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On indecomposable groups and groups with hypercentral-by-finite proper subgroups

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Abstract. The properties of indecomposable nonperfect groups are investigated. It is shown that an indecomposable solvable group is a p-group. The characterization of minimal non-"hypercentral-by-finite" groups are obtained.

0. Introduction

A $\overline{ZAF}$-group G is a group which is not hypercentral-by-finite, while all proper subgroups of G are hypercentral-by-finite. The group constructed by Čarin [1] and the groups of Heineken–Mohamed type [2–8] (i.e. the non-nilpotent groups with all proper subgroups nilpotent and subnormal) are examples of $\overline{ZAF}$-groups. The class of $\overline{ZAF}$-groups contains the $\overline{NF}$-groups (respectively the $\overline{AF}$-groups), i.e. the groups which are minimal non-"nilpotent-by-finite" (respectively minimal non-"abelian-by-finite"). The $\overline{AF}$-groups are independently described by V.V. Belyaev [9] and B. Bruno [10]. As it is proved in [9] each locally finite $\overline{AF}$-group G is either an indecomposable metabelian group or the Čarin group. After a while in [11] it was proved that the periodic indecomposable metabelian groups are related in the some sence to the groups of Heineken–Mohamed type and (as it is well-known [3–5]) there exist an uncountable family of pair-wise nonisomorphic p-groups of Heineken–Mohamed type. The $\overline{NF}$-groups are studied in [12–14].

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Recall that a group G is called indecomposable if any two proper subgroups of G generate a proper subgroup of G, and is called decomposable otherwise. The decomposable groups are related to the groups which have a proper factorization. According to [17] we say that G has a proper factorization if there are proper subgroups A and B such that $G = AB$.

Recall also one construction from [9], which is a generalization of the construction from [1]. Let p and q be distinct primes, $\mathbb{Z}_q$ the field with q elements, $\mathbb{Z}_q(\alpha)$ will indicate the subfield of the algebraic closure of $\mathbb{Z}_q$ generated by α. If ϵ_i is a primitive p^i-th root of 1 $(i = 0, 1, 2, \dots)$, put $F_i = \mathbb{Z}_q(\epsilon_i)$ and $F = \bigcup_{i=0}^{\infty} F_i$. Let A be the additive group of F, B be the multiplicative group which contains the p^i-th roots of 1 where $i = 0, 1, 2, \dots$. The rule

$$bab^{-1} = b^{p^m} \cdot a$$

where $a \in A$, $b \in B$ and $b^{p^m} \cdot a$ is the product of the elements b^{p^m} and a in the field F, m is some nonnegative integer, defines the action of B on A. Constructed in this manner the group $G = A \rtimes B$ is called a Čarin group.

Throughout this paper p will always denote a prime number, G', G'', ... will indicate the terms of derived series of G and by $\mathbb{C}_{p^\infty}$ stands for the quasicyclic p-group. For any group G, $F(G)$ means the Fitting subgroup of G, $\Phi(G)$ the Frattini subgroup of G, and $Z(G)$ the center of G.

Most of the standard notation used comes from [18] and [19].

1. In this part we establish some properties of indecomposable groups which we shall need in the sequel.

The following theorem gives the answer to Question 1 [17] for nonperfect groups.

Theorem 1.1. *Let G be an infinite nonperfect nonabelian group. The following are equivalent:*

(1) *G is an indecomposable group;*

(2) *G has no proper factorization;*

(3) *G is countable, the commutator subgroup G' of G is not properly supplemented in G and the quotient group G/G' is a p-quasicyclic group for some prime p.*

Proof. (1) ⇒ (2) is clear.

(2) ⇒ (1). Suppose the group G has no proper factorization, but $G = \langle A, B\rangle$ for some proper subgroups A, B of G. Then since $G'A \neq G$ and $G'B \neq G$, we conclude that $G = (G'A)(G'B)$, a contradiction.

(1) ⇒ (3) is immediate (see [11]).

