

## More on finite rank elements

By GORAZD LEŠNJAK (Maribor)

**Abstract.** Recently, B. AUPETIT and H. DU T. MOUTON have shown that the trace and determinant on a semisimple Banach algebra could be defined in a purely spectral and analytic way. Here, these notions are based on an equivalent approach using the representation theory which was given by M. BREŠAR and P. ŠEMRL. Applying their point of view it is possible to extend several (in)equalities on rank, trace and determinant.

### 1. Introduction

Different concepts of finite dimensional or finite rank elements of Banach algebras have been introduced by several authors ([7], [1], [6], [5], [4], ...). AUPETIT and MOUTON [2] have recently given the definition of rank for elements of a unital semisimple complex Banach algebra  $\mathcal{A}$ . They proved that the subset of  $\mathcal{A}$  consisting of all elements with finite rank coincides with the socle of  $\mathcal{A}$ , i.e. the sum of all minimal left (respectively, of all minimal right) ideals of  $\mathcal{A}$ . We denote this set by  $\text{soc}(\mathcal{A})$ . If there are no minimal one-sided ideals in  $\mathcal{A}$  then by convention  $\text{soc}(\mathcal{A}) = \{0\}$ . In their paper [3] BREŠAR and ŠEMRL gave the following definition of rank which enabled them to unify different approaches by proving their equivalence.

*Definition.* The element  $0 \in \text{soc}(\mathcal{A})$  has rank 0. An element  $a \in \text{soc}(\mathcal{A}) \setminus \{0\}$  has rank one if it belongs to some minimal left ideal of  $\mathcal{A}$ . An element  $a \in \text{soc}(\mathcal{A})$  has rank  $n > 1$  if it belongs to the sum of  $n$  minimal

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left ideals, but does not belong to any sum of less than  $n$  minimal left ideals.

In order to clarify the structure of finite rank elements they applied the representation theory. The following shortened version of the main theorem in [3] is the starting point for our considerations.

**Theorem 1.** *Let  $n \in \mathbb{N}$ . For an element  $a$  in a semisimple unital complex Banach algebra  $\mathcal{A}$  the following assertions are equivalent:*

- (A)  $a$  has rank  $n$ ;
- (B) the left ideal  $\mathcal{A}a$  is a sum of  $n$  minimal left ideals, but is not a sum of less than  $n$  minimal left ideals of  $\mathcal{A}$ ,
- (C)  $\sigma(xa)$  contains at most  $n$  nonzero points for every  $x \in \mathcal{A}$  and there exists  $x_0 \in \mathcal{A}$  such that  $\sigma(x_0a)$  contains  $n$  nonzero points;
- (D)  $a$  satisfies
  - (1) there exists primitive ideals  $P_1, \dots, P_k$  ( $P_i \neq P_j$  whenever  $i \neq j$ ) of  $\mathcal{A}$  such that  $a \in P$  for every primitive ideal  $P \notin \{P_1, \dots, P_k\}$ ,
  - (2) if  $\pi_i, i = 1, \dots, k$ , are continuous irreducible representations of  $\mathcal{A}$  on Banach spaces  $X_i$  such that  $\text{Ker } \pi_i = P_i$ , then  $\pi_i(a)$  are finite rank operators and  $n = \text{rank } \pi_1(a) + \dots + \text{rank } \pi_k(a)$ .

Here we wrote  $\sigma(a)$  for the spectrum of  $a$ . Recall that an element  $a$  of an algebra  $\mathcal{B}$  is *indecomposable* if it cannot be written in the form  $a = b + c$  with nonzero elements  $b$  and  $c$  of  $\mathcal{B}$  satisfying  $b\mathcal{B}c = 0$ . In [3] it is shown that a nonzero indecomposable element of  $\text{soc}(\mathcal{A})$  belongs to all primitive ideals of  $\mathcal{A}$  except one. The last assertion of the above theorem and Lemma 2.6 in [3] can be combined in the following characterization.

