paths of the nonlinear connection* $\overset{\circ}{N}{}^i_j$ *(the same with the geodesics of* $F^n = \big(M, F(x,y)\big)$ *parametrized by arc length) are* h*-paths of* $C\Gamma(\overset{\circ}{N}{}^i_j) = (L^i_{jk}, C^i_{jk})$ *iff* $2\alpha_k = \frac{\beta_k F^2}{1+\beta F^2}$.

From the Theorem 2.2 is clear that starting with the Cartan connection $C\overset{\circ}{\Gamma}$ of the Finsler space $F^n = \big(M, F(x,y)\big)$ the deflection tensor of the canonical d-connection of GL-space (M, a_{ij}) vanishes iff

$$a_{ij} = e^{\sigma - \log(1+F^2\beta)}(g_{ij} + \beta \overset{\circ}{y}_i \overset{\circ}{y}_j) \tag{2.12}$$

where $\sigma = \sigma(x,y)$ with $\sigma_k = 0$. We can write (2.12) after changing notation to the following form

$$a_{ij} = e^{\sigma}\big(\theta g_{ij} + (1-\theta)\overset{\circ}{\ell}_i \overset{\circ}{\ell}_j\big) \tag{2.13}$$

where $\theta = \theta(x,y)$, $\theta \neq 0$ and $\sigma_k = 0$.

These results have suggested we give up use at the start of the Cartan nonlinear connection $\overset{\circ}{N}{}^i_j$ and look for the Cartan connection of more general metrics than (2.13).

Remark. For the GL-manifold $(M, \overline{g}_{ij})$ the deflection tensor of $C\Gamma(\overset{\circ}{N}{}^i_j) = (L^i_{jk}, C^i_{jk})$ is null iff $\beta_k = 0$. Then a conformal change of this metric leads to a new GL-space for which the horizontal lift of the geodesics of its associated Finsler space (parametrized by arc length) are h-paths, in less restrictive conditions.

3. The Cartan connection for some special GL-space

We consider the GL-space (M, a_{ij}) with metric tensor given by

$$a_{ij} = \theta g_{ij} + \big(\psi(F^2)e^{2\alpha} - \theta\big)\overset{\circ}{\ell}_i \overset{\circ}{\ell}_j \tag{3.1}$$

where

$$\theta = \theta(x,y), \quad \alpha = \alpha(x) \quad \text{are smooth scalar functions}, \quad \theta \neq 0,$$

and $\psi : (0,\infty) \to (0,\infty)$ is a smooth function such that $\psi(t) + t\psi'(t) \neq 0$ $\forall\, t \in (0,\infty)$.

It is easy to check that the reciprocal components of a_{ij} are given by

$$a^{ij} = \frac{1}{\theta}\, g^{ij} + \left(\frac{1}{\psi}\, e^{-2\alpha} - \frac{1}{\theta}\right) \overset{\circ}{\ell}{}^i \overset{\circ}{\ell}{}^j. \tag{3.2}$$

Using the angular metric tensor $\overset{\circ}{h}_{ij} = g_{ij} - \overset{\circ}{\ell}_i \overset{\circ}{\ell}_j$ we can rewrite (3.1) and (3.2) in the following simple forms:

$$a_{ij} = \theta \overset{\circ}{h}_{ij} + \psi e^{2\alpha} \overset{\circ}{\ell}_i \overset{\circ}{\ell}_j \tag{3.1$'$}$$

$$a^{ij} = \frac{1}{\theta}\, \overset{\circ}{h}{}^{ij} + \frac{1}{\psi}\, e^{-2\alpha} \overset{\circ}{\ell}{}^i \overset{\circ}{\ell}{}^j \tag{3.2$'$}$$

where $\overset{\circ}{h}{}^{ij} = g^{ip} g^{jq} \overset{\circ}{h}_{pq} = g^{ij} - \overset{\circ}{\ell}{}^i \overset{\circ}{\ell}{}^j$.

The metric tensor a_{ij} is quite general and many other particular metrics used in different papers can be obtained from it.