(3) ⇒ (1). Suppose the contrary and let G be a countable group with not properly supplemented subgroup G' and $G/G' \simeq \mathbb{C}_{p^\infty}$, but $G = \langle H, T\rangle$ for some proper subgroups H, T of G. Then

$$\overline{G} = G/G' = (HG'/G')\,(TG'/G') \simeq \mathbb{C}_{p^\infty},$$

whence we conclude that

$$\overline{G} = TG'/G' \text{ and } HG' \leq G'$$

in consequence of which $G = TG' = T$, a contradiction.

Lemma 1.2. *If G is an indecomposable group then $[G, G'] = G'$.*

Indeed, if G is a noncyclic group then this follows from quasicyclity of quotient group G/G'.

Corollary 1.3. *Any nonperfect noncyclic p-group of finite exponent is decomposable.*

Lemma 1.4. *An indecomposable periodic solvable group G is a countable (i.e. is finite or countable infinite) p-group for some prime p.*

Proof. If G is cyclic then the result follows from [11]. Thus, suppose that G is noncyclic group satisfying the conditions of lemma. Then $G/G' \simeq \mathbb{C}_{p^\infty}$ for some prime p. If, further, G is not p-group there is a positive integer m such that the quotient groups $G^{(m)}/G^{(m+1)} = P_1 \times Q_1$ with the Sylow p-subgroup P_1 and a p'-subgroup Q_1 and therefore $G/G^{(m+1)} = Q_1 \rtimes P$ for some p-subgroup P. We have a contradiction with indecomposability of G. It follows that all factors $G^{(i)}/G^{(i+1)}$ of the derived series of G are p-groups; this completes the proof.

Lemma 1.5. *Let G be an indocomposable locally finite group. If every proper subgroup of G is almost locally solvable, then G is nonsimple.*

Proof. Let G be a group satisfies the assumptions. If, further, G is simple then by Corollary A1 [20] G must be linear. Since the locally finite simple groups which are linear must be of Lie type (see [21]), the group G must be decomposable, a contradiction. Hence G is nonsimple and lemma is proved.

Lemma 1.6. *Let G be an indecomposable group. Then the following statement are equivalent.*

(1) *G is nonperfect p-group with every proper subgroup nilpotent;*

(2) *G is a non-nilpotent group with all subgroups subnormal and the commutator subgroup G' of G is nilpotent.*

Proof. (1) $\Rightarrow$ (2). If G is an indecomposable nonperfect group whose proper subgroups are nilpotent and K is any proper subgroup of G then by Theorem 1.1 $G'K$ is also a proper subgroup of G. Hence K is a subnormal subgroup of G.

(2) $\Rightarrow$ (1). Suppose that G is an indecomposable non-nilpotent group with all subgroups subnormal and the commutator subgroup G' of G is nilpotent. Then $G/G' \simeq \mathbb{C}_{p^\infty}$ and, further, G is a p-group by Lemma 1.4. Note that

$$K/K \cap G' \simeq G'K/G'$$

is an abelian group of finite exponent for every proper subgroup K of G and so by Proposition 1.2 [22] the subgroups $G'K$ and K are nilpotent.

Relative to Corollary 1 [2] we argue

Lemma 1.7. *Any torsion-free (and consequently non-periodic) group G with every proper subgroup nilpotent (respectively hypercentral) is also nilpotent (respectively hypercentral).*

Proof. Since nilpotence and hypercentrality are properties of countable character [19, p. 355], we have that G with a noncountable group G with all subgroups nilpotent (respectively hypercentral) is nilpotent (respectively hypercentral). Therefore suppose that G is countable. If G is torsion-free then by Lemma 2 [23] G coincides with the isolator

$$I_G(K) = \{x \in G \mid \exists n \in \mathbb{N} : x^n \in K\}$$

of some proper subgroup K of G and so (see [24]) G is nilpotent (respectively hypercentral).

Suppose now that G is not torsion-free. Then as stated above the quotient group $\overline{G} = G/\tau G$ of G (here τG is the periodic part of G) is nilpotent (respectively hypercentral). Further, if G is indecomposable then $\overline{G}/\overline{G}' \simeq \mathbb{C}_{p^\infty}$ and consequently the isolator $I_{\overline{G}}(\overline{G}')$ of $\overline{G}'$ coinsides with $\overline{G}$, a contradiction. Thus, $G = \langle A, B\rangle$ for some proper subgroups A, B of G.