**Proposition 2.** *An element  $a$  in a semisimple unital complex Banach algebra  $\mathcal{A}$  has rank  $n$  if and only if there exist (unique) indecomposable nonzero elements  $a_1, a_2, \dots, a_k$  in  $\mathcal{A}$  such that*

- i)  $a = a_1 + a_2 + \dots + a_k$ ,
- ii)  $a_i \mathcal{A} a_j = 0$  whenever  $i \neq j$ ,
- iii)  $\pi_i(a) = \pi_i(a_i)$  is a finite rank operator for  $i = 1, 2, \dots, k$ , with  $\pi_i$  being the continuous irreducible representation of  $\mathcal{A}$  on a Banach space  $X_i$  such that  $\text{Ker } \pi_i = P_i$  is the primitive ideal not containing  $a_i$ .
- iv)  $n = \text{rank } \pi_1(a_1) + \text{rank } \pi_2(a_2) + \dots + \text{rank } \pi_k(a_k)$ .

AUPETIT and MOUTON [2] defined the rank by the condition (C) in Theorem 1. In the same article they gave definitions of the trace of  $a$  and

the determinant of  $1 + a$  for an element  $a \in \text{soc}(\mathcal{A})$  by introducing first the multiplicity of an element in  $\sigma(a)$ . Then they deduced the validity of basic properties which are known to hold true for these notions in matrix algebras. For example, they verified the subadditivity of rank ([2], Theorem 2.14), the additivity of trace and the multiplicity of determinant ([2], Theorem 3.3). Their intricate arguments are based on the fact that spectrum is an analytic multifunction.

It is the aim of this note to prove and extend their results by exploiting the above definition and the representation given by the above theorem. For example, the subadditivity of rank is an immediate consequence of the definition. As this paper rests heavily on the results in [3] we follow the notation given there wherever it is possible.

### 2. Preliminaries

We begin with some definitions and basic facts from [2]. The rank of an element  $a$  in  $\text{soc}(\mathcal{A})$  will be denoted by  $\text{rank}(a)$  and the cardinality of the set  $S$  by  $\#S$ .

Let us fix an element  $a \in \text{soc}(\mathcal{A})$  with  $\text{rank}(a) = n$ . In [2], Theorem 2.2, it is shown that the set  $E(a) = \{x \in \mathcal{A} : \#\sigma(xa) \setminus \{0\} = n\}$  is a dense open subset of  $\mathcal{A}$ . Moreover, for any open subset  $\Delta$  of  $\mathbb{C}$  whose boundary is an oriented regular contour  $\Gamma$  satisfying  $\Gamma \cap \sigma(a) = \emptyset$  there exists a neighbourhood  $U$  of  $1 \in \mathcal{A}$  such that the finite number  $\#\sigma(xa) \cap \Delta$  is the same for all  $x \in E(a) \cap U$ . In the case when  $\Delta$  is a disc with  $\alpha \in \sigma(a)$  as its centre and diameter small enough this value is called (see [2]) the *multiplicity* of  $a$  at  $\alpha$  and denoted by  $m(\alpha, a)$ . To simplify calculations it will be convenient to have  $m(\alpha, a) = 0$  for an  $\alpha \notin \sigma(a)$ .

*Definition* [2]. The trace of an element  $a \in \text{soc}(\mathcal{A})$  is the number

$$\text{tr}(a) = \sum_{\lambda \in \sigma(a)} \lambda m(\lambda, a)$$

and the determinant of  $1 + a$  is given by

$$\det(1 + a) = \prod_{\lambda \in \sigma(a)} (1 + \lambda)^{m(\lambda, a)}.$$

Let us return to Proposition 2 above and fix an index  $i \in \{1, 2, \dots, k\}$ . The operator  $\pi_i(a) = \pi_i(a_i)$  in  $\mathcal{B}(X_i)$  has finite rank  $r_i$ . Thus, in every