Examples. 1. For $\psi = 1$, $\alpha = 0$, $\theta = c$, $g_{ij}(x,y) = \gamma_{ij}(x)$ (Riemannian metric)

$$a_{ij} = c\gamma_{ij} + (1 - c) \overset{\circ}{\ell}_i \overset{\circ}{\ell}_j \tag{3.3}$$

which has been used in [7], [8], [21] in problems of post-Newtonian estimation, and also in [22].

2. For $\psi(t) = \frac{1}{c^2}\, t + 1$ ($c \in \mathbb{R}^*$), $\alpha = 0$, $\theta = 1$ the metric tensor is given by

$$a_{ij} = g_{ij} + \frac{1}{c^2}\, \overset{\circ}{y}_i \overset{\circ}{y}_j \tag{3.4}$$

which have been used in [15], [9], [21], [18] for applications in relativistic geometric optics.

3. For $\alpha = 0$, $\theta(x,y) = \varphi'\big(F^2(x,y)\big)$, $\psi = 2\varphi''(t)t + \varphi'(t)$ we obtain

$$a_{ij} = \varphi' g_{ij} + 2\varphi'' \overset{\circ}{y}_i \overset{\circ}{y}_j \tag{3.5}$$

which is nothing but the metric tensor of a φ-Lagrange [3], [4] which have been used for applications in Biology and Physics.

In particular for $\varphi(t) = t^{m/2}$ we get

$$a_{ij} = \frac{m}{2} F^{m-2}\big(g_{ij} + (m-2)\overset{\circ}{\ell}_i\overset{\circ}{\ell}_j\big) \tag{3.6}$$

which is the metric tensor for m-homogeneous Lagrange space [2], [12], [13].

4. If $\theta = 1$, $\psi = 1$, $2\alpha(x) = \ell\mathrm{n}\big(1 - \overline{\alpha}(x)\big)$ thus

$$a_{ij} = g_{ij} - \overline{\alpha}(x)\overset{\circ}{\ell}_i\overset{\circ}{\ell}_j \tag{3.7}$$

and we get the GL-space studied in [14].

5. For $\theta = \overline{\theta}(x,y)e^{2\alpha(x)}$, $\overline{\theta} \neq 0$

$$a_{ij} = e^{2\alpha(x)}[\overline{\theta}g_{ij} + (\psi - \overline{\theta})\overset{\circ}{\ell}_i\overset{\circ}{\ell}_j] \tag{3.8}$$

which is a conformal change of metric $\overline{g}_{ij} = \overline{\theta}g_{ij} + (\psi - \overline{\theta})\overset{\circ}{\ell}_i\overset{\circ}{\ell}_j$ more general as in [11].

The central problem of this paper is to determine the Cartan connection of the GL-space (M, a_{ij}) where a_{ij} is given by (3.1) or (3.1)$'$.

Following [6], [16] the Cartan connection is a triple $C\Gamma{=}(N^i_j, L^i_{jk}, C^i_{jk})$ which must verify the following axioms (Matsumoto's axioms).

1° $a_{ij|k} = 0$ 2° $a_{ij}|_k = 0$

3° $D^i_j = L^i_{j0} - N^i_j = 0$ 4° $T^i_{jk} = L^i_{jk} - L^i_{kj} = 0$

5° $S^i_{jk} = C^i_{jk} - C^i_{kj} = 0.$

We shall use the following notations:

$$y_i := a_{ij}y^i, \qquad \|y\|^2 := y_iy^i = a_{ij}y^iy^j = \mathcal{E} \tag{3.9}$$

$$\ell_i := y_i/\|y\|, \tag{3.10}$$

$\ell^i := y^i/\|y\|$ (normalized supporting element)

$$h_{ij} := a_{ij} - \ell_i\ell_j \quad \text{(angular metric tensor).} \tag{3.11}$$