Moreover, the image $\overline{A}$ of A (respectively $\overline{B}$ of B) in $\overline{G}$ is contained in a proper normal subgroup $\overline{A_1}$ (respectively $\overline{B_1}$) of $\overline{G}$. Then if A_1 and B_1 are the inverse images of $\overline{A_1}$ and $\overline{B_1}$ in G, respectively, $G = A_1B_1$ is a product of two nilpotent (respectively hypercentral) normal subgroups and consequently G is nilpotent (respectively hypercentral).

We consider the question on linearity of indecomposable groups. In wiev of well-known theorem of Zassenhaus [19, Th. 15.1.3] any group of matrices (over field) with subnormal (respectively hypercentral) proper subgroups is solvable. From the results of Mal'cev [25], Kargapolov [26–27] and Theorem 8 [23], Lemmas 1.4, 1.6, 1.7 we conclude the following

Corollary 1.8. *Let G be an indecomposable locally solvable periodic group of matrices (over field). Then G is either the cyclic p-group $\mathbb{C}_{p^n}$ or $\mathbb{C}_{p^\infty}$.*

Thus, neither the groups of Heineken–Mohamed type nor the minimal non-hypercentral groups are not linear (over field).

Proposition 1.9. *Let G be a countable group with the hypercentral commutator subgroup G' and the quasicyclic quotient group G/G'. Then the group G that satisfies the normalizer condition is an indecomposable p-group.*

Proof. Without restricting generality, suppose G is a metabelian group. Suppose the assertion is false and G is decomposable. Then clearly $G = G'V$ for some proper subgroup V of G, whence

$$\overline{G} = G/G' \cap V = \overline{G}' \rtimes \overline{V}.$$

It is easy to see that $\overline{V} \simeq \mathbb{C}_{p^\infty}$, $\overline{1} \neq N_{\overline{G'}}(\overline{V}) \leq Z(\overline{G})$ and every proper homomorphic image of $\overline{G}$ has a nontrivial centre. This means [19, Example 12.2.2] that $\overline{G}$ is hypercentral. But then by Lemma 1.18 [18, p. 63] the group $\overline{G}$ is abelian, a contradiction. Thus G is indecomposable. Further, it is easy to see that G is a p-group.

Corollary 1.10. *The quotient group G/G' of a nonabelian countable hypercentral group G is not quasicyclic.*

Proposition 1.11. *If the commutator subgroup G' of nonabelian indecomposable p-group G is abelian (respectively nilpotent of finite exponent) then G satisfies the normalizer condition.*

Proof. Pick an arbitrary proper subgroup K of G. Clearly without loss generality we may assume that $G' \nsubseteq K$ and $K \nsubseteq G'$. Obviously $G'K$ is a proper subgroup of G and there is an element a of G such that $G'K = G'\langle a\rangle$. Then the subgroup $G'\langle a\rangle$ is hypercentral (see [18, Proposition 1.7] and [28], respectively) and $N_G(K) \geq N_{G'K}(K) \neq K$, as desired.

The following lemma is obvious.

Lemma 1.12. *Let G be a group in which every proper subgroup satisfies the normalizer condition. Then the one of the following statements holds.*

(1) *G satisfies the normalizer condition.*

(2) *G is a finitely generated group with the simple quotient group G/N for some normal subgroup N of G.*

2. In this section we establish some properties of groups without a proper factorization (see [17, Question 1]).

It is well-known that there are finite nonsolvable groups without proper factorization. The following lemma is due to [29].

Lemma 2.1. *Let G be a nonabelian finite group. The following statements are equivalent.*

(1) *G has no proper factorization.*

(2) *$F(G) = \Phi(G) = Z(G)$ and the quotient group $G/Z(G)$ is a simple group without proper factorization.*

Proof is immediate.

Lemma 2.2. *Let G be a nonabelian finitely generated group. If G is a decomposable group without proper factorization then G has a simple quotient group. Further, if $\Phi(G) = 1$ then G is simple.*

Proof. Suppose $G = \langle A, B\rangle$ for some proper subgroups A and B of G. Without restricting generality, let A and B be maximal subgroups of G. Then it is easy to see that $H \triangleleft G$ implies $H \leq A \cap B$. If F is a subgroup of G generated by all normal subgroups of G then the quotient group G/F is simple, and this completes the proof.

