Publ. Math. Debrecen
53 / 1-2 (1998), 191–206

On associative algebras which are sum of two almost commutative subalgebras

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Abstract. The following theorem is proved: if an associative algebra A over an arbitrary field can be decomposed into a sum $A = B + C$ with almost commutative subalgebras B and C (an algebra is called here almost commutative if it has a commutative ideal of finite codimension) then the algebra A possesses a nilpotent ideal I such that the quotient algebra A/I is almost commutative.

1. Introduction

In the paper of K.I. Beidar and A.V. Mikhalev [4] the following problem was stated: whether a sum $R = A+B$ of two associative PI-rings A and B is a PI-ring? There are positive answers to this question for some classes of rings A and B which are near to commutative [4], [5] (every sum of two commutative rings is a PI-ring [2]).

Any associative algebra over an arbitrary field which has a commutative ideal of finite codimension (we will call a such algebra almost commutative) is a PI-algebra and the question about the structure of a sum of two such algebras is of interest. In this paper, the following result is obtained: every sum of two almost commutative algebras contains a nilpotent ideal such that the quotient algebra on this ideal is almost commutative, in particular, every such sum is a PI-algebra. Similar question in group theory i.e. the question about structure of the product of two almost abelian

Mathematics Subject Classification: 16D70.
Key words and phrases: sum of algebras commutative ideal, nilpotent subalgebra.
This work was written while the author visited the University of Freiburg with the aid of Deutsche Forschungsgemeinschaft (DFG). The author is grateful to DFG for its support and to Professor O.H. Kegel for the hospitality and help.

(finite-by-abelian) groups is open although it was proved in some cases that this product is almost soluble (see [7], [6] and others).

All considered algebras and rings are associative, the ground field F is arbitrary. The centre of an algebra (or a ring) A is denoted by $Z(A)$. For F-subspaces X and Y of an algebra A, as usual, $[X,Y] = \{xy-yx \mid x \in X, y \in Y\}$; for a subset S of A and for a subalgebra B of A we will denote by $\mathrm{Ann}^l_B(S)$ and $\mathrm{Ann}^r_B(S)$ the left and corresponding the right annulator of S in the subalgebra B.

The following statement is the main result of this paper:

Theorem. *Let A be an associative algebra over an arbitrary field which is decomposable into a sum $A = B + C$, where B and C are almost commutative subalgebras of A. Then the algebra A contains a nilpotent ideal I such that the quotient algebra A/I is almost commutative.*

Previously, we prove a series of lemmas, some these results can be of interest out of this work. In particular, the Proposition 2 which is used in the proof of the Theorem is an extension (for algebras over a field) of the result of O.H. Kegel about sum of two nilpotent associative rings [8].

Lemma 1 (See for example [9])**.** *Let A be an associative algebra over an arbitrary field and B a subalgebra of A with $\dim A/B < \infty$. Then B contains an ideal I of algebra A such that $\dim A/I < \infty$.*

Lemma 2. *Let I be an one-sided or two-sided commutative ideal of a ring R. Then R has an ideal J such that $J^2 = 0$ and $(I + J)/J \subseteq Z(R/J)$.*

Proof. Let I be for example a right ideal of the ring R and $i, i_1 \in I$, $r \in R$ any elements. Then

$$i_1(ir - ri) = i(i_1r) - (i_1r)i = 0$$

because $i_1r \in I$ and $[I, I] = 0$. Hence $I[I,R] = 0$. Let $T = \mathrm{Ann}^r_R(I)$. Clearly, T is an ideal of the ring R and $[I,R] \subseteq T$. For any element $t \in T$ it holds $(ir - ri)t = irt = 0$ (because $rt \in T$, $IT = 0$) and therefore $[I,R]T = 0$. Now let $J = \mathrm{Ann}^l_T(T)$. It is obvious that J is an ideal of the ring R and $J^2 = 0$. As $[I,R] \subseteq J$, we have $(I + J)/J \subseteq Z(R/J)$. The case of the left ideal can be considered analogously.

Lemma 3. *Let A be an associative algebra and I an ideal of the algebra A. If J is an ideal of the subalgebra I then it holds:*

(1) if subalgebra J is nilpotent then J lies in a nilpotent ideal J_I of the algebra A and $J_I \subseteq I$;

(2) if subalgebra J is finite dimensional then J lies in an ideal J_I of the algebra A such that $J_I \subseteq I$ and J_I contains a nilpotent ideal T of the algebra A with $\dim J_I/T < \infty$.

Proof. (1) See for example [1, Lemma 1.1.5].

(2) Let J_I be the smallest ideal of the algebra A which contains J and lies in the ideal I of A. Since $J_I^3 \subseteq J$ (see [1, Lemma 1.1.5]), J_I^3 is a finite dimensional ideal of the algebra A. If $J_I^3 = 0$ then we set $T = J_I$ and the statement (2) is proved. Let $J_I^3 \neq 0$ and $T = \mathrm{Ann}^l_{J_I}(J_I^3)$. Clearly, T is an ideal of the algebra A and $(T \cap J_I^3)^2 = 0$. Further, $T/(T \cap J_I^3) \simeq T + J_I^3/J_I^3$ is a nilpotent algebra as a subalgebra of the nilpotent algebra J_I/J_I^3 and therefore the ideal T is nilpotent. Since $\dim J_I^3 < \infty$, we have, clearly, $\dim J_I/T < \infty$. The statement (2) and the Lemma are proved.