The following properties are immediate

$$y_i = \varphi e^{2\alpha} \overset{\circ}{y}_i; \quad \|y\|^2 = \psi F^2 e^{2\alpha} \tag{3.12}$$

$$\ell_i = a_{ij}\ell^j; \qquad \ell_i \ell_j = \psi e^{2\alpha} \overset{\circ}{\ell}_i \overset{\circ}{\ell}_j \tag{3.13}$$

$$h_{ij} = \theta \overset{\circ}{h}_{ij}; \qquad h^i_j := a^{ik} h_{kj} = \overset{\circ}{h}{}^i_j. \tag{3.14}$$

Theorem 3.1. *In the GL-space (M, a_{ij}) with a_{ij} given by (3.1) there exists a unique d-connection $C\Gamma = (N^i_j, L^i_{jk}, C^i_{jk})$ which satisfies the Matsumoto's axioms.*

Proof. We look for $C\Gamma$ which verifies $1^\circ - 5^\circ$. From 1° and 3° we get $\|y\|^2_{|k} = 0$ and making use of (3.12) we obtain

$$F^2_{|k} = \frac{-2\alpha_k \psi F^2}{\psi + \psi' F^2}, \tag{3.15}$$

where $\alpha_k := \partial_k \alpha$.

From (3.1) using (3.13) we get the following expression for the metric tensor of F^n

$$g_{ij} = \frac{1}{\theta}\, a_{ij} + \Big(\frac{e^{-2\alpha}}{\psi} - \frac{1}{\theta}\Big)\, \ell_i \ell_j. \tag{3.16}$$

We have, making use of 1° and 3°

$$\begin{aligned} g_{ij|k} &= -\frac{\theta_k}{\theta}\, a_{ij} + \frac{\theta_k}{\theta^2} \ell_i \ell_j - \frac{2e^{-2\alpha}}{\psi} \alpha_k \ell_i \ell_j - \frac{\psi' e^{-2\alpha}}{\psi^2} F^2_{|k} \ell_i \ell_j \\ &= -\frac{\theta_k}{\theta^2} h_{ij} - \frac{2e^{-2\alpha}}{\psi} \alpha_k \ell_i \ell_j + \frac{\psi' e^{-2\alpha}}{\psi^2} \frac{2\psi F^2}{\psi + \psi' F^2} \alpha_k \ell_i \ell_j. \end{aligned}$$

From (3.13), (3.14) and the equality above it follows

$$g_{ij|k} = -\frac{\theta_k}{\theta}\, \overset{\circ}{h}_{ij} - 2\alpha_k \overset{\circ}{\ell}_i \overset{\circ}{\ell}_j + \frac{2F^2\psi'}{\psi + F^2\psi'} \alpha_k \overset{\circ}{\ell}_i \overset{\circ}{\ell}_j.$$

Therefore

$$L^p_{ik} g_{pj} + L^p_{jk} g_{ip} = \delta_k g_{ij} + \frac{1}{\theta} \theta_k \overset{\circ}{h}_{ij} + \frac{2}{1 + A} \alpha_k \overset{\circ}{\ell}_i \overset{\circ}{\ell}_j \tag{3.17}$$

where we have put

$$A := \frac{\psi'(F^2)}{\psi(F^2)}\, F^2. \tag{3.18}$$

Using the Christoffel process and 4° we get

$$\begin{aligned} L^i_{jk} &= \frac{1}{2}\, g^{ip}(\delta_j g_{pk} + \delta_k g_{jp} - \delta_p g_{jk}) \\ &+ \frac{1}{2\theta}\, g^{ip}\, (\theta_j \overset{\circ}{h}_{pk} + \theta_k \overset{\circ}{h}_{jp} - \theta_p \overset{\circ}{h}_{jk}) \\ &+ \frac{1}{1+A}\, (\alpha_j \overset{\circ}{\ell}_k \overset{\circ}{\ell}{}^i + \alpha_k \overset{\circ}{\ell}_j \overset{\circ}{\ell}{}^i - \overset{\circ}{\ell}_j \overset{\circ}{\ell}_k \alpha^i). \end{aligned} \tag{3.19}$$

Introducing the generalized Christoffel symbols

$$\gamma^i_{jk} = \frac{1}{2}\, g^{ip}(\partial_j g_{pk} + \partial_k g_{jp} - \partial_p g_{jk})$$