Obviously, any non-finitely generated group with a proper subgroup of finite index has a proper factorization. Then we state the following

Corollary 2.3. *Let G be a non-finitely generated group. If G contains a nontrivial normal finite subgroup then either G has a proper factorization or the centre $Z(G)$ of G is nontrivial.*

Corollary 2.4. *(i) An abelian group G has no proper factorization if and only if either G is a cyclic p-group or G is a quasicyclic p-group.*

(ii) A nonperfect nonabelian finite group has a proper factorization.

We shall prove the following theorem.

Theorem 2.5. *An indecomposable solvable group G is a locally finite p-group.*

For the proof of 2.5 we need the following lemma.

Lemma 2.6. *Let G be a locally finite group and $M \neq \{0\}$ be a $\mathbb{Z}G$-module which is torsion-free as a group. Then for any finite set π of primes, there is a $\mathbb{Z}G$-submodule N of M such that the quotient module M/N is periodic as a group and, for all p in π, contains an element of order p.*

Proof of 2.6 is analogous with proof of Lemma 2.3 [14]. We notice only that the group ring $\mathbb{Q}G$ is a (Von Neymann) regular ring by Theorem 1.5 [30, p. 68].

Proof of Theorem 2.5. Suppose that G is a solvable group with derived lenght $n+1$, the quotient group $G/G^{(n)}$ is periodic and $G^{(n)}$ has an element of infinite order. Let T be the torsion subgroup of $G^{(n)}$. Put $H = G/T$. Obviously $H/H' \simeq \mathbb{C}_{p^\infty}$ for some prime p. Now $H^{(n)}$ and $H/H^{(n)}$ satisfy the hypotheses of Lemma 2.6 (with $M = H^{(n)}$ and $G = H/H^{(n)}$); hence there exist N normal in H, $N \leq H^{(n)}$ such that the quotient group $H^{(n)}/N$ is periodic and contains the elements of order r and q for two distinct primes r and q different from p. Clearly, H/N is an indecomposable periodic solvable p-group by Lemma 1.4, a contradiction. The proof of Theorem 2.5 is complete.

Corollary 2.7. *Any non-periodic solvable group has a proper factorization.*

3. This section contains several characterizations of $\overline{ZAF}$-groups.

Remark 3.1. An abelian-by-(periodic abelian) R-group is abelian.

Remark 3.2. If G is a $\overline{ZAF}$-group then the one of the following holds:

(1) G is a finitely generated group with a normal subgroup N such that the quotient group G/N is simple.
(2) G is a locally graded group.

Indeed, if each homomorphic image of a finitely generated $\overline{ZAF}$-group G is nonsimple then the group G is hypercentral-by-finite, a contradiction. On the other hand, if G is not finitely generated then it is readily verified that G is locally nilpotent-by-finite.

Lemma 3.3. *Let G be a $\overline{ZAF}$-group. Then each normal subgroup of G is an extension of a hypercentral group, which is normal in G, by a finite abelian group. If, further, the group G is nonperfect and indecomposable then every subgroup of G is hypercentral-by-(finite abelian).*

Proof. Pick an arbitrary normal subgroup N of G. It is now easy to verify that there is a hypercentral subgroup H of N that is a normal subgroup of G with $|N : H| < \infty$. Put $\overline{G} = G/H$. Then $\overline{N} = N/H$ is a finite normal subgroup of $\overline{G}$ and consequently

$$|\overline{G} : C_{\overline{G}}(\overline{N})| = |N_{\overline{G}}(\overline{N}) : C_{\overline{G}}(\overline{N})| < \infty.$$

Further, since $\overline{G}$ contains no proper subgroup of finite index, we have $\overline{G} = C_{\overline{G}}(\overline{N})$ and $\overline{N}$ is abelian.

Suppose now that the group G is indecomposable and nonperfect. Then $G'K$ is a proper subgroup of G for each subgroup K of G and consequently $G'K$ contains a hypercentral subgroup F of finite index which is normal in G. Moreover,

$$K/K \cap F \simeq KF/F \leq G'K/F$$

and as stated above $G'K/F$ is abelian; this completes the proof.