Lemma 4. *Let R be an associative ring and I any commutative ideal of R. If the quotient ring R/I is commutative or nilpotent then the ring R contains some nilpotent ideal with the commutative quotient ring.*

Proof. We may restrict ourselves by Lemma 2 only to the case $I \subseteq Z(R)$. First, let the quotient ring R/I be commutative. For any elements $i \in I$, $r_1, r_2 \in R$ we have

$$ir_1r_2 - r_2ir_1 = 0 = i[r_1, r_2]$$

(because $ir_1 \in I \subseteq Z(R)$) and therefore $I[R, R] = 0$. Let J denote the annulator of the ideal I in I. Then J is an ideal of the ring R with $J^2 = 0$ and $[R, R] \subseteq J$ (because $[R, R] \subseteq I$). Thus the quotient ring R/J is commutative and and the proof is complete in the case of commutative quotient ring R/I. Now let the quotient ring R/I be nilpotent. If $(R/I)^2{=}0$ then this case follows from the above considered case. Let the statement of Lemma be true for an arbitrary ring R with $(R/I)^n = 0$, $n \geq 2$, prove it for a ring R with condition $(R/I)^{n+1} = 0$. Denote $N = R^2 + I$. Clearly, N/I is an ideal of the quotient ring R/I and $(N/I)^n = 0$. By inductive assumption the subring N contains some nilpotent ideal T such that the quotient ring N/T is commutative. By Lemma 3 T lies in some nilpotent ideal S of the ring R with $S \subseteq N$. Then the quotient ring $\overline{R} = R/S$ contains a commutative ideal $\overline{N} = N/S$ such that $(\overline{R}/\overline{N})^2 = 0$. As was proved above the ring $\overline{R}$ contains some nilpotent ideal $\overline{J} = J/S$ such that $\overline{R}/\overline{J}$ is commutative. It is obvious that J is a nilpotent ideal of the ring R and the quotient ring R/J is commutative. The proof is complete.

Lemma 5. *Let A be an almost commutative associative algebra and I a commutative ideal of A with $\dim A/I < \infty$. Then:*

(1) $[A,A]I$ lies in some nilpotent ideal of the algebra A;

(2) for some nilpotent ideal J the quotient algebra A/J contains a finite dimensional ideal T/J such that the quotient algebra A/T is commutative.

Proof. (1) If $I \subseteq Z(A)$ then we have for any elements $a_1, a_2 \in A$ and $i \in I$

$$(a_1a_2 - a_2a_1)i = (a_1i)a_2 - a_2(a_1i) = 0,$$

because $a_1i \in I \subseteq Z(A)$ and therefore $[A,A]I = 0$. Now if $I \nsubseteq Z(A)$ then going to the quotient algebra A/J on some nilpotent ideal J with $I \subseteq Z(A/J)$ (it exists by Lemma 2) we get $[A,A]I \subseteq J$.

(2) We can assume, without loss of generality, by Lemma 2 that $I \subseteq Z(R)$. Clearly, $T = \mathrm{Ann}_A(I)$ is an ideal of the algebra A and by part 1 of this Lemma $T \supseteq [A,A]$. Let denote $J = T \cap I$. Obviously, $J^2 = 0$ and T/J is a finite dimensional ideal of the algebra A/J. At that the quotient algebra $(A/J)/(T/J) \simeq A/T$ is commutative.

For convenience and shortness we introduce the following:

Definition 1. An associative algebra A over an arbitrary field will be called an *NCF*-algebra if it contains a nilpotent ideal with almost commutative quotient algebra.

An ideal of an associative algebra will be called an *NCF*-ideal if it is an *NCF*-algebra.

In particular, every nilpotent, commutative and finite dimensional algebras are *NCF*-algebras by this definition.

Proposition 1. *The following statements hold:*

(1) every subalgebra and every quotient algebra of an NCF-algebra are NCF-algebras;

(2) if A and B are NCF-algebras then the direct product $A \times B$ is an NCF-algebra;

(3) every extension of an NCF-algebra by other NCF-algebra is an NCF-algebra.

Proof. The statements (1) and (2) of the Proposition are obvious. Prove the statement (3), i.e. show that an algebra A is an *NCF*-algebra if it contains an *NCF*-ideal B such that A/B is also an *NCF*-algebra. Consider some cases previously:

(a) The quotient algebra A/B is finite dimensional.

Let I be a nilpotent ideal of the *NCF*-algebra B such that B/I is almost commutative. By part 1 of Lemma 3 I lies in some nilpotent ideal of the algebra A and the latest lies in B. Then we can assume, without loss of generality, $I = 0$ i.e. B is commutative. Since $\dim A/B < \infty$, the algebra A is almost commutative by Lemma 1 and therefore it is an *NCF*-algebra.

(b) The ideal B is finite dimensional.

The right annulator $C = \mathrm{Ann}^r_A(B)$ is an ideal of the algebra A, and since $C/(B \cap C) \simeq C + B/B$ is an *NCF*-algebra, the ideal C is an *NCF*-algebra in view of equality $(B \cap C)^2 = 0$. It follows from the inequality $\dim A/C < \infty$ and part (a) of this proof that A is an *NCF*-algebra.

(c) The ideal B is commutative.