direct sum decomposition  $X_i = U_i \oplus V_i$  with  $U_i$  finite dimensional, containing  $\text{Im}(\pi_i(a_i))$  and such that  $\text{Ker}(\pi_i(a_i)) \supset V_i$ , the subspace  $U_i$  is  $\pi_i(a_i)$ -invariant and closed. Hence, the restriction  $A_i = \pi_i(a_i)|_{U_i}$  is an endomorphism of the finite dimensional vector space  $U_i$  satisfying  $\sigma(A_i) \setminus \{0\} = \sigma(\pi_i(a_i)) \setminus \{0\}$ . It can be identified with an appropriate matrix, say  $[A_i]$ , which depends on the basis chosen in  $U_i$ . The rank (spectrum, trace, determinant) of this operator is equal to the rank (spectrum, trace, determinant) of the corresponding matrix with no regard to the basis of  $U_i$ . If it suits our purposes we can replace  $U_i$  with any finite dimensional subspace  $W_i$  of  $X_i$  containing  $U_i$ .

Recall that for an endomorphism  $A$  of a finite dimensional complex linear space (or any corresponding matrix  $[A]$ ) the *algebraic multiplicity*  $n(\lambda, A)$  of an element  $\lambda$  in  $\sigma(A)$  (or in  $\sigma([A])$ ) is equal to the exponent at the factor  $(t - \lambda)$  in its characteristic polynomial. Let us define  $n(\lambda, A) = 0$  for  $\lambda \notin \sigma(A)$ . We write  $\text{Tr}(A)$  and  $\text{Det}(I + A)$  for the usual trace of  $A$  and the usual determinant of  $I + A$  where  $I$  stands for the identity.

**Proposition 3.** *Let  $a$  be an element of the socle of  $\mathcal{A}$  and  $a = a_1 + \dots + a_k$  its unique decomposition as given in the Proposition 2. Then the following equalities hold:*

- (a)  $\text{rank}(a) = \sum_i \text{rank}(a_i) = \sum_i \text{rank}(\pi_i(a_i)) = \sum_i \text{rank}(A_i) = \sum_i \text{rank}[A_i]$ ,  
 (b) for each nonzero  $\lambda$  in  $\sigma(a)$  one has

$$\begin{aligned} m(\lambda, a) &= \sum_i m(\lambda, a_i) = \sum_i m(\lambda, \pi_i(a_i)) = \sum_i m(\lambda, A_i) \\ &= \sum_i m(\lambda, [A_i]) = \sum_i n(\lambda, [A_i]) = \sum_i n(\lambda, A_i), \end{aligned}$$

- (c)  $\text{tr}(a) = \sum_i \text{tr}(a_i) = \sum_i \text{tr}(\pi_i(a_i)) = \sum_i \text{Tr}(A_i) = \sum_i \text{Tr}[A_i]$ ,  
 (d) the determinant of  $1 + a$  is equal to the following products:

$$\prod_i \det(1 + a_i) = \prod_i \det(1 + \pi_i(a_i)) = \prod_i \text{Det}(I + A_i) = \prod_i \text{Det}([I + A_i]).$$

PROOF. In each assertion the equality at the right hand side is well known. Also, the equalities in (a) are rewritten from the above proposition.

To proceed, take any  $x \in E(a)$ . Then  $xa = x(a_1 + \dots + a_k) = xa_1 + \dots + xa_k$  is the decomposition of  $xa$  guaranteed by Proposition 2. By Theorem 2.2 in [3] it follows that  $\sigma(xa) \setminus \{0\} = \bigcup_i (\sigma(xa_i) \setminus \{0\})$ . This

further implies  $\text{rank}(a) = \#\sigma(xa) \setminus \{0\} \leq \sum_i \#\sigma(xa_i) \setminus \{0\} \leq \sum_i \text{rank}(a_i) = \text{rank}(a)$  which shows that  $x \in E(a_i)$  for all  $i \in \{1, \dots, k\}$  and that the sets  $\sigma(xa_i) \setminus \{0\}$  are disjoint.