Lemma 3.4. *If G is a nonperfect $\overline{ZAF}$-group then the commutator subgroup G' of G is hypercentral and $G/G' \simeq \mathbb{C}_{p^\infty}$.*

Proof. Since G is a nonperfect $\overline{ZAF}$-group, the quotient group G/G' is obviously indecomposable and so $G/G' \simeq \mathbb{C}_{p^\infty}$ (see [11]).

Suppose now that the commutator subgroup G' of G is non-hypercentral. Then G' contains a subgroup F of finite index which is normal in G. Put $\overline{G} = G/F$. Clearly, $|\overline{G'}| < \infty$ and $\overline{G}/\overline{G'} \simeq \mathbb{C}_{p^\infty}$, whence by Lemma 1.15 [18] $\overline{G}$ is abelian, a contradiction.

Corollary 3.5. *Any nonperfect $\overline{ZAF} - p$-group G is indecomposable.*

Indeed, it is easy to see that the quotient group G/G'' is an $\overline{AF}$-group and hence (see [9] or [10]) it is indecomposable.

Corollary 3.6. *Any nonperfect $\overline{ZAF} - p$-group G is a minimal non-hypercentral group if and only if G satisfies the normalizer condition.*

Proof. Part "if" follows from Remark 3.2 and Lemma 1.12.

"Only if". Let K be an arbitrary proper subgroup of G. Then K is hypercentral by Lemma 3.3 and Lemma 2 [31, p. 396], as desired.

Lemma 3.7. *Let $G = K \rtimes Q$ be a $\overline{ZAF}$-group, $Q \simeq \mathbb{C}_{p^\infty}$ and K a hypercentral subgroup. Then $Z(K) = K' = \Phi(K)$ and K is a q-group for a prime q different from p.*

Proof. Let A be a arbitrary proper G-invariant subgroup of K. Then AQ contains a normal hypercentral subgroup F of finite index and as follows from $|Q : Q \cap F| < \infty$ we conclude $Q \leq F$ and $AQ = AF$ is hypercentral. Thus, $Q \leq C_G(A)$. Since G is nonabelian, the subgroup T generated by all proper G-invariant subgroups of K is a proper G-invariant subgroup of K.

Suppose, first, that a subgroup K is abelian. If, further, K is nonperiodic then without loss of generality we can assume that K is torsion-free. Since by Theorem of Zaĭtsev [32] K/T is an abelian q-group of exponent q for some prime q. From $[a, t] = b$ with some $1 \neq b$, $a \in K$ and $t \in Q$ we conclude that $1 = [a^q, t] = b^q$, a contradiction. Hence K is a periodic group and so K is an abelian q-group of exponent q. Consequently $\Phi(K) = 1$. Moreover, Corollary 3.5 implies that p and q are distinct. Since $K = [K, Q] \times C_K(Q)$, we have $C_K(Q) = 1$ and so $T = 1$. Therefore K is a minimal G-invariant subgroup of G.

Suppose next that K is nonabelian. Since obviously $K' \leq T$ and as proved before K/K' is minimal G-invariant abelian subgroup of exponent q, we have $T = K' = \Phi(K) = Z(K)$. The proof is completed.

The following lemma is due to [14].

Lemma 3.8. *Any nonperfect $\overline{ZAF}$-group G is periodic.*

Proof. Let G be a $\overline{ZAF}$-group. Clearly $G/G' \simeq \mathbb{C}_{p^\infty}$. Suppose that it is not periodic; then G' is not periodic. By Lemma 3.4 G' is hypercentral and application of [19, 12.2.6] shows that G'/G'' is not periodic. Put $H = G/G''$ and let T/G'' be the torsion part of H'. Obviously, T is properly contained in G'. Thus $K = G/T$ is an $\overline{AF}$-group and so by Theorem 2.1 [14] K is periodic, a contradiction. Thus G must be periodic, and the proof is completed.