In the *NCF*-algebra A/B there exists a nilpotent ideal N/B such that quotient algebra $(A/B)/(N/B) \simeq A/N$ is almost commutative. Without loss of generality one can assume in view of Lemma 4 and part 1 of Lemma 3 that the ideal N is commutative. Denote by S/N any commutative ideal of finite codimension in algebra A/N. By Lemma 4 and part 1 of Lemma 3 we can assume also S is commutative. Since $\dim A/S < \infty$, we obtain that A is an *NCF*-algebra.

Now prove the statement (3) in general case. Let N be any nilpotent ideal of the subalgebra B such that the quotient algebra B/N is almost commutative. By part 1 of Lemma 3 one can assume without loss of generality $N = 0$ i.e. the ideal B is almost commutative. Analogously, by part 2 of Lemma 5 and part 1 of Lemma 3 we can consider the subalgebra B has a finite dimensional ideal T with commutative quotient algebra B/T. Similarly, one can assume the subalgebra T of A lies in some finite dimensional ideal T_B of algebra A such that $T_B \subseteq B$ and B/T_B is commutative. Then A/T_B is an *NCF*-algebra in view of part (c) of this proof. Since $\dim T_B < \infty$, the algebra A is an *NCF*-algebra by part (b) of this proof. The proof is complete.

It follows from Lemma 1 and Proposition 1 the next statement:

Corollary 1. *If an associative algebra A has an NCF-subalgebra B and $\dim A/B < \infty$ then A is an NCF-algebra.*

Lemma 6 ([2], [3, Th. 2.2])**.** *Let R be an associative ring which is decomposable into a sum $R = A + B$ of two commutative subrings A and B. Then R has an ideal I with $I^2 = 0$ and commutative quotient ring R/I.*

Lemma 7. *Let A be an associative algebra over an arbitrary field F, B and C commutative subalgebras of A and let I be an ideal of A which lies in the F-subspace $B + C$. Then I is an NCF-ideal.*

Proof. Let $I_B = \{b \in B \mid \text{there exists } i \in I \text{ of the form } i = b + c,\ c \in C\}$ i.e. I_B is a projection of the ideal I into subalgebra B. Analogously, define the projection I_C of I on subalgebra C. Obviously, it holds for elements $i_1, i_2 \in I$, $i_1 = b_1 + c_1$, $i_2 = b_2 + c_2$, where $b_i \in B$, $c_i \in C$, $i = 1, 2$ the equality

$$i_1 i_2 = (b_1 + c_1)(b_2 + c_2) = i_1 c_2 + c_1 i_2 + b_1 b_2 - c_1 c_2.$$

Thus $b_1 b_2 - c_1 c_2 \in I$, and hence I_B, I_C are subalgebras of B and corresponding C. It is easy to see that $I_B + I_C$ is a subalgebra of A, and since the subalgebras I_B and I_C are both commutative, $I_B + I_C$ is an *NCF*-algebra by Lemma 6. Then the ideal I which lies in $I_B + I_C$ is an *NCF*-algebra. The proof is complete.

For convenience we give the following definition:

Definition 2. An associative algebra A over an arbitrary field F decomposable into a sum $A = B + C$ of two almost commutative subalgebras B and C will be called a minimal BM-counter-example if A satisfies the following conditions:

(1) A is not an *NCF*-algebra;

(2) the subalgebras B and C have commutative ideals B_0 and corresponding C_0 such that $\dim B/B_0 + \dim C/C_0 < \infty$ and the number $\dim A/(B_0 + C_0)$ is the smallest;

(3) the algebra A has not nonzero ideals which lie in the F-subspace $B_0 + C_0$ from the condition (2).

Lemma 8. *Let $A = B + C$ be a minimal BM-counter-example. Then for every nonzero ideal I of A the quotient algebra A/I is an NCF-algebra. Besides, the algebra A has not nonzero NCF-ideals.*

PROOF. Let $\dim A/(B_0 + C_0) = n$ where B_0 and C_0 are the commutative ideals of subalgebra B and corresponding C from Definition 2. If $n = 0$ then the algebra A is a sum of two commutative subalgebras B_0 and C_0 and hence it is an *NCF*-algebra by Lemma 6. This contradicts to the choice of the algebra A and therefore $n \geq 1$. Let I be a nonzero ideal of the algebra A such that A/I is not an *NCF*-algebra. Denote

$$\overline{A} = A + I/I, \quad \overline{B} = B + I/I, \quad \overline{C} = C + I/I,$$

$$\overline{B_0} = B_0 + I/I, \quad \overline{C_0} = C_0 + I/I.$$

Let $m = \dim \overline{A}/(\overline{B}_0 + \overline{C}_0)$. By the Definition 2 it holds $I \not\subseteq B_0 + C_0$ and hence $m < n$. Denote by $\overline{T}$ the sum of all ideals of the algebra $\overline{A}$ which lie in the F-subspace $\overline{B}_0 + \overline{C}_0$. The ideal $\overline{T}$ is an *NCF*-algebra by Lemma 7 and therefore the quotient algebra $\overline{A}/\overline{T}$ is not an *NCF*-algebra in view of the choice of A and Proposition 1.

Since the F-subspace $(\overline{B}_0 + \overline{C}_0)/\overline{T}$ does not contain nonzero ideals of the algebra $\overline{A}/\overline{T}$ and its codimension in $\overline{A}/\overline{T}$ is equal to m, $m < n$, it contradicts to the choice of A. The obtained contradiction proves that A/I is an *NCF*-algebra.

Now let J be a nonzero *NCF*-ideal of the algebra A. As has just been proved A/J is an *NCF*-algebra, and then A is an *NCF*-algebra by Proposition 1. The latest is impossible and hence A has not nonzero *NCF*-ideals. The proof is complete.