Write  $\Delta_r(z)$  for the open disc of radius  $r$  with its centre at  $z \in \mathbb{C}$  and let  $0 < \rho < \frac{1}{2} \min\{|\alpha - \beta| : \alpha, \beta \in \sigma(a) \cup \{0\}\}$ . From Theorem 2.4 in [2] we deduce the existence of an open neighbourhood  $U$  of 1 such that for every  $x \in E(a) \cap U$  one has  $\sigma(xa) \subset \bigcup_{\lambda \in \sigma(a)} \Delta_\rho(\lambda)$  and for every  $\lambda \in \sigma(a) \setminus \{0\}$  the equality  $m(\lambda, a) = \#\sigma(xa) \cap \Delta_\rho(\lambda)$  holds true. Again we have  $\sigma(xa) \cap \Delta_\rho(\lambda) = \bigcup_i (\sigma(xa_i) \cap \Delta_\rho(\lambda))$  for each nonzero  $\lambda$  in  $\sigma(a)$ . The sets on the right hand side are disjoint, thus  $\#\sigma(xa) \cap \Delta_\rho(\lambda) = \sum_i \#(\sigma(xa_i) \cap \Delta_\rho(\lambda))$  which shows that  $m(\lambda, a) = \sum_i m(\lambda, a_i)$ . It follows directly from definitions that for every  $i \in \{1, \dots, k\}$  the multiplicities  $m(\lambda, a_i)$ ,  $m(\lambda, \pi_i(a_i))$ ,  $m(\lambda, A_i)$  and  $m(\lambda, [A_i])$  coincide for nonzero  $\lambda$ .

It remains to prove  $m(\lambda, T) = n(\lambda, T)$  for any matrix  $T$  and any nonzero  $\lambda$ . We can assume that  $T$  is an upper triangular Jordan matrix. Moreover, suppose that  $\lambda_1, \dots, \lambda_k$  are all its nonzero eigenvalues with algebraic multiplicities  $n(\lambda_j, T) = n_j$  for  $1 \leq j \leq k$ . Arrange corresponding Jordan blocks in the upper  $n = n_1 + \dots + n_k$  rows. For small positive  $r$  we now define  $M_r$  to be a block-diagonal matrix having the following form. Upper  $n$  rows have only diagonal terms different from zero:

$$m_{j,j} = \begin{cases} 1 + r \exp(2\pi i j/n) / \lambda_1, & 1 \leq j \leq n_1, \\ 1 + r \exp(2\pi i j/n) / \lambda_2, & n_1 < j \leq n_1 + n_2, \\ \dots, & \dots \\ 1 + r \exp(2\pi i j/n) / \lambda_k, & n - n_k < j \leq n. \end{cases}$$

The structure of  $M_r$  below these rows corresponds to the Jordan cells of  $T$  having zero diagonal, if there are any. Let  $n_0 = \text{rank } T - n$  stands for the number of nonzero rows in these cells of  $T$ . With each of these Jordan cells of  $T$  we associate a block of the same size and at the same location, with nonzero elements only on the first subdiagonal. If  $p$  is the index of the  $n + j$ -th nonzero row of  $T$  we define  $m_{p+1,p} = r \exp(2\pi i j/n_0)$ . It is easy to verify that for  $r$  small enough the product  $M_r T$  is upper triangular matrix of the same rank as  $T$  having rank  $T$  distinct nonzero eigenvalues. Hence, matrices  $M_r$  belong to  $E(T)$  for small values of  $r$ . For  $r < \rho < \frac{1}{2} \min\{|\alpha - \beta| : \alpha, \beta \in \sigma(T) \cup \{0\}\}$  and  $1 \leq j \leq k$  we have

$$m(\lambda_j, T) = \#\sigma(M_r T) \cap \Delta_\rho(\lambda_j) = n_j = n(\lambda_j, T),$$

which finishes the proof of assertion (b).