Theorem 3.9. *Let G be a decomposable nonperfect group. Then the following statements are equivalent.*

(1) *G is a $\overline{ZAF}$-group.*

(2) *$G = M \rtimes Q$, $Q \simeq \mathbb{C}_{q^\infty}$, M is a p-group, p and q are the distinct primes, $Z(M) = M' = \Phi(M)$, Q acts trivially on the Frattini subgroup $\Phi(M)$ and irreducibly on $M/\Phi(M)$, and, further, $G/\Phi(M)$ is a Čarin group.*

(3) *G is a $\overline{NF}$-group.*

Proof. The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are almost obvious. Therefore we prove only (1) $\Rightarrow$ (3). Since by assumption a nonperfect $\overline{ZAF}$-group G is decomposable, there are two nontrivial proper subgroups U and V of G such that $G = \langle U, V\rangle$. Then by Lemma 3.4, for example, $G = G'V$. It follows by Corollary 3.5 that G is not p-group. Since V contains a hypercentral subgroup K of finite index then $|G : G'K| < \infty$ and $G = G'K$. By Lemma 3.8 G (and so K) is periodic and by Lemma 3.4 there is a p-subgroup K_1 of K such that $G = G'K_1$. If is easy to see that G' is a r-group for some prime r. An application of Lemma 3.7 completes the proof.

Remark 3.10. The Theorem 1 of [2] implies that if p and q are the primes from Theorem 3.9 then $q \neq 2$ and the order of p modulo q is an even number.

Remark 3.11. An example of the decomposable nonperfect $\overline{NF}$-group (and consequently $\overline{ZAF}$-group) which is not a $\overline{AF}$-group is constructed in [12].

Lemma 3.12. *Any indecomposable nonperfect $\overline{ZAF}$-group G is a p-group.*

Proof. By Lemma 3.4 $G/G' \simeq \mathbb{C}_{p^\infty}$. Put $\overline{G} = G/G''$. It is easy to see that $\overline{G}$ is an indecomposable $\overline{AF}$-group and so it is a p-group. Therefore a hypercentral subgroup G' (and so G) is a p-group, too.

Theorem 3.13. *Let G be a nonperfect group. Then the following statements are equivalent.*

(1) *G is an indecomposable $\overline{ZAF}$-group.*

(2) *G is a countable p-group and has an infinite normal subgroup N not supplemented nontrivially in G with $G/N \simeq \mathbb{C}_{p^\infty}$, $N^p \neq N$ and the quotient group G/G'' is a minimal non-hypercentral group.*

Proof. (1) ⇒ (2). By Lemma 3.4 G' is a hypercentral subgroup and $G/G' \simeq \mathbb{C}_{p^\infty}$. In view of indecomposability of G the commutator subgroup G' is not supplemented nontrivially in G and G is a p-group. From [9] and [11] it follows that $(G')^p \neq G'$. An application of the Proposition 1.7 [18] completes the proof of this part.

(2) ⇒ (1). By Theorem 1.1 the group G is indecomposable. If K is an arbitrary proper subgroup of G then $G'K$ is a proper subgroup of G. Obviously $G'K$ (and as consequence K) is a hypercentral-by-finite, but G is not almost hypercentral. This completes the proof of Theorem.

Note that, it follows from what is proved before that, in particular, every nonperfect minimal non-nilpotent group is a countable solvable p-group of Heineken–Mohamed type.

4. In this section we are concerned with the perfect $\overline{ZAF}$-groups.

The next result is due to [14, Proposition 3.1].

Proposition 4.1. *A perfect $\overline{ZAF}$-group G must be countable hyperabelian p-group. Moreover, G is the join of an ascending sequence of hypercentral normal subgroups and all proper subgroups of G are hypercentral and ascendant (hence G satisfies the normalizer condition).*

This runs along the same lines as proof of Proposition 3.1 [14], replasing "nilpotent" by "hypercentral" and "subnormal" by "ascendant". Moreover, by Lemma 1.5 G do not have infinite simple images. Since hypercentrality is a property of countable character [19, p. 355] then G is countable. Finilly, from Lemma 1.7 follows that G is p-group.

From Proposition 4.1 it follows that a non-"locally nilpotent" $\overline{ZAF}$-group is not perfect. Whether or not there are perfect $\overline{ZAF}$-groups remains an open question.

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