Lemma 9. *Let R be an associative ring which is decomposable into a sum $R = A + B$ of two subrings A and B and let R_0 be a subring of R with $R_0 \supseteq B$. If R_0 contains an ideal A_0 of the subring A then R_0 contains some ideal J of the ring R such that $J \supseteq A_0$.*

PROOF. Consider the subring $J = A_0 + BA_0 + A_0B + BA_0B$ of the ring R. Clearly, $J \subseteq R_0$ and $A_0 \subseteq J$. As A_0 is an ideal of the subring A, J obviously, is an ideal of the ring R.

Lemma 10. *Let $A = B+C$ be a minimal BM-counter-example where subalgebras B and C satisfy all conditions of the Definition 2 and let $A_1 = B + C_1$ ($C_1 \subseteq C$) be a subalgebra of A. If A_1 is not an NCF-algebra then for some NCF-ideal J of subalgebra A_1 the quotient algebra A_1/J is a minimal BM-counter-example.*

Proof. Let $B_0 \subseteq B$ and $C_0 \subseteq C$ be commutative ideals of the sublagebra B and corresponding C from the Definition 2 and let $n = \dim A/(B_0 + C_0)$. Denote by C_1' the subalgebra in C which is generated by C_1 and $C \cap B$. Clearly,

$$B + C_1 = B + C_1', \qquad C_1' \cap B = C \cap B,$$

and therefore one can assume, without loss of generality, that $C_1' = C_1$ and hence $C_1 \cap B = C \cap B$. It follows from the latest equality that $C_0 \cap A_1 = C_0 \cap C_1$. Indeed, let $x \in C_0 \cap A_1$. Then $x = c_0, c_0 \in C_0, x = b + c_1$ for some $c_1 \in C_1$, $b \in B$. Hence $b = c_0 - c_1 \in B \cap C = B \cap C_1$ and $x \in C_1$. But then $x \in C_0 \cap C_1$ and $C \cap A_1 \subseteq C_0 \cap C_1$, because the element x has been chosen in any way. The inclusion $C_0 \cap C_1 \subseteq C_0 \cap A_1$ is obvious.

Since $A_1 \cap (B_0 + C_0) \supseteq B_0$, it holds $A_1 \cap (B_0 + C_0) = B_0 + (C_0 \cap A_1)$ and therefore as proved above

$$A_1 \cap (B_0 + C_0) = B_0 + (C_0 \cap C_1).$$

Now denote by J the sum of all ideals of the algebra A_1 which lie in $B_0 + (C_0 \cap C_1)$. By Lemma 7 J is an *NCF*-ideal of the algebra A_1 and A_1/J is not an *NCF*-algebra (because A_1 is not an *NCF*-algebra). It is easy to see that A/J is a minimal BM-counter-example (in particular, $\dim A_1/((B_0 + C_0) \cap A_1) = n$). The proof is complete.

Lemma 11. *If I is an one-sided finite dimensional ideal of an associative algebra A then A has a nilpotent ideal J such that $(I + J)/J$ lies in some finite dimensional (two-sided) ideal of the quotient algebra A/J.*

Proof. Let I be for example a right ideal of the algebra A and $S = \mathrm{Ann}_A^r(I)$. Clearly, S is an ideal of A and $\dim A/S < \infty$. Further, $T = \mathrm{Ann}_A^l(S)$ is also an ideal of A, $T \supseteq I$ and $J = T \cap S$ is an ideal of the algebra A with $J^2 = 0$. It is easy to see that $\dim T/J < \infty$ and $I + J/J \subseteq T/J$. The case of a left ideal of A can be considered analogously. The Lemma is proved.

Lemma 12. *Let A be an associative algebra over an arbitrary field F and a is an element in A such that* $\dim A/\operatorname{Ann}^r_A(a) < \infty$ ($\dim A/\operatorname{Ann}^l_A(a) < \infty$). *Then the element a belongs to a finite dimensional right (corresponding left) ideal of the algebra A.*

Proof. Denote by C the right annulator $\operatorname{Ann}^r_A(a)$ and let $\dim A/C < \infty$. Choose a complete system of representatives $\{h_1, \dots , h_n\}$ of the congruence classes of A by C. We will show that F-subspace $I = \langle a, ah_1, \dots , ah_n\rangle$ is a right ideal of the algebra A. If g is any element in A then g is of the form $g = c + \sum_{i=1}^n \alpha_i h_i$ where $c \in C$, $\alpha_i \in F$, $i = 1, \dots , n$. Hence

$$ag = a\left(c + \sum_{i=1}^n \alpha_i h_i\right) = \sum_{i=1}^n \alpha_i a h_i \in I,$$

because $ac = 0$. Further, denote $h_i g = c_i + \sum_{j=1}^n \beta_{ij} h_j$ where $c_i \in C$, $\beta_{ij} \in F$, $i, j = 1, \dots , n$. Then we obtain

$$(ah_i)g = a(h_i g) = a\left(c_i + \sum_{j=1}^n \beta_{ij} h_j\right) = \sum_{j=1}^n \beta_{ij} a h_j \in I.$$

Therefore I is a right ideal of the algebra A, $a \in I$ and $\dim I < \infty$. Analogously, one can consider the case $\dim A/\operatorname{Ann}^l_A(a) < \infty$. The proof is complete.