The trace of a matrix is the sum of all its diagonal entries and it is well known that it is equal to the sum of all its eigenvalues repeated according to their algebraic multiplicities. For nonzero elements in the spectrum of a matrix  $T$  these are equal to the numbers  $m(\lambda, T)$ . Thus, the trace  $\text{Tr } T$  of this matrix is equal to its trace  $\text{tr } T$  given by the definition above. This fact together with the following computation proves assertion (c):

$$\begin{aligned} \text{tr}(a) &= \sum_{\lambda \in \sigma(a)} \lambda m(\lambda, a) = \sum_{\lambda} \lambda \sum_{1 \leq i \leq k} m(\lambda, a_i) \\ &= \sum_i \sum_{\lambda} \lambda m(\lambda, a_i) = \sum_i \text{tr}(a_i). \end{aligned}$$

The last assertion follows easily from the fact that for any matrix its determinant is equal to the product of its eigenvalues repeated according to their algebraic multiplicities. For a matrix of the form  $I + T$  this gives the equality  $\text{Det}(I + T) = \prod_{\lambda \in \sigma(T)} (1 + \lambda)^{n(\lambda, T)}$ . But this expression is equal to  $\prod_{\lambda \in \sigma(T)} (1 + \lambda)^{m(\lambda, T)} = \det(I + T)$ . The rest of the proof is a simple verification based on the assertion (a):

$$\begin{aligned} \det(1 + a) &= \prod_{\lambda \in \sigma(a)} (1 + \lambda)^{m(\lambda, a)} = \prod_{\lambda} (1 + \lambda)^{\sum_i m(\lambda, a_i)} \\ &= \prod_{\lambda} \prod_i (1 + \lambda)^{m(\lambda, a_i)} = \prod_i \prod_{\lambda} (1 + \lambda)^{m(\lambda, a_i)} = \prod_i \det(1 + a_i). \quad \square \end{aligned}$$

The main theorem of [3] and the proposition above justify viewing elements of  $\text{socle}$  as finite direct sums of finite rank endomorphisms or just as finite direct sums of matrices. These can be identified in a natural way with block diagonal matrices.

### 3. Ranks, traces and determinants

The importance of matrix calculus in applied mathematics is just one of the many reasons why there are so many equalities and inequalities involving ranks, traces and determinants of matrices. Moreover, some of these relations inspired great efforts to prove their validity in more general settings. The propositions above allow us to extend these relations to the  $\text{socle}$  of  $\mathcal{A}$ . To this end, we first define  $\sim$  to be a common substitution for any of the symbols  $=$ ,  $>$  and  $\geq$ .

**Theorem 4.** *Let  $\mathcal{A}$  be a unital semisimple complex Banach algebra,  $n \in \mathbb{N}$  and  $F$  a polynomial in  $m + i + j \in \mathbb{N}$  variables. Furthermore, let  $p_1, \dots, p_m, q_1, \dots, q_i, r_1, \dots, r_j$  be polynomials with zero constant term in  $n$  noncommuting variables and denote by  $f_1, \dots, f_i, g_1, \dots, g_j$  entire functions having zero at the origin. The relation*

$$F(\text{rank } p_1(\mathbf{a}), \dots, \text{rank } p_m(\mathbf{a}), f_1(\text{tr } q_1(\mathbf{a})), \dots, f_i(\text{tr } q_i(\mathbf{a})), \det(1 + g_1(r_1(\mathbf{a}))), \dots, \det(1 + g_j(r_j(\mathbf{a})))) \sim 0$$

is valid for all  $n$ -tuples  $\mathbf{a} = (a_1, a_2, \dots, a_n)$  of elements in the socle of  $\mathcal{A}$  if and only if the same relation holds true for all  $n$ -tuples of square matrices of the same size with no regard to their size.