(3.19) can be rewritten in the following form

$$\begin{aligned} L^i_{jk} &= \gamma^i_{jk} - N^r_j \overset{\circ}{C}{}^i_{rk} - N^r_k \overset{\circ}{C}{}^i_{rj} + g^{ip} N^r_p \overset{\circ}{C}_{rjk} \\ &+ \frac{1}{2\theta}(\theta_j \overset{\circ}{h}{}^i_k + \theta_k \overset{\circ}{h}{}^i_j - \theta^i \overset{\circ}{h}_{jk}) \\ &+ \frac{1}{1+A}(\alpha_j \overset{\circ}{\ell}_k \overset{\circ}{\ell}{}^i + \alpha_k \overset{\circ}{\ell}_j \overset{\circ}{\ell}{}^i - \overset{\circ}{\ell}_j \overset{\circ}{\ell}_k \alpha^i) \end{aligned} \tag{3.20}$$

with $\theta^i := g^{ip}\theta_p$.

Transvecting by y^k and then by y^j and using again 3° we get

$$\begin{aligned} N^i_j &= \gamma^i_{j0} - N^r_0 \overset{\circ}{C}{}^i_{rj} + \frac{1}{2\theta}\, \theta_0 \overset{\circ}{h}{}^i_j \\ &+ \frac{1}{1+A}\, (F\alpha_j \overset{\circ}{\ell}{}^i + \alpha_0 \overset{\circ}{\ell}_j \overset{\circ}{\ell}{}^i - F\overset{\circ}{\ell}_j \alpha^i) \end{aligned} \tag{3.21}$$

$$N^i_0 = \gamma^i_{00} + \frac{1}{1+A}(2F\alpha_0 \overset{\circ}{\ell}{}^i - F^2\alpha^i) \tag{3.22}$$

where $\alpha^i := g^{ij}\alpha_j$ and

$$\theta_0 = y^k(\partial_k\theta - N^i_k\dot{\theta}_i) = \partial_0\theta - \gamma^i_{00}\dot{\theta}_i - \frac{1}{1+A}\, (2\alpha_0\dot{\theta}_0 - F^2\alpha^i\dot{\theta}_i). \tag{3.23}$$

Therefore, making use of (3.22) a direct calculation gives

$$N^i_j = \gamma^i_{j0} - \overset{\circ}{C}{}^i_{jk}\gamma^k_{00} + \frac{F^2}{1+A}\overset{\circ}{C}{}^i_{jk}\alpha^k + \frac{1}{1+A}\Big(\alpha_j y^i + \frac{1}{F^2}\,\alpha_0\overset{\circ}{y}_j y^i - \overset{\circ}{y}_j\alpha^i\Big) + \frac{\theta_0}{2\theta}\overset{\circ}{h}{}^i_j.$$

That is

(3.24)
$$N^i_j = \overset{\circ}{N}{}^i_j + \frac{1}{1+A}(\delta^i_j\alpha_0 + \alpha_j y^i - \overset{\circ}{y}_j\alpha^i + F^2\overset{\circ}{C}{}^i_{jk}\alpha^k) + \left(\frac{\theta_0}{2\theta} - \frac{\alpha_0}{1+A}\right)\overset{\circ}{h}{}^i_j$$

where

$$\overset{\circ}{N}{}^i_j = \gamma^i_{j0} - \overset{\circ}{C}{}^i_{jk}\gamma^k_{00}$$

is the Cartan nonlinear connection of the Finsler space $F^n = \big(M, F(x,y)\big)$.

L^i_{jk} given by (3.19) with (N^i_j) from (3.24) verifies $1°, 3°, 4°$.

Now, from $2°$ and $5°$ we get by standard calculatin

$$C^i_{jk} = \frac{1}{2}\,a^{ik}(\dot{\partial}_j a_{pk} + \dot{\partial}_k a_{jp} - \dot{\partial}_p a_{jk}).$$

Remark. 1. $B^i_j := \delta^i_j\alpha_0 + \alpha_j y^i - \overset{\circ}{y}_i\alpha^i + F^2\overset{\circ}{C}{}^i_{jk}\alpha^k$ from (3.24) is just the tensor which perturbs the Cartan nonlinear connection of a Finsler space after a conformal change of metric tensor (see [10]).