Let A be a nilpotent algebra, $A \neq 0$. The number $n = n(A)$ such that $A^n = 0$, $A^{n-1} \neq 0$ will be called the index of nilpotency of A and denoted by $n(A)$. The index of nilpotency of the zero algebra we assume to be equal 1.

An associative algebra A will be called almost nilpotent if it has a nilpotent ideal of finite codimension. By $\bar{n}(A)$ will be denoted the smallest nilpotency index of all nilpotent ideals of A of finite codimension in A.

Lemma 13. *If I is a right (left) almost nilpotent ideal of an algebra A then A has a nilpotent ideal J such that $I + J/J$ is a finite dimensional right (corresponding left) ideal of A.*

Proof. Let I be, for example, a right ideal. Let B be any nilpotent ideal of the subalgebra I with $\dim I/B < \infty$ such that $n(B) = \bar{n}(I)$. If $\bar{n}(I) = 1$ then $\dim I < \infty$, that is $B = 0$, and Lemma is proved. First consider the case $\dim I/\operatorname{Ann}_I^r(g) < \infty$ for every element $g \in I$. Let the statement of Lemma be true for algebras with $\bar{n}(I) < k$, prove it for algebras with $\bar{n}(I) = k$. Choose a complete system of representatives $\{g_1, \dots, g_m\}$ of the congruence classes of I by B. Using Lemma 12 one can easy show that there exists a finite dimensional right ideal N of the algebra I with $\{g_1, \dots, g_m\} \subseteq N$. Obviously, $I = B + N$. Then $T = \operatorname{Ann}_B^r(N)$ is a nilpotent ideal of the subalgebra I and $\dim I/T < \infty$. Analogously, $I_0 = \operatorname{Ann}_I^r(I)$ is a nilpotent right ideal of the algebra A and $I_0 \supseteq T \cap B^{k-1}$. Then as is well known (see for example [1, Lemma 1.1.2]) I_0 lies in some nilpotent ideal S of the algebra A. At that the quotient algebra $\overline{A} = A/S$ has the right almost nilpotent ideal $\overline{I} = I + S/S$. Since $T \cap B^{k-1} \subseteq S$ then the ideal $\overline{T} = T + S/S$ of the subalgebra $\overline{I}$ has the nilpotency index $\leq k-1$ and therefore $\bar{n}(\overline{I}) \leq k-1$. By inductive assumption there exists in $\overline{A}$ some nilpotent ideal $\overline{J} = J/S$ such that $\overline{I} + \overline{J}/\overline{J}$ is a finite dimensional right ideal of the algebra $\overline{A}/\overline{J}$. Then J is a nilpotent ideal of the algebra A and $I + J/J$ is a finite dimensional right ideal of the quotient algebra A/J.

Now let $I_1 = \{i \in I \mid \dim I/\operatorname{Ann}_I^r(i) < \infty\}$. It is easy to see that I_1 is a right ideal of the algebra A, $I_1 \cap B$ is a nilpotent ideal in I_1 and $\dim I_1/(I_1 \cap B) < \infty$. Besides, $B^{k-1} \subseteq I_1$, because $B^{k-1}B = 0$ and $\dim I/B < \infty$. As has just been proved there exists in A a nilpotent ideal U such that $I_1 + U/U$ is a finite dimensional right ideal of the algebra A/U. One can assume without loss of generality (by Lemma 11) that the algebra A has a finite dimensional ideal M such that $M \supseteq B^{k-1}$. By inductive assumption (induction on $\overline{n}(I)$)the quotient algebra A/M contains a nilpotent ideal V/M such that $(I + V/M)/(V/M)$ is a finite dimensional right ideal of the algebra $(A/M)/(V/M)$. Let $V_1 = \operatorname{Ann}_V^r(M)$. It is easy to see that V_1 is a nilpotent ideal of the algebra A and $I + V_1/V_1$ is a finite dimensional right ideal of the algebra A/V_1.

The case of the left ideal I can be considered analogously. The Lemma is proved.

The statements below follow from Lemmas 11 and 13.

Corollary 2. *Let A be an associative algebra and I a right (left) almost nilpotent ideal of the algebra A. Then I lies in some almost nilpotent ideal of the algebra A.*

Corollary 3. *If an associative algebra A has an almost nilpotent ideal I with almost nilpotent quotient algebra A/I then the algebra A is almost nilpotent.*

Proof. One can assume, without loss of generality, that $\dim I < \infty$ (in view of Lemma 3). Let J/I be a nilpotent ideal of the quotient algebra A/I such that $\dim A/J < \infty$ and let $C = \mathrm{Ann}^r_J(I)$. Obviously, C is an ideal of the algebra A and $\dim J/C < \infty$ (and hence $\dim A/C < \infty$). Since $(C \cap I)^2 = 0$ and $C/(C \cap I) \simeq C + I/I$ is a nilpotent algebra then the ideal C is nilpotent. The proof is complete.