PROOF. Suppose that a relation of the form above holds for square matrices and take any elements  $a_1, \dots, a_n$  of the  $\text{soc}(\mathcal{A})$ . Let  $\mathcal{P} = \{P_1, \dots, P_k\}$  be the set of all primitive ideals in  $\mathcal{A}$  with the property that for each of them at least one of  $a_1, \dots, a_n$  does not belong to it. Denote by  $\pi_1, \dots, \pi_k$  the corresponding continuous irreducible representations on some Banach spaces  $X_1, \dots, X_k$ . According to Proposition 2, elements  $a_1, \dots, a_n$  can be decomposed in finite sums of indecomposable elements with uniquely determined nonzero components. Hence, for  $p = 1, \dots, n$  we can write  $a_p = a_{p,1} + \dots + a_{p,k}$  knowing that for each  $i \in \{1, \dots, k\}$  the endomorphism  $\pi_i(a_p) = \pi_i(a_{p,i})$  of the Banach space  $X_i$  has a finite rank. Let  $X_i = U_i \oplus V_i, i = 1, \dots, k$ , be direct sum decompositions with  $U_i$  finite dimensional, containing all the images  $\text{Im}(\pi_i(a_{p,i}))$  for  $p = 1, \dots, n$  and such that  $V_i \subset \bigcap_p \text{Ker}(\pi_i(a_{p,i}))$ . All these subspaces are  $\pi_i(a_{p,i})$ -invariant and we can proceed as in Preliminaries to obtain endomorphisms of the finite dimensional spaces  $U_i$  representing our elements. Furthermore, if bases are chosen in all these spaces we get the corresponding matrices.

Let  $p$  be a polynomial in  $n$  noncommuting variables with zero constant term. Then  $p(\mathbf{a})$  belongs to the socle of  $\mathcal{A}$ . Homomorphisms of algebras commute with polynomials in one or several variables, hence for each  $i \in \{1, \dots, k\}$  we have  $\pi_i(p(\mathbf{a})) = p(\pi_i(a_1), \dots, \pi_i(a_n)) = p(\pi_i(a_{1,i}), \dots, \pi_i(a_{n,i}))$ . This further implies that the spaces  $U_i$  are invariant for elements of the form  $\pi_i(p(\mathbf{a}))$  and, consequently, also for entire functions of such elements. Moreover, if  $f$  is an entire function with  $f(0) = 0$ , the matrix corresponding to the endomorphism  $\pi_i(f(p(\mathbf{a})))|_{U_i}$  is equal to the matrix  $f(p(\pi_i(a_1)|_{U_i}, \dots, \pi_i(a_n)|_{U_i}))$ . It follows that  $f(p(\mathbf{a}))$  can be

represented by a direct sum of matrices, each of them representing the same function of a  $n$ -tuple of the corresponding indecomposable parts of elements in  $\mathbf{a}$ .

By Proposition 2 the values of ranks, traces and determinants for the elements of socle coincide with the values of ranks, traces and determinants of the corresponding direct sums of matrices. Hence, if a relation of the form above is valid for square matrices, then it holds also for direct sums of matrices and, consequently, for the elements of socle which they are representing.

The reverse implication is trivial. □

The relations in the following corollary are well known for matrices. The proof that they are valid for finite rank elements of unital semisimple algebras has already been given in [2]. Here, we get them by substituting some simple polynomials of one or two variables and entire functions  $f(z) = z$  or  $f(z) = e^z - 1$  in the general form considered above.

**Corollary 5.** *Let  $a$  and  $b$  be elements in the socle of a unital semisimple complex Banach algebra  $\mathcal{A}$ . Then the following equalities and inequalities are valid:*

- (a)  $\text{rank}(a + b) \leq \text{rank}(a) + \text{rank}(b)$ ,
- (b)  $\text{tr}(a + b) = \text{tr}(a) + \text{tr}(b)$ ,
- (c)  $\text{tr}(ab) = \text{tr}(ba)$ ,
- (d)  $\det(e^a) = e^{\text{tr}(a)}$ ,
- (e)  $\det((1 + a)(1 + b)) = \det(1 + a) \det(1 + b)$ ,
- (f)  $\det(e^{a+b}) = \det(e^a) \det(e^b) = \det(e^a e^b)$ .

At the end let us point out that relations involving also spectral radius but of more general form (see [2], p. 130 and p. 134) as considered here could be derived from their validity for square matrices.

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GORAZD LEŠNJAK  
FACULTY OF ELECTRICAL ENGINEERING AND COMPUTER SCIENCE  
UNIVERSITY OF MARIBOR  
SMETANOVA UL. 17  
2000 MARIBOR, P.O. BOX 218  
SLOVENIA

*E-mail:* Gorazd.Lesnjak@Uni-Mb.Si

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