In fact, this theorem gives us the change of the Cartan connection $C\overset{\circ}{\Gamma} = (\overset{\circ}{N}{}^i_j, L^i_{jk}, \overset{\circ}{C}{}^i_{jk})$ after a distorted conformal change of the metric tensor g_{ij}

$$g_{ij} \to \theta g_{ij} + (\psi - \theta)\overset{\circ}{\ell}_i\overset{\circ}{\ell}_j \to e^{\alpha(x)}\big(\theta g_{ij} + (\psi - \theta)\overset{\circ}{\ell}_i\overset{\circ}{\ell}_j\big).$$

2. The d-connection $C\Gamma = (N^i_j, L^i_{jk}, C^i_{jk})$ has the coefficients given as follows

(3.25)
$$L^i_{jk} = \frac{1}{2}\,a^{ip}(\delta_j a_{pk} + \delta_k a_{jp} - \delta_p a_{jk})$$

(3.25)′
$$C^i_{jk} = \frac{1}{2}\,a^{ip}(\dot{\partial}_j a_{pk} + \dot{\partial}_k a_{jp} - \partial_p a_{jk})$$

with N^i_j from (3.24).

We can also express these coefficients in terms of the Cartan connection of the Finsler space $\big(M, F(x,y)\big)$.

3. The following formulae hold:

i) $g_{ij|k} = -\frac{\theta_k}{\theta}\overset{\circ}{h}_{ij} - \frac{2}{1+A}\alpha_k\overset{\circ}{\ell}_i\overset{\circ}{\ell}_j$

ii) $\overset{\circ}{\ell}_{i|k} = -\frac{1}{1+A}\alpha_k\overset{\circ}{\ell}_i$, $\overset{\circ}{\ell}{}^i_{|k} = \frac{1}{1+A}\alpha_k\overset{\circ}{\ell}{}^i$

iii) $\overset{\circ}{h}_{ij|k} = -\frac{\theta_k}{\theta}\overset{\circ}{h}_{ij}$.

4. The d-connection obtained in Theorem 3.1 will be called *the Cartan connection of the GL*-space (M, a_{ij}).

From Theorem 3.1 we get

Theorem 3.2. *The Cartan connection* $C\Gamma = (N^i_j, L^i_{jk}, C^i_{jk})$ *of the GL-space* $(M, \overline{g}_{ij})$ *where* $\overline{g}_{ij} = \theta g_{ij} + (\psi - \theta)\overset{\circ}{\ell}_i\overset{\circ}{\ell}_j$ *has the following coefficients:*

$$L^i_{jk} = F^i_{jk} + \frac{1}{2\theta}(\theta_j\overset{\circ}{h}{}^i_k + \theta_k\overset{\circ}{h}{}^i_j - \theta^i\overset{\circ}{h}_{jk}) - \frac{\theta_0}{2\theta}\overset{\circ}{C}{}^i_{jk} \tag{3.26}$$

$$\begin{aligned} C^i_{jk} = \overset{\circ}{C}{}^i_{jk} &+ \frac{1}{2\theta}(\dot\theta_j\overset{\circ}{h}{}^i_k + \dot\theta_k\overset{\circ}{h}{}^i_j - \dot\theta^i\overset{\circ}{h}_{jk}) \\ &+ \frac{\psi' F}{\psi}\overset{\circ}{\ell}_j\overset{\circ}{\ell}_k\overset{\circ}{\ell}{}^i + \frac{\psi - \theta}{F\psi}\Big(1 + \frac{\dot\theta_0}{2\theta}\Big)\overset{\circ}{h}_{jk}\overset{\circ}{\ell}{}^i \end{aligned} \tag{3.27}$$

$$N^i_j = \overset{\circ}{N}{}^i_j + \frac{\theta_0}{2\theta}\overset{\circ}{h}{}^i_j. \tag{3.28}$$