Proposition 2. *If an associative algebra A over an arbitrary field is decomposable into a sum $A = B + C$ with almost nilpotent subalgebras B and C of A then the algebra A is almost nilpotent.*

Proof. Let the statement of the Proposition be false. Choose among all counter-examples to the Proposition an algebra $A = B + C$ with the smallest sum $\bar{n}(B) + \bar{n}(C)$. Clearly, $\bar{n}(B) \geq 2$ and $\bar{n}(C) \geq 2$ (if, for example, $\bar{n}(B) = 1$ then $\dim B < \infty$ and therefore the algebra A is almost nilpotent in contradiction to the our assumption). Denote by B_0 and C_0 some nilpotent ideals of the subalgebra B and corresponding of the subalgebra C such that $\dim B/B_0 + \dim C/C_0 < \infty$ and $n(B_0) = \bar{n}(B)$, $n(C_0) = \bar{n}(C)$. Let $I = B_0^{\bar{n}(B)-1}$. It is easy to see that $IB_0 = B_0 I = 0$ and $A_0 = B + IC$ is a subalgebra from A of the form $A_0 = B + C_1$ where $C_1 = C \cap A_0$. Note that B_0 is a right nilpotent ideal of the subalgebra A_0. Then the right ideal B_0 lies as is well known in some (two-sided) nilpotent ideal S of the subalgebra A_0. The almost nilpotent subalgebra $C_1 + S/S$ is of finite codimension in A_0/S and by Lemma 1 the quotient algebra A_0/S is almost nilpotent. Then the algebra A_0 is almost nilpotent.

It is eeasy to see that $I + IC$ is a right ideal of the algebra A, and since $I + IC \subseteq A_0$, the subalgebra $I + IC$ is almost nilpotent. Further, $I + IC$ lies by Corollary 2 in some almost nilpotent ideal T of the algebra A. The quotient algebra $\overline{A} = A/T$ is decomposable into a sum $\overline{A} = \overline{B} + \overline{C}$ where $\overline{B} = B + T/T, \overline{C} = C + T/T$. Since $I \subseteq T$, we have $\bar{n}(\overline{B}) < \bar{n}(B)$ and therefore the quotient algebra A/T is almost nilpotent by choice of the algebra A. In view of Corollary 3 the algebra A is almost nilpotent. It contradicts to the choice of A and the proof is complete.

Lemma 14. *Let A be an associative algebra without nonzero NCF-ideals decomposable into a sum $A = B + C$ of subalgebras B and C which contain commutative ideals $B_0 \subseteq B$ and $C_0 \subseteq C$ such that $\dim B/B_0 + \dim C/C_0 < \infty$. If $A_1 = B + B_0C$ is an NCF-subalgebra of A then the subalgebra $C_1 = C \cap A_1$ is almost nilpotent.*

Proof. Since B_0 is an ideal of subalgebra B, the subspaces B_0C and A_1 are obvius subalgebras of A. Let $g = b + c$ be any element in the F-subspace $B_0C \cap (B_0 + C_0)$ where $b \in B_0, c \in C_0$. It is easy to see that $C_0c = cC_0$ is a two-sided ideal of the algebra C and cC_0 lies in the subalgebra $A_1 = B + B_0C = B + C_1$ Then there exists by Lemma 9 an ideal S of algebra A such that $cC_0 \subseteq S$ and $S \subseteq A_1$. Let A_1 be an *NCF*-subalgebra. Then S is an *NCF*-ideal of the algebra A and by conditions of Lemma $S = 0$. Hence $cC_0 = C_0c = 0$, that is, $c \in \mathrm{Ann}_{C_0}(C_0)$. In view of choice of the element $g = b + c$ this means $(B_0 + B_0C) \cap C_0 \subseteq \mathrm{Ann}_{C_0}(C_0)$. Further, it is easy to see that $\dim C_1/(A_1 \cap C_0) < \infty$ because $\dim C/C_0 < \infty$, and since $\dim B/B_0 < \infty$, we have

$$\dim C_1/((B_0 + B_0C) \cap C_0) < \infty.$$

Obviously, $(\mathrm{Ann}_{C_0}(C_0))^2 = 0$, hence $((B_0 + B_0C) \cap C_0)^2 = 0$ and therefore C_1 is an almost nilpotent subalgebra of the algebra C. The proof is complete.

Lemma 15. *Let $A = B+C$ be a minimal BM-counter-example where subalgebras B and C satisfy conditions of Definition 2. Then both subalgebras B and C are not almost nilpotent.*

Proof. Let the statement of Lemma be false. Then there exist minimal BM-counter-examples of the form $A = B + C$ such that one of the subalgebras B or C is almost nilpotent (by Proposition 2 both subalgebras B and C can not be almost nilpotent simultaneously). Choose among all such counter-examples an algebra $A = B + C$ with almost nilpotent subalgebra B which has the smallest number $n = \bar{n}(B)$. Let B_0' and C_0 be commutative ideals of finite codimension of the algebra B and corresponding C which satisfy conditions of Definition 2. Take a nilpotent ideal N of subalgebra B with $\dim B/N < \infty$ and $n(N) = \bar{n}(B)$ and set $B_0 = B_0' \cap N$. Obviously, B_0 is a commutative nilpotent ideal of finite codimension in B and $n(B_0) = \bar{n}(B)$. It is easy to see that $A_1 = B + B_0C$ is a subalgebra of A. Show that A_1 is an *NCF*-algebra. Denote $I = (B_0)^{\bar{n}(B)-1}$. Then I is a right nilpotent ideal of the subalgebra A_1 and hence I lies in certain nilpotent ideal S of A_1. The quotient algebra A_1/S is decomposable into a sum

$$A_1/S = (B+S)/S + (B_0C+S)/S$$

and $B_0^{\bar{n}(B)-2} + S/S$ is a right nilpotent ideal in A_1/S. Therefore $B_0^{\bar{n}(B)-2} + S/S$ lies in some nilpotent ideal S_1/S of the algebra A_1/S. Repeating this considering one can show that B_0 lies in some nilpotent ideal T of the algebra A_1. Since $\dim B/B_0 < \infty$, the quotient algebra A_1/T is an *NCF*-algebra by Corollary 1 and hence A_1 is an *NCF*-algebra. Obviously, $A_1 = B + C_1$ where $C_1 = C \cap A_1$. The subalgebra C_1 is almost nilpotent (see Lemma 14) and therefore the subalgebra A_1 is almost nilpotent by Proposition 2 as a sum of two almost nilpotent subalgebras B and C_1. Then the right ideal $B_0 + B_0C$ of the algebra A is almost nilpotent and lies by Corollary 2 in some almost nilpotent ideal T_1 of the algebra A. But $T_1 = 0$ by Lemma 8 and hence $B_0 = 0$. It follows from this $\dim A/C < \infty$ and A is an *NCF*-algebra in view of Corollary 1. This contradicts to the choice of algebra A. The proof is complete.

Lemma 16. *Let $A = B + C$ be a minimal BM-counter-example, let B_0 and C_0 be commutative ideals of the subalgebras B and corresponding C from Definition 2. Then $B + B_0C$ and $B + CB_0$ are subalgebras of A and at least one of these subalgebras is not an NCF-algebra.*

Proof. It is easy to see that $A_1 = B + B_0C$ and $A_2 = B + CB_0$ are subalgebras of A because B_0 is an ideal of the subalgebra B. One can immediately check up that B_0C, CB_0 and $A_0 = B + B_0C + CB_0 + CB_0^2C$ are also subalgebras of the algebra A. Suppose the Lemma is false and both subalgebras A_1 and A_2 are *NCF*-algebras. Obviously, it holds $A_1 = B + C_1$, $A_2 = B + C_2$ where $C_1 = C \cap A_1$ and $C_2 = C \cap A_2$. Denote $N_i = C_i \cap \operatorname{Ann}_{C_0}(C_0)$, $i = 1, 2$. Repeating the considerations from the proof of Lemma 14 one can show that $\dim C_i/N_i < \infty$, $i = 1, 2$. Obviously, N_i is an ideal of $C_i, N_i^2 = 0$, $i = 1, 2$. It is easy to see that $A_0 = B + C_3$ where $C_3 = C \cap A_0$. Show that C_3 is almost nilpotent. Since $A_0 = A_1 + A_2 + A_1A_2$, we have $A_0 = B + C_1 + C_2 + C_1C_2$. It follows from this equality that $C_3 = C_1 + C_2 + C_1C_2$. Really, the inclusion $C_1 + C_2 + C_1C_2 \subseteq C_3$ is obvious. Now let $c \in C_3 = C \cap A_0$ be any element. Then $c = b + x_1 + y_1 + \sum_{i=2}^{k} x_iy_i$ where $x_i \in C_1$, $y_i \in C_2$, $i = 1, \dots, k$.

Then $b \in C$ and hence $b \in C \cap B$. Since $C \cap B \subseteq C_1$ then $c \in C_1 + C_2 + C_1C_2$ and therefore $C_3 \subseteq C_1 + C_2 + C_1C_2$, because the element c was chosen in any way. Thus $C_3 = C_1 + C_2 + C_1C_2$.

Choose a complete system of representatives $\{x_1, \dots, x_m\}$ of the congruence classes of C_1 by N_1 and analogous system $\{y_1, \dots, y_n\}$ of C_2 by N_2. Then

$$N_3 = N_1 + N_2 + \sum_{i=1}^{m} x_i N_2 + \sum_{j=1}^{n} y_j N_1 \subseteq \mathrm{Ann}_{C_0}(C_0),$$

and obviously $\dim C_3/N_3 < \infty$. Since $N_3^2 = 0$, the subalgebra C_3 of A_0 is almost nilpotent.

Show that $A_0 = B + C_3$ is not an *NCF*-subalgebra. Really, let A_0 be conversely an *NCF*-algebra. Then $J = B_0^2 + CB_0^2 + B_0^2C + CB_0^2C$ is an ideal of the algebra A which lies in A_0 and hence J is an *NCF*-ideal. By the conditions of the present Lemma and by Lemma 8 $J = 0$ and hence $B_0^2 = 0$. As a sum of two almost nilpotent subalgebras B and C_3 the subalgebra A_0 is almost nilpotent by Proposition 2. But then $B_0 + B_0C (\subseteq A_0)$ is an almost nilpotent right ideal of the algebra A. By Corollary 2 $B_0 + B_0C$ lies in some almost nilpotent ideal of the algebra A. In view of conditions of this Lemma and Lemma 8 the latest ideal equals zero and hence $B_0 = 0$. Then obviously $\dim A/C < \infty$ and A is an *NCF*-algebra by Corollary 1. This contradicts to the conditions of Lemma and hence $A_0 = B + C_3$ is not an *NCF*-algebra. By Lemma 10 the quotient algebra A_0/J_0 is a minimal BM-counter-example for a some *NCF*-ideal J_0. On other hand $A_0/J_0 = (B + J_0)/J_0 + (C_3 + J_0)/J_0$ where the subalgebra $C_3 + J_0/J_0$ is almost nilpotent and therefore A_0/J_0 is not an BM-counter-example by Lemma 15. The obtained contradiction proves the statement of Lemma.