Corollary 3.3. *The coefficients of the Cartan connection for a m-homogeneous Lagrange manifold are given as follows*

$$L^i_{jk} = F^i_{jk}, \quad N^i_j = \overset{\circ}{N}{}^i_j \tag{3.29}$$

$$\begin{aligned} C^i_{jk} = \overset{\circ}{C}{}^i_{jk} &+ \frac{m-2}{2F}\,(\delta^i_j\overset{\circ}{\ell}_k + \delta^i_k\overset{\circ}{\ell}_j) \\ &+ \frac{m-2}{(m-1)F}\overset{\circ}{h}_{jk}\overset{\circ}{\ell}{}^i - \frac{m-2}{2F}\overset{\circ}{\ell}_j\overset{\circ}{\ell}_k\overset{\circ}{\ell}{}^i \end{aligned} \tag{3.30}$$

(see [4], [12]*).*

Corollary 3.4. *The Cartan connection of GL-space $(M, \overline{g}_{ij})$, $\overline{g}_{ij} = \theta g_{ij} + (1-\theta)\overset{\circ}{\ell}_i\overset{\circ}{\ell}_j$ with $\theta = const.$ are as follows:*

$$L^i_{jk} = F^i_{jk}, \quad N^i_j = \overset{\circ}{N}{}^i_j \tag{3.31}$$

$$C^i_{jk} = \overset{\circ}{C}{}^i_{jk} + \frac{1-\theta}{F^2}\, h_{jk} y^i. \tag{3.32}$$

Proposition 3.5. *The Cartan connection given in Theorem 3.2 has the property $C^i_{j0} = 0$ iff θ is 0-homogeneous and $\psi = const.$ $(n \geq 2)$.*

Proof. Transvecting in (3.27) by y^k we get:

$$0 = C^i_{j0} = -\frac{1}{2\theta}\dot{\theta}_0 h^i_j + \frac{\psi' F^2}{\psi}\overset{\circ}{\ell}_j\overset{\circ}{\ell}{}^i.$$

Transvecting again by y^j we obtain $\psi' = 0$ and then $\dot{\theta}_0 = 0$, therefore θ is 0-homogeneous and $\psi = \text{const.}$

Let us see now about geodesics of GL-space $(M, \overline{g}_{ij})$ with

$$\overline{g}_{ij} = \theta g_{ij} + (\psi - \theta)\overset{\circ}{\ell}_i\overset{\circ}{\ell}_j.$$

The energy functional is given by

$$\overline{\mathcal{E}} = \psi(F^2)F^2.$$

We remark that $(M, \overline{\mathcal{E}})$ is a φ-Lagrange space with $\varphi(t) = \psi(t)\cdot t$ and from a result of [4] the geodesics of the Finsler space $\big(M, F(x,y)\big)$ parametrized by arc length coincide with the extremals of energy integral and with the paths of $\overset{\circ}{N}{}^i_j$.

From (3.28) we obtain that $\overset{\circ}{N}{}^i_j$ and N^i_j have the same paths and from the deflection free property these are the h-paths of GL-space $(M, \overline{g}_{ij})$.

Therefore, we can state:

Theorem 3.6. *For the GL-space $(M, \overline{g}_{ij})$ the geodesic of the associate Finsler space (parametrized by arc length), the extremals of the action integral and the paths of nonlinear connection N^i_j (or $\overset{\circ}{N}{}^i_j$) coincide. The horizontal lifts of these curves are h-paths.*

Remark. For the metric $\overline{g}_{ij}(x,y) = \gamma_{ij}(x) + \beta(x,y)\overset{\circ}{\ell}_i\overset{\circ}{\ell}_j$ where $\gamma_{ij}(x)$ is a Riemannian metric used in the geometric relativistic optic (see [17]) we can make a conformal change $a_{ij} = e^{-\ell n\left(1+\beta(x,y)\right)}\left(\gamma_{ij} + \beta(x,y)\overset{\circ}{\ell}_i\overset{\circ}{\ell}_j)\right)$ and the new GL-space (M, a_{ij}) have the properties of these from Theorem 3.2 and Theorem 3.6.

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