Proof of the Theorem. Let the statement of the Theorem be false. Choose among all counter-examples to the Theorem a such associative algebra $A = B + C$ over a field F which is not *NCF*-algebra and its F-subspace $B_0 + C_0$ is of the smallest codimension in A where B_0 and C_0 are commutative ideals of subalgebras B and corresponding C. Denote by J_0 the sum of all ideals of the algebra A which lie in $B_0 + C_0$. Then J_0 is an *NCF*-ideal by Lemma 7 and A/J_0 is obviously a BM-counter-example. Thus one can assume, without loss of generality, that $J = 0$ and A is a minimal BM-counter-example.

Note that $A_1 = B + B_0C$ and $A_2 = B + CB_0$ are subalgebras of A, and by Lemma 16 at least one of these subalgebras is not an *NCF*-algebra. Let A_1, for example, be not an *NCF*-algebra. Then the quotient algebra A_1/J_1 for some *NCF*-ideal J_1 of A_1 is a minimal BM-counter-example by Lemma 10. Denote $I_1 = \mathrm{Ann}_{B_0}(B_0)$. Obviously, I_1 is a right nilpotent ideal of the subalgebra $A_1 = B + B_0C$. Then I_1 lies in some nilpotent ideal S_1 of this algebra, and since the quotient algebra A_1/J_1 has not nonzero *NCF*-ideals (see Lemma 8), $S_1 \subseteq J_1$ and hence $I_1 \subseteq J_1$. We have $[B_0, B] \subseteq B_0$ (B_0 is a commutative ideal in B) and repeating the consideration from the proof of Lemma 2 one can show that $[B_0, B] \subseteq \mathrm{Ann}_{B_0}(B_0) = I_1$. But then A_1/J_1 is a sum of almost commutative subalgebra $C_1 + J_1/J_1$ and finite dimensional over its center subalgebra $B + J_1/J_1$. Therefore we can assume, without loss of generality, that in the initial minimal BM-counter-example $A = B + C$ holds the inclusion $B_0 \subseteq Z(B)$ (otherwise we can replace A by A_1/J_1).

Now consider the right ideal $D_0 = \mathrm{Ann}_B(B_0)$ of the subalgebra $A_1 = B + B_0C$ (which is not an *NCF*-algebra by our choice). It is easy to see that $T_0 = D_0 + A_1D_0$ is an ideal of the algebra A_1 and $T_0 \subseteq \mathrm{Ann}^l_{A_1}(B_0)$. Further, denote $I_0 = \mathrm{Ann}^r_{A_1}(T_0)$. Obviously, I_0 is an ideal of the subalgebra A_1, $I_0 \supseteq B_0$ and $(I_0 \cap T_0)^2 = 0$. The quotient algebra $A_1/(I_0 \cap T_0)$ is not an *NCF*-algebra, because in the contrary case the algebra A_1 were also an *NCF*-algebra (in view of nilpotency of the ideal $I_0 \cap T_0$). The latest is impossible. The quotient algebra A_1/I_0 contains an almost commutative subalgebra $C_1 + I_0/I_0$ of finite codimension in A_1/I_0 (because $I_0 \supseteq B_0$) and therefore by Corollary 1 A_1/I_0 is an *NCF*-algebra. Then the quotient algebra $\overline{A}_1 = A_1/T_0$ is not an *NCF*-algebra, because in the contrary case the algebra $A_1/(I_0 \cap T_0)$ were also an *NCF*-algebra in view of embedding $A_1/(I_0 \cap T_0)$ into the product $(A_1/I_0) \times (A_1/T_0)$ and by Proposition 1. Note that $[B, B] \subseteq D_0 \subseteq T_0$. Really, we have for any elements $b_1, b_2 \in B$ and $b_0 \in B_0$

$$(b_1b_2 - b_2b_1)b_0 = b_1b_2b_0 - b_2b_1b_0 = b_1(b_2b_0) - (b_2b_0)b_1 = 0$$

because $b_2b_0 \in B_0 \subseteq Z(B)$. Hence the quotient algebra $\overline{A}_1 = A_1/T_0$ is a sum of the commutative subalgebra $\overline{B} = B + T_0/T_0$ and almost commutative subalgebra $\overline{C}_1 = C_1 + T_0/T_0$ where $C_1 = C \cap A_1$. It easy to see that certain quotient algebra $\overline{A}_1/\overline{S}_1$ is a minimal BM-counter-example for some *NCF*-ideal $\overline{S}_1$ from $\overline{A}_1$. We can assume, without loss of generality, that the subalgebra B in original BM-counter-example $A = B + C$ is commutative. Repeating the above considerations in respect to one of the subalgebras $C + C_0B$ or $C + BC_0$ we can show that there exist minimal BM-counter-examples of the form $A = B + C$ with commutative subalgebras B and C. It is impossible in view of [2] (see Lemma 6). The obtained contradiction proves the Theorem.

Remark. Let $A = B + C$ be the associative algebra from the main theorem and $B_0 \subseteq B$, $C_0 \subseteq C$ be some commutative ideals of B and respectively C of finite codimensions $p = \dim B/B_0$, $q = \dim C/C_0$. By this theorem the algebra A contains a nilpotent ideal I with almost commutative quotient algebra A/I. Let K/I be any commutative ideal of A/I of finite codimension. Then using the main theorem and Lemma 6 one can show that there exist two functions $f(x, y)$ and $g(x, y)$ such that the nilpotency index $n(I) \leq f(p, q)$ and $\dim A/K \leq g(p, q)$.

The author is grateful to the referee of this paper for his outpointing at the existence of the estimations from the Remark.

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(Received June 19, 1997 revised September 24, 